Open Problems for Catalan Number Analogues

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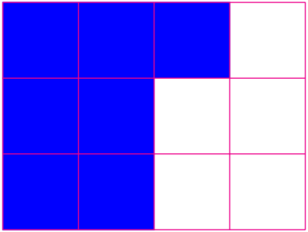
Fibonomial coefficients

Open problems

For integers $0 \le k \le n$, the binomial coefficient $\binom{n}{k}$ has the following combinatorial interpretation. An integer partition λ *fits in a $k \times l$ rectangle*, $\lambda \subseteq k \times l$, if its Ferrers diagram has at most k rows and at most l columns.

Ex. $\lambda = (3, 2, 2) \subseteq 3 \times 4$:



Proposition

We have

$$\binom{n}{k} = \#\{\lambda \subseteq k \times (n-k)\}$$

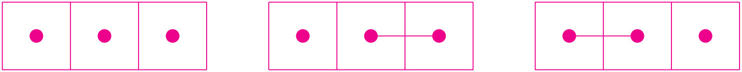
where # denotes cardinality

The *Fibonacci numbers* are defined by $F_0 = 0$, $F_1 = 1$ and

$$F_n = F_{n-1} + F_{n-2} \text{ for } n \geq 2.$$

The F_n have the following combinatorial interpretation. Let $\mathcal{T}_n$ be the set of tilings of a row of n boxes with disjoint dominos (covering two boxes) and monominos (covering one box).

Ex. The tilings in $\mathcal{T}_3$ are



Proposition

We have

$$F_n = \#\mathcal{T}_{n-1}.$$

The nth *Fibotorial* is

$$F_n^! = F_1 F_2 F_3 \dots F_n.$$

The *Fibonomial coefficients* are

$$\binom{n}{k}_F = \frac{F_n^!}{F_k^! F_{n-k}^!}.$$

The Fibonomial coefficients are integers and so one would like a combinatorial interpretation. Call a tiling $T \in \mathcal{T}_n$ *special* if it begins with a domino.

Theorem (S and Savage)

We have

$$\binom{n}{k}_F = \sum_{\lambda \subseteq k \times (n-k)} (\#\ \text{of tilings of the rows of}\ \lambda)$$

$$\cdot (\#\ \text{of special tilings of the columns of}\ k \times (n-k)/\lambda).$$

(a) FiboCatalan numbers (Lou Shapiro)

The *Catalan numbers* are

$$C_n = \frac{1}{n+1}\binom{2n}{n}.$$

They count the number of $\lambda \subseteq n \times n$ using only squares above the main diagonal. Define *FiboCatalan numbers* by

$$C_{n,F} = \frac{1}{F_{n+1}}\binom{2n}{n}_F.$$

Shapiro asked

(1) Is $C_{n,F}$ an integer for all n?

(2) If so, find a natural combinatorial interpretation.

The answer to (1) is "yes" since

$$C_{n,F} = \binom{2n-1}{n-2}_F + \binom{2n-1}{n-1}_F.$$

Problem (2) is still open.

(b) Lucas sequences (S and Savage)

The *Lucas sequence* of polynomials in variables s, t is defined by $\{0\} = 0$, $\{1\} = 1$ and, for $n \geq 2$,

$$\{n\} = s\{n-1\} + t\{n-2\}.$$

Ex. The first few polynomials in the Lucas sequence are

n	0	1	2	3	4
$\{n\}$	0	1	s	$s^2 + t$	$s^3 + 2st$

Specializations of this sequence include the Fibonacci numbers, the nonnegative integers, and others. The polynomial $\{n\}$ counts tilings with monominos weighted by s and dominos weighted by t. Define the nth *Lucatorials* and *LucaCatalans* by

$$\{n\}! = \{1\}\{2\}\{3\}\ldots\{n\} \quad \text{and} \quad C_{\{n\}} = \frac{\{2n\}!}{\{n\}!\{n+1\}!}.$$

There are polynomials in s, t with nonnegative integral coefficients. What do they count?

(c) q-analogue (N. Bergeron)

The standard *q-analogue* of the nonnegative integer n is

$$[n] = 1 + q + q^2 + \cdots + q^{n-1}.$$

The sequence of polynomials $[F_n]$ satisfies $[F_0] = 0$, $[F_1] = 1$, and, for $n \geq 2$,

$$[F_n] = [F_{n-1}] + q^{F_{n-1}}[F_{n-2}].$$

So this is not a specialization of the Lucas sequence. Define *q-Fibotorials* and *q-FiboCatalan numbers* by

$$[F_n]^! = [F_1][F_2]\ldots[F_n] \quad \text{and} \quad C_{[n]} = \frac{[F_{2n}]^!}{[F_n]^![F_{n+1}]^!}.$$

There are polynomials in q with integral coefficients. What do they count?

(d) rational FiboCatalan numbers (N. Bergeron)

Let a, b be relatively prime positive integers. The corresponding *rational Catalan numbers* are

$$C_{a,b} = \frac{1}{a+b}\binom{a+b}{a}.$$

The $C_{a,b}$ count $\lambda \subseteq a \times b$ only using squares above the main diagonal.

Ex. Note that when $a = n$ and $b = n+1$ then

$$C_{n,n+1} = \frac{1}{2n+1}\binom{2n+1}{n} = C_n.$$

Define *rational FiboCatalan numbers* by

$$C_{a,b,F} = \frac{1}{F_{a+b}}\binom{a+b}{a}_F.$$

These are integers. What do they count?

(e) Coxeter-FiboCatalan numbers (Armstrong)

Let W be a finite Coxeter group with degrees $d_1 < \cdots < d_n$. The *Coxeter-Catalan number* for W is

$$\mathrm{Cat}(W) = \prod_{i=1}^{n} \frac{d_n + d_i}{d_i}.$$

The integer $\mathrm{Cat}(W)$ counts the number of W-noncrossing partitions.

Ex. Note that when $W = A_{n-1}$ then

$$d_1 = 2,\ d_2 = 3,\ \ldots,\ d_{n-1} = n$$

and

$$\mathrm{Cat}(A_{n-1}) = \frac{(n+2)(n+3)\ldots(2n)}{(2)(3)\ldots(n)} = C_n.$$

Define the *Coxeter-FiboCatalan number* for W by

$$\mathrm{Cat}_F(W) = \prod_{i=1}^{n} \frac{F_{d_n+d_i}}{F_{d_i}}.$$

These are integers. What do they count?

THANKS FOR
LISTENING!
(AND COUNTING!)