Factoring rook polynomials

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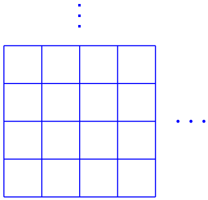
March 14, 2014

Outline

Consider tiling the first quadrant of the plane with unit squares:

$\mathcal{Q} =$ 

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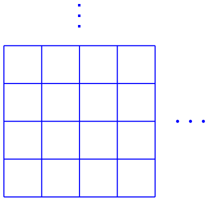

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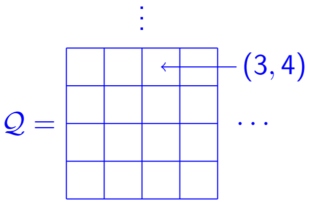

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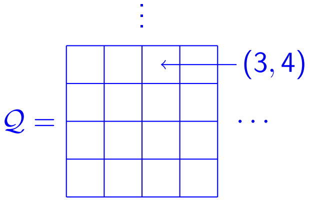

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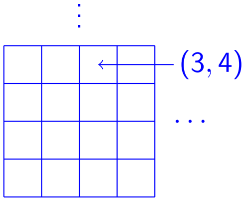
$\mathcal{Q} =$ 

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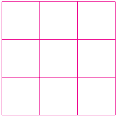
$B_3 =$ 

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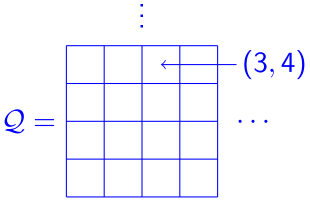

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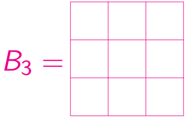

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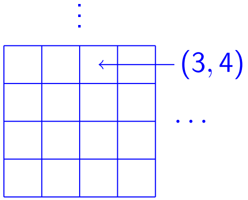

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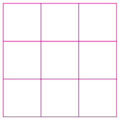

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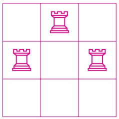

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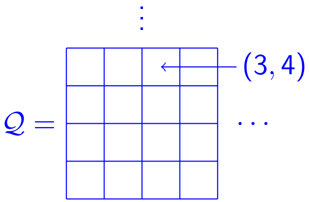

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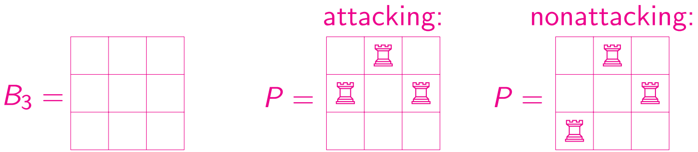

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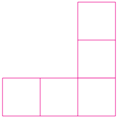
$B = (1, 1, 3) =$ 

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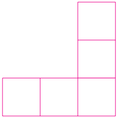

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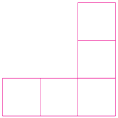
$B = (1, 1, 3) =$ 

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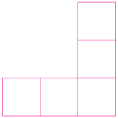

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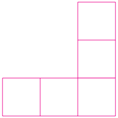

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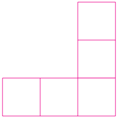

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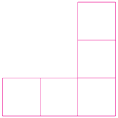

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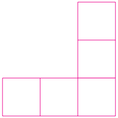
$B = (1, 1, 3) =$  $r_0(B) = 1,\ r_1(B) = 5,\ r_2(B) = 4,$ $r_3(B) = 0.$

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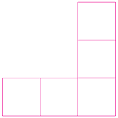

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$$= (x+1)x(x+1) = (x+b_1)(x+b_2-1)(x+b_3-2).$$

$$\sum_{k=0}^{n} r_k(b_1, \ldots, b_n) x{\downarrow}_{n-k} = \prod_{j=1}^{n} (x + b_j - j + 1). \qquad (1)$$

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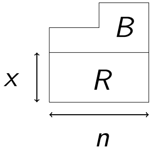

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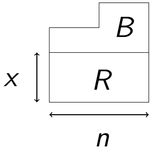

Claim: both sides of (1) equal $r_n(B_x)$.

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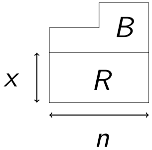

Claim: both sides of (1) equal $r_n(B_x)$. Placing rooks left to right

$$\sum_{k=0}^{n} r_k(b_1,\dots,b_n)x{\downarrow_{n-k}} = \prod_{j=1}^{n}(x+b_j-j+1). \qquad (1)$$

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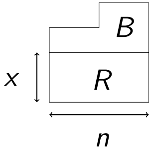

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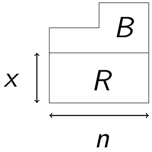

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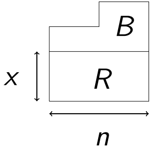
$B_x =$ 

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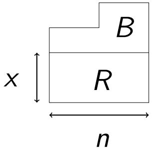
$B_x =$ 

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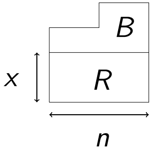

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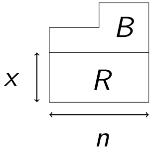
$B_x =$ 

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Outline

Call boards B and B' *rook equivalent,* $B \equiv B'$, if $r_k(B) = r_k(B')$ for all $k \geq 0$.

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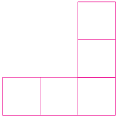
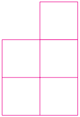
$B = (1, 1, 3) =$ 

$B' = (2, 3) =$ 

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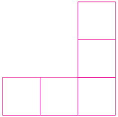
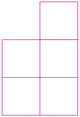
$B = (1, 1, 3) =$  $B' = (2, 3) =$ 

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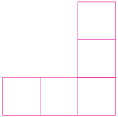
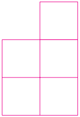
$B = (1, 1, 3) =$  $B' = (2, 3) =$ 

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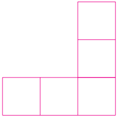
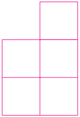
$B = (1, 1, 3) =$  $B' = (2, 3) =$ 

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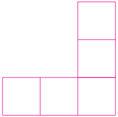
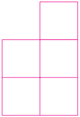
$B = (1, 1, 3) =$  $\qquad B' = (2, 3) =$ 

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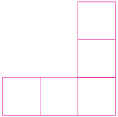
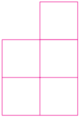
$B = (1, 1, 3) =$ 

$B' = (2, 3) =$ 

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Theorem (Foata-Schützenberger)

Every Ferrers board is rook equivalent to a unique increasing board.

The *root vector* of $B = (b_1, \ldots, b_n)$ is

$$\zeta(B) = (0 - b_1, 1 - b_2, \ldots, n - 1 - b_n)$$

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So if $B = (b_1, \ldots, b_n)$ and $B' = (b'_1, \ldots, b'_n)$ then

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Ex. $B = (0, 1, 1, 3)$ so $\zeta(B) = (0, 1, 2, 3) - (0, 1, 1, 3) = (0, 0, 1, 0)$.
Now $\zeta' = (0, 1, 0, 0)$

The *root vector* of $B = (b_1, \ldots, b_n)$ is

$$\zeta(B) = (0 - b_1, 1 - b_2, \ldots, n-1-b_n) = (0, 1, \ldots, n-1) - (b_1, b_2, \ldots, b_n)$$

The entries of $\zeta(B)$ are exactly the zeros of $\sum_k r_k(B) x{\downarrow}_{n-k}$.
So if $B = (b_1, \ldots, b_n)$ and $B' = (b'_1, \ldots, b'_n)$ then

$$B \equiv B' \iff \zeta(B) \text{ is a rearrangement of } \zeta(B').$$

Ex. $B = (1, 1, 3)$ so $\zeta(B) = (0, 1, 2) - (1, 1, 3) = (-1, 0, -1)$.
$B' = (0, 2, 3)$ so $\zeta(B') = (0, 1, 2) - (0, 2, 3) = (0, -1, -1)$ $\therefore B \equiv B'$.

Every Ferrers board B is rook equivalent to a unique increasing board.

Proof sketch. Pad B with zeros so that $\zeta = \zeta(B)$ starts with 0 and has all entries ≥ 0. Let $m = \max \zeta(B)$. Rearrange ζ to form

$$\zeta' = (0, 1, 2, \ldots, m, \zeta'_{m+1}, \ldots, \zeta'_n)$$

where $\zeta'_{m+1} \geq \cdots \geq \zeta'_n$. Then $\exists$ increasing B' with $\zeta(B') = \zeta'$. $\square$

Ex. $B = (0, 1, 1, 3)$ so $\zeta(B) = (0, 1, 2, 3) - (0, 1, 1, 3) = (0, 0, 1, 0)$.
Now $\zeta' = (0, 1, 0, 0)$ so $B' = (0, 1, 2, 3) - (0, 1, 0, 0) = (0, 0, 2, 3)$.

Outline

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   (a) Compute $r_k(B_n)$ for any $0 \leq k \leq n$.

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2. Let $T_n = (0, 1, 2, \ldots, n-1)$.
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   (d) Find the unique increasing board equivalent to T_n.

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THANKS FOR LISTENING!