Pattern-avoiding polytopes and Bruhat orders I

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Introduction to polytopes

Pattern-avoiding Birkhoff polytopes and weak Bruhat order

The dimension of $B_n(132, 312)$

A *polytope* is the convex hull of (smallest convex body containing) a set of points $v_1, \ldots, v_k \in \mathbb{R}^n$, written

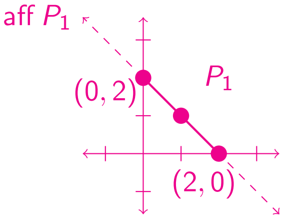
$$P = \text{conv}\{v_1, \ldots, v_k\}.$$

All our polytopes will be *integral*, meaning $v_1, \ldots, v_k \in \mathbb{Z}^n$.

(1) Dimension. The *affine span* of P, aff P, is the smallest affine subspace containing P. The *dimension* of P is

$$\dim P = \dim \text{aff } P.$$

Ex. If $v_1 = (2, 0)$ and $v_2 = (0, 2)$ then $P_1 = \text{conv}\{v_1, v_2\}$ is



So $\dim P_1 = 1$.

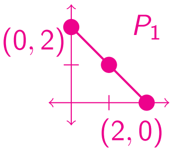
(2) Volume. The *(relative) volume* of polytope P is

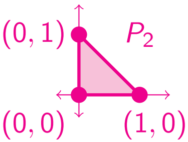
$$\operatorname{vol} P = \text{volume with respect to the lattice } \mathbb{Z}^n \cap \operatorname{aff} P.$$

A *simplex* is $\Sigma = \operatorname{conv}\{v_1, \ldots, v_{k+1}\}$ with $\dim \Sigma = k$. Call Σ *unimodular* if $\operatorname{vol} \Sigma$ is minimum with respect to $\mathbb{Z}^n \cap \operatorname{aff} \Sigma$. A unimodular simplex has volume $\operatorname{vol} \Sigma = 1/(dim \Sigma)!$ The *normalized volume* of polytope P is

$$\operatorname{Vol} P = (\dim P)! \operatorname{vol} P.$$

Ex. Let P_1 be as before and $P_2 = \operatorname{conv}\{(0,0),(1,0),(0,1)\}$.





So $\operatorname{vol} P_1 = 2$, and $\operatorname{vol} P_2 = 1/2$. Both P_i are simplices with P_2 unimodular and P_1 not. Also $\operatorname{Vol} P_1 = 2$ and $\operatorname{Vol} P_2 = 1$.

(3) h^*-polynomials. The *mth dilate* of polytope P is

$$mP = \{mv \mid v \in P\}.$$

The *Ehrhart polynomial* of P is

$$\mathcal{L}_P(m) = |mP \cap \mathbb{Z}^n|.$$

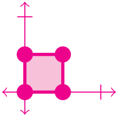
Theorem (Ehrhart-Stanley)

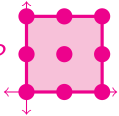
If P is integral then $\mathcal{L}_P(m)$ is a polynomial in m and for some d

$$\sum_{m \geq 0} \mathcal{L}_P(m) t^m = \frac{\sum_{j=0}^{d} h_j^* t^j}{(1-t)^{\dim P + 1}}$$

where $\sum_j h_j^* t^j \in \mathbb{Z}_{\geq 0}[t]$ is called the h^*-polynomial of P, $h^*(P;t)$.

Ex. Let $P = \text{conv}\{(0,0),(1,0),(0,1),(1,1)\}$.

P 

$2P$ 

So $\mathcal{L}_P(m) = (m+1)^2$.

Let $\mathfrak{S}_n$ be the nth symmetric group. If $\sigma = \sigma_1 \ldots \sigma_n \in \mathfrak{S}_n$ and $\pi = \pi_1 \ldots \pi_k \in \mathfrak{S}_k$ then σ *contains the pattern* π if there is a subsequence of σ order isomorphic to π. Otherwise σ *avoids* π.
Ex. $\sigma = 2415376$ contains $\pi = 312$ because of the subsequence 413 but avoids $\pi = 321$ since it has no subsequence $s_i > s_j > s_k$.
For any set of permutations Π, let

$$\mathrm{Av}_n(\Pi) = \{\sigma \in \mathfrak{S}_n \mid \sigma \text{ avoids every } \pi \in \Pi\}.$$

If M_σ is the permutation matrix of σ then the *Birkhoff polytope* is

$$B_n = \mathrm{conv}\{M_\sigma \mid \sigma \in \mathfrak{S}_n\} \subseteq \mathbb{R}^{n \times n}.$$

(1) $\dim B_n = (n-1)^2$,
(2) $\mathrm{vol}\, B_n$ has only been calculated for $n \leq 10$,
(3) $h^*(B_n; t)$ is symmetric and unimodal.
Define the Π*-avoiding Birkhoff polytope* by

$$B_n(\Pi) = \mathrm{conv}\{M_\sigma \mid \sigma \in \mathrm{Av}_n(\Pi)\} \subseteq B_n.$$

Here we study $B_n(132, 312)$; other Π are in our paper.

Let $Q_n(132, 312)$ be $\mathrm{Av}_n(132, 312)$ partially orderd by weak Bruhat order, that is, we have a cover $\pi \lessdot \sigma$ if for some i,

$$\sigma = \pi(i, i+1) \text{ where } \pi_i < \pi_{i+1}.$$

Let $M(n)$ be the poset of shifted Young diagrams contained in $(n, \ldots, 2, 1)$ ordered by inclusion.

Proposition

For all n we have

$$Q_n(132, 312) \cong M(n-1).$$

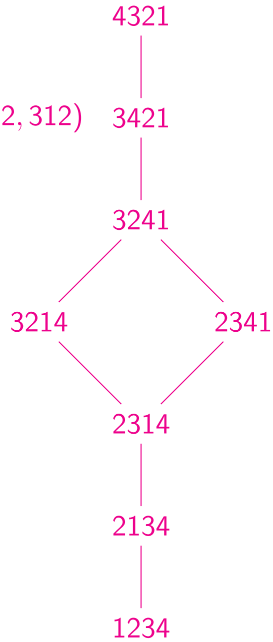
Proof sketch. The map $\phi : Q_n(132, 312) \to M(n-1)$ given by

$$\phi(\sigma) = \mathrm{Des}\,\sigma$$

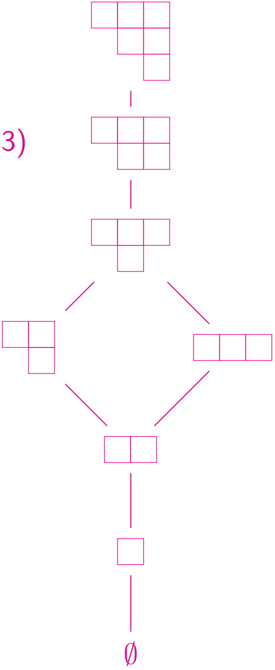
is an isomorphism where $\mathrm{Des}\,\sigma$ is the descent set of σ. □

Ex.

$Q_4(132, 312)$



$M(3)$



Let $\Delta(Q_n(132, 312))$ be the order complex of all chains Γ in $Q_n(132, 312)$. Since $Q_n(132, 312) \cong M(n-1)$ which is a distributive lattice, $\Delta(Q_n(132, 312))$ is shellable. Consider the map $f : \Delta(Q_n(132, 312)) \to B_n(132, 312)$ defined by

$$f(\sigma_1 < \cdots < \sigma_k) = \mathrm{conv}\{M_{\sigma_1}, \ldots, M_{\sigma_k}\}.$$

Proposition

$$\mathcal{T}_n(132, 312) = \{f(\Gamma) \mid \Gamma \in \Delta(Q_n(132, 312))\}$$

is a set of unimodular simplices in $B_n(132, 312)$.

Proof sketch. Induct on maximal chains using the shelling order. □

From the previous result, for Γ a maximal chain in $Q_n(132, 312)$,

$$\dim B_n(132, 312) \geq \dim \Gamma = |(n-1, \ldots, 2, 1)| = \binom{n}{2}.$$

Theorem

$$\dim B_n(132, 312) = \binom{n}{2}.$$

□

THANKS FOR
LISTENING!

AND PLEASE STAY
FOR THE NEXT TALK!