Factoring rook polynomials

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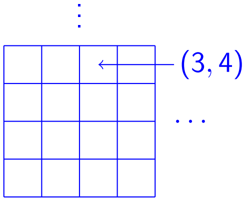
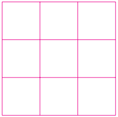
Consider tiling the first quadrant of the plane with unit squares:

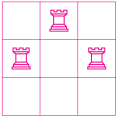
$\mathcal{Q} =$ 

Let (c, d) be the square in column c and row d. A *board* is a finite set of squares $B \subseteq \mathcal{Q}$.

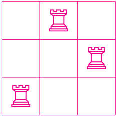
Ex. Let B_n be the $n \times n$ chess board. For example,

$B_3 =$ 

attacking:

$P =$ 

nonattacking:

$P =$ 

A placement P of rooks on B is *attacking* if there is a pair of rooks in the same row or column. Otherwise it is *nonattacking*.

Define the *rook numbers* of B to be

$$r_k(B) = \text{number of ways of placing } k \text{ nonattacking rooks on } B.$$

For any board B we have $r_0(B) = 1$ and $r_1(B) = \#B$ (cardinality).

Ex. We have

$$\begin{aligned} r_n(B_n) &= (\text{\# of ways to place a rook in column 1}) \\ &\quad \cdot(\text{\# of ways to then place a rook in column 2}) \cdots \\ &= n \cdot (n-1) \cdots \\ &= n! \end{aligned}$$

There is a bijection between placements P counted by $r_n(B_n)$ and permutations π in the symmetric group $\mathfrak{S}_n$ where $(c, d) \in P$ if and only if $\pi(c) = d$.

Ex. Let

$$D_n = B_n - \{(1,1), (2,2), \ldots, (n,n)\}.$$

Then

$$\begin{aligned} r_n(D_n) &= \text{\# of permutations } \pi \in \mathfrak{S}_n \text{ with } \pi(c) \neq c \text{ for all } c \\ &= \text{the } n\text{th derangement number.} \end{aligned}$$

A *partition* is a weakly increasing sequence $(b_1, \ldots, b_n)$ of nonnegative integers. A *Ferrers board* is $B = (b_1, \ldots, b_n)$ consisting of the lowest b_j squares in column j of $\mathcal{Q}$ for all j. If x is a variable and $n \geq 0$ then the corresponding *falling factorial* is

$$x{\downarrow_n} = x(x-1)\cdots(x-n+1).$$

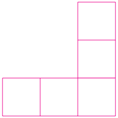
Theorem (Factorization Theorem: Goldman-Joichi-White)

For any Ferrers board $B = (b_1, \ldots, b_n)$ we have

$$\sum_{k=0}^{n} r_k(B) x{\downarrow_{n-k}} = \prod_{j=1}^{n} (x + b_j - j + 1).$$

Ex.

$B = (1, 1, 3) =$



$r_0(B) = 1,\ r_1(B) = 5,\ r_2(B) = 4,$
$r_3(B) = 0.$

$$\sum_{k=0}^{3} r_k(B) x{\downarrow_{3-k}} = 1 \cdot x{\downarrow_3} + 5 \cdot x{\downarrow_2} + 4 \cdot x{\downarrow_1} = x^3 + 2x^2 + x$$

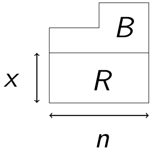
$$= (x+1)x(x+1) = (x + b_1)(x + b_2 - 1)(x + b_3 - 2).$$

$$\sum_{k=0}^{n} r_k(b_1, \ldots, b_n) x{\downarrow_{n-k}} = \prod_{j=1}^{n} (x + b_j - j + 1). \qquad (1)$$

◂ 1 ◂ 2

Proof. It suffices to prove (1) for x a positive integer. Consider

$B_x =$



Claim: both sides of (1) equal $r_n(B_x)$. Placing rooks left to right

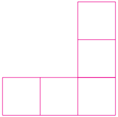
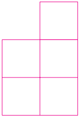
$$\begin{aligned} r_n(B_x) &= \prod_{j=1}^{n} (\#\text{ of unattacked squares in column } j) \\ &= (x + b_1)(x + b_2 - 1) \ldots = \text{RHS of (1)}. \end{aligned}$$

$$\begin{aligned} r_n(B_x) &= \sum_{k=0}^{n} (\#\text{ of ways to put } k \text{ rooks on } B \text{ and } n-k \text{ on } R) \\ &= \sum_{k=0}^{n} r_k(B) \cdot x(x-1)\ldots(x-n+k+1) = \text{LHS of (1)}. \quad \square \end{aligned}$$

Call boards B and B' *rook equivalent*, $B \equiv B'$, if $r_k(B) = r_k(B')$ for all $k \geq 0$. Note that $B \equiv B'$ implies

$$\#B = r_1(B) = r_1(B') = \#B'.$$

Ex.

$B = (1, 1, 3) =$  $B' = (2, 3) =$ 

For B, B': $r_0 = 1$, $r_1 = 5$, $r_2 = 4, r_k = 0$ for $k \geq 3$ so $B \equiv B'$.

A Ferrers board $B = (b_1, \ldots, b_n)$ is *increasing* if $b_1 < \cdots < b_n$. In the example above, B' is increasing but B is not.

Theorem (Foata-Schützenberger)

Every Ferrers board is rook equivalent to a unique increasing board.

The *root vector* of $B = (b_1, \ldots, b_n)$ is

$$\zeta(B) = (0-b_1, 1-b_2, \ldots, n-1-b_n) = (0, 1, \ldots, n-1)-(b_1, b_2, \ldots, b_n)$$

The entries of $\zeta(B)$ are exactly the zeros of $\sum_k r_k(B)x{\downarrow}_{n-k}$.
So if $B = (b_1, \ldots, b_n)$ and $B' = (b'_1, \ldots, b'_n)$ then

$$B \equiv B' \iff \zeta(B) \text{ is a rearrangement of } \zeta(B').$$

Ex. $B = (1, 1, 3)$ so $\zeta(B) = (0, 1, 2) - (1, 1, 3) = (-1, 0, -1)$.
$B' = (0, 2, 3)$ so $\zeta(B') = (0, 1, 2)-(0, 2, 3) = (0, -1, -1) \quad \therefore B \equiv B'$.

Every Ferrers board B is rook equivalent to a unique increasing board.

Proof sketch. Pad B with zeros so that $\zeta = \zeta(B)$ starts with 0 and has all entries ≥ 0. Let $m = \max \zeta(B)$. Rearrange ζ to form

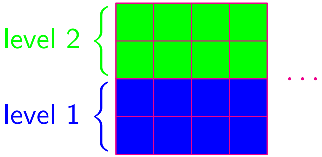
$$\zeta' = (0, 1, 2, \ldots, m, \zeta'_{m+1}, \ldots, \zeta'_n)$$

where $\zeta'_{m+1} \geq \cdots \geq \zeta'_n$. Then $\exists$ increasing B' with $\zeta(B') = \zeta'$. $\square$

Ex. $B = (0, 1, 1, 3)$ so $\zeta(B) = (0, 1, 2, 3) - (0, 1, 1, 3) = (0, 0, 1, 0)$.
Now $\zeta' = (0, 1, 0, 0)$ so $B' = (0, 1, 2, 3) - (0, 1, 0, 0) = (0, 0, 2, 3)$.

Fix a positive integer m. Partition the rows of $\mathcal{Q}$ into levels where the *ith level* consists of rows $(i-1)m+1, (i-1)m+2, \ldots, im$.

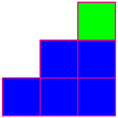
Ex. If $m = 2$ then

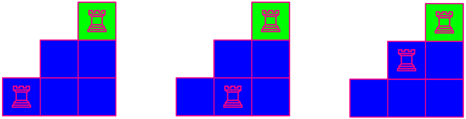


An *m-level rook placement* on B is a set P of rooks with no two in the same level or column. A 1-level rook placement is just an ordinary placement. The *m-level rook numbers* of B are

$$r_{k,m}(B) = \text{number of } m\text{-level rook placements on } B \text{ with } k \text{ rooks.}$$

Ex. If $m = k = 2$ and

$B = (1, 2, 3) =$ 

$\therefore r_{2,2}(B) = 3 :$ 

The m-level rook placements are related to $C_m \wr \mathfrak{S}_n$ where C_m is the order m cyclic group and $\mathfrak{S}_n$ is the nth symmetric group, e.g.,

$$r_{n,m}(\overbrace{mn,\ldots,mn}^{n}) = (mn)(mn-m)\cdots(m) = m^n n! = \#(C_m \wr \mathfrak{S}_n).$$

Define the *m-falling factorials* by

$$x\downarrow_{n,m} = x(x-m)(x-2m)\cdots(x-(n-1)m).$$

A *singleton board* is $B = (b_1,\ldots,b_n)$ with at most one b_j in each of the open intervals $(0,m),(m,2m),(2m,3m),\ldots$.

Theorem (Briggs-Remmel)

If B is a singleton board then

$$\sum_{k=0}^{n} r_{k,m}(B)x\downarrow_{n-k,m} = \prod_{j=1}^{n}(x+b_j-(j-1)m).$$

◂ 1 ◂ 2

Given an integer m, define the *mod m floor function* by

$\lfloor n \rfloor_m =$ largest multiple of m which is less than or equal to n.

Ex. $\lfloor 17 \rfloor_3 = 15$ since $15 \le 17 < 18$.

Define a *zone*, $z = z(B)$, of a Ferrers board $B = (b_1, \ldots, b_n)$ to be a maximal subsequence $(b_i, \ldots, b_j)$ with

$$\lfloor b_i \rfloor_m = \cdots = \lfloor b_j \rfloor_m.$$

Given a zone $z = (b_i, \ldots, b_j)$ define its *remainder* to be

$$\rho(z) = \sum_{t=i}^{j} (b_t - \lfloor b_t \rfloor_m).$$

Ex. If $m = 3$ then $B = (1, 1, 2, 3, 5, 7)$ has zones
$\therefore z = (1, 1, 2), \ z' = (3, 5), \ z'' = (7)$.
Also $\rho(z) = 1 + 1 + 2 = 4, \ \rho(z') = 0 + 2 = 2, \ \rho(z'') = 1$.

Theorem (Barrese-Loehr-Remmel-S)

Let $B = (b_1, \ldots, b_n)$ be any Ferrers board. Then

$$\sum_{k=0}^{n} r_{k,m}(B) x\downarrow_{n-k,m} = \prod_{j=1}^{n} (x + \lfloor b_j \rfloor_m - (j-1)m + \epsilon_j)$$

where

$$\epsilon_j = \begin{cases} \rho(z) & \text{if } b_j \text{ is the last column in zone } z, \\ 0 & \text{else.} \end{cases}$$

$$\sum_{k=0}^{n} r_{k,m}(B)x\!\downarrow_{n-k,m} = \prod_{j=1}^{n} \begin{cases} x + \lfloor b_j \rfloor_m - (j-1)m + \rho(z) & \text{if } b_j \text{ last in } z, \\ x + \lfloor b_j \rfloor_m - (j-1)m & \text{else.} \end{cases}$$

Ex. Recall that if $m = 3$ and $B = (1, 1, 2, 3, 5, 7)$ then we have zones $z = (1, 1, 2)$, $z' = (3, 5)$, $z'' = (7)$, and remainders $\rho(z) = 1 + 1 + 2 = 4$, $\rho(z') = 0 + 2 = 2$, $\rho(z'') = 1$. Thus

$$\sum_{k=0}^{n} r_{k,m}(B)x\!\downarrow_{n-k,m} = (x + 0 - 0 + 0)(x + 0 - 3 + 0)(x + 0 - 6 + 4) \cdot (x + 3 - 9 + 0)(x + 3 - 12 + 2)(x + 6 - 15 + 1).$$

BLRS implies Goldman-Joichi-White: If $m = 1$ then it is clear that the LHS of both equations are the same. Also $\lfloor b_j \rfloor_1 = b_j$ for all j. So $\rho(z) = 0$ for all z. Thus the RHS's also agree.

BLRS imples Briggs-Remmel: Clearly the LHS's are the same. If B is singleton, then $\lfloor b_j \rfloor_m = b_j$ for every b_j in a zone except possibly the last. For the last b_j, $\lfloor b_j \rfloor_m + \rho(z) = \lfloor b_j \rfloor_m + \rho(b_j) = b_j$. So RHS's agree factor by factor.

1. ***m*-level rook equivalence.** Say B, B' are *m-level rook equivalent* if $r_{k,m}(B) = r_{k,m}(B')$ for all k. Call $B = (b_1, \ldots, b_n)$ *m-increasing* if $b_1 > 0$ and $b_j \geq b_{j-1} + m$ for $j \geq 2$. Note that B is 1-increasing if and only if B is increasing.

Theorem (BLRS)

Every Ferrers board is m-level rook equivalent to a unique m-increasing board.

2. A p, q-analogue. Permutation $\pi = a_1 \ldots a_n \in \mathfrak{S}_n$ has *inversion set* and *inversion number*

$$\operatorname{Inv} \pi = \{(i,j) \mid i < j \text{ and } a_i > a_j\}, \quad \text{and} \quad \operatorname{inv} \pi = \# \operatorname{Inv} \pi.$$

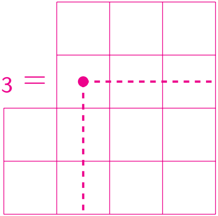
If B is a board then the *hook* of $(i,j) \in B$, $H_{i,j}$, is all cells directly south or directly east of (i,j). If P is a rook placement on B then the *Rothe diagram* of P is the skew diagram

$$R(P) = B \setminus \cup_{(i,j) \in P} H_{i,j}$$

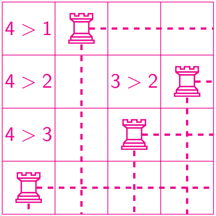
If P_π is the permutation matrix of π then $\operatorname{inv} \pi = \# R(P_\pi)$. BLRS have a generalization of the factor theorem with two parameters p, q keeping track of inversions and non-inversions.

Ex. $\pi = 4132 \implies \operatorname{Inv} \pi = \{(1,2),(1,3),(1,4),(3,4)\}, \operatorname{inv} \pi = 4.$

$H_{2,3} =$



$R(P_\pi) =$



3. Counting equivalence classes. Write $\zeta \geq 0$ if ζ is a nonnegative sequence. In this case, the *multiplicity vector* of ζ is

$$n(\zeta) = (n_0, n_1, \dots) \text{ where } n_i = \text{the number of } i\text{'s in } \zeta.$$

Theorem (Goldman-Joichi-White)

If Ferrers board B has $\zeta = \zeta(B) \geq 0$ and $n(\zeta) = (n_0, n_1, \dots)$ then

$$\#\text{ of Ferrers boards equivalent to } B = \prod_{i \geq 0} \binom{n_i + n_{i+1} - 1}{n_i - 1}.$$

The *m-root vector* of $B = (b_1, \dots, b_n)$ is

$$\zeta_m(B) = (0 - b_1, m - b_2, 2m - b_3, \dots, (n-1)m - b_n).$$

Theorem (BLRS)

Let B be singleton with $\zeta = \zeta_m(B) \geq 0$ and $n(\zeta) = (n_0, n_1, \dots)$.

$$\#\text{ of singleton boards equivalent to } B = \prod_{i \geq 0} \binom{n_{im} + \cdots + n_{im+m} - 1}{n_{im} - 1, n_{im+1}, \dots, n_{im+m}}.$$

It would be interesting to find a result holding for all Ferrers B.

4. File placements. A *file placement* F on B is a placement of rooks with no two in the same column. Fix $m \geq 1$ and let the *m-weight* of F be

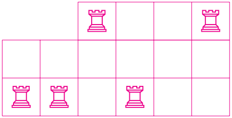
$$\mathrm{wt}_m F = 1\!\downarrow_{y_1,m} \cdot 1\!\downarrow_{y_2,m} \cdots$$

where y_i is the number of rooks of F in row $i \geq 1$. Let

$$f_{k,m}(B) = \sum_F \mathrm{wt}_m F$$

where the sum is over all file placements F of k rooks on B.

Ex. $F =$



F has $y_1 = 3,\ y_2 = 0,\ y_3 = 2.$
If $m = 4$ then

$$\mathrm{wt}_4 F = 1\!\downarrow_{3,4} \cdot 1\!\downarrow_{0,4} \cdot 1\!\downarrow_{2,4}$$

$$= (1)(-3)(-7) \cdot (1)(-3) = -63.$$

Theorem (BLRS)

For any Ferrers board $B = (b_1, \ldots, b_n)$

$$\sum_{k=0}^{n} f_{k,m}(B) x\!\downarrow_{n-k,m} = \prod_{j=1}^{n} (x + b_j - (j-1)m).$$

5. Higher q,t-Catalan numbers. The *m-triangluar board* is

$$\Delta_{n,m} = (0, m, 2m, \dots, (n-1)m).$$

If $B = (b_1, \dots, b_n) \subseteq \Delta_{n,m}$ then $\zeta_m(B) = (z_1, \dots, z_n)$ gives the heights of the columns of $\Delta_{n,m} \setminus B$. Define $\mathrm{area}_m(B) = \#B$ and

$$\mathrm{dinv}_m(B) = \sum_{k=0}^{m-1} \#\{i < j \ : \ 0 \le z_i - z_j + k \le m\}.$$

The *higher q,t-Catalan numbers* are

$$C_{n,m}(q,t) = \sum_{B \subseteq \Delta_{n,m}} q^{\mathrm{dinv}_m(B)} t^{\mathrm{area}_m(\Delta_{n,m} \setminus B)}.$$

We also have

$$C_{n,m}(q,t) = \sum_{B \subseteq \Delta_{n,m}} q^{\mathrm{area}_m(\Delta_{n,m} \setminus B)} t^{\mathrm{bounce}_m(B)}$$

for another statistic $\mathrm{bounce}_m(B)$. Using the $C_{n,m}(q,t)$, BLRS derives a formula for the number of boards m-weight equivalent to a given board as a product of binomial coefficients.

6. Hyperplane arrangements. Given $\pi \in \mathfrak{S}_n$ the corresponding *inversion arrangement* is the set of hyperplanes in $\mathbb{R}^n$

$$\mathcal{A}(\pi) = \{x_i = x_j \mid (i,j) \in \operatorname{Inv} \pi\}.$$

If $\pi = a_1 \ldots a_n$ then its *non-inversion board* is

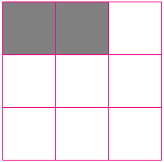
$$B(\pi) = \{(i,j) \mid i < j \text{ and } a_i < a_j\} \subseteq B_n.$$

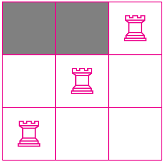
Theorem (Hultman, Lewis-Morales)

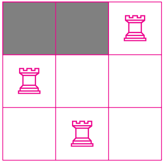
For all $\pi \in \mathfrak{S}_n$, the number of regions of the arrangement $\mathcal{A}(\pi)$ equals the rook number $r_n(B_n \setminus B(\pi))$.

Barrese, Hultman and S are looking for a type B analogue.

Ex. If $\pi = 213$ then $\operatorname{Inv} \pi = \{(1,2)\}$ and $\mathcal{A}(\pi) = \{x_1 = x_2\}$. So the non-inversions of π are $(1,3), (2,3)$ and

$B(\pi) =$ 





7. Bibliography.

Kenneth Barrese, Nicholas Loehr, Jeffrey B. Remmel, and Bruce E. Sagan, *m*-level rook placements, preprint, 2013.

Karen S. Briggs and Jeffrey B. Remmel, *m*-rook numbers and a generalization of a formula of Frobenius to $C_m \wr \mathcal{S}_n$, *J. Combin. Theory Ser. A*, 113(6):1138–1171, 2006.

D. Foata and M. P. Schützenberger, On the rook polynomials of Ferrers relations, in *Combinatorial theory and its applications, II (Proc. Colloq., Balatonfüred, 1969)*, pages 413–436, North-Holland, Amsterdam, 1970.

Jay R. Goldman, J. T. Joichi, and Dennis E. White, Rook theory. I. Rook equivalence of Ferrers boards, *Proc. Amer. Math. Soc.*, 52:485–492, 1975.

James Haglund, The *q*,*t*-Catalan numbers and the space of diagonal harmonics, volume 41 of *University Lecture Series*, American Mathematical Society, Providence, RI, 2008, with an appendix on the combinatorics of Macdonald polynomials.

THANKS FOR
LISTENING!