Partially Ordered Sets and their Möbius Functions: Exercises
Encuentro Colombiano de Combinatoria 2014

Lecture I: The Möbius Inversion Theorem

1. (a) Prove that if $S \subseteq T$ in B_n then $[S,T] \cong B_{|T-S|}$.
   (b) Prove that if $c|d$ in D_n then $[c,d] \cong D_{d/c}$.

2. Prove that if the prime factorization of n is $n = p_1^{m_1} \cdots p_k^{m_k}$ then

$$D_n \cong C_{m_1} \times \cdots \times C_{m_k}.$$

3. A *partition* of a set S is a family π of nonempty sets $B_1, \ldots, B_k$ called *blocks* such that $\uplus_i B_i = S$ (disjoint union). We write $\pi = B_1/\ldots/B_k \vdash S$ and often leave out set braces and commas in the blocks. The *partition lattice* is

$$\Pi_n = \{\pi \ : \ \pi \vdash [n]\}.$$

   with the partial order $B_1/\ldots/B_k \leq C_1/\ldots/C_l$ if for each B_i there is a C_j with $B_i \subseteq C_j$.

   (a) Draw Π_3.
   (b) Show Π_n has $\hat{0} = 1/2/\ldots/n$ and $\hat{1} = 12\ldots n$.
   (c) Prove that if $\pi = B_1/\ldots/B_k$ then $[\pi, \hat{1}] \cong \Pi_k$.
   (d) Prove that if $\pi = B_1/\ldots/B_k$ then $[\hat{0}, \pi] \cong \Pi_{|B_1|} \times \cdots \times \Pi_{|B_k|}$.
   (e) Combine the two previous results to show that if $\pi \leq \sigma$ then $[\pi, \sigma]$ is isomporphic to a product of partition lattices.

4. For any poset, P, prove that the map $I(P) \to M(P)$ by $\alpha \mapsto M^\alpha$ preserves addition and scalar multiplication.

5. Show that if $f : P \to Q$ is an isomorphism of posets and $x, y \in P$ then

$$\mu_P(x,y) = \mu_Q(f(x), f(y)).$$

6. Prove that if $d \in D_n$ with prime factorization $d = p_1^{m_1} \cdots p_k^{m_k}$ then

$$\mu(d) = \begin{cases} (-1)^k & \text{if } m_1 = \ldots = m_k = 1, \\ 0 & \text{if } m_i \geq 2 \text{ for some } i. \end{cases}$$

7. Prove the Dual Möbius Inversion Theorem: Consider two functions $f, g : P \to \mathbb{R}$. Then

$$f(x) = \sum_{y \geq x} g(y)$$

   for all $x \in P$ if and only if

$$g(x) = \sum_{y \geq x} \mu(x,y) f(y)$$

   for all $x \in P$.

8. (Research Problem) Let $\alpha = a_1 \ldots a_k$ and $\beta = b_1 \ldots b_k$ be sequences of distinct integers. Call α and β *order isomorphic* and write $\alpha \equiv \beta$ if, for all indices i and j,

$$a_i < a_j \iff b_i < b_j.$$

Intuitively, α and β are order isomorphic if the elements appearing in α are in the same relative order as those appearing in β. To illustrate, the sequences 132, 253 and 587 are all order isomorphic since, in each one, the smallest element is first, the largest element is second, and the middle element is third.

Let $\mathfrak{S}$ denote the set of all permutations $\alpha = a_1 \ldots a_n$ of $[n]$ for all $n \geq 0$. Call $\alpha \in \mathfrak{S}$ a *pattern* in $\beta \in \mathfrak{S}$ if there is a subsequence β' of β with $\alpha \cong \beta'$. For example, if our pattern is $\alpha = 132$ then the permutation $\beta = 425613$ contains two copies of α, namely $\beta' = 253$ and $\beta' = 263$. Partially order $\mathfrak{S}$ by letting $\alpha \leq \beta$ if α is a pattern in β. Herbert Wilf asked to determine the Möbius function of $\mathfrak{S}$. Partial results are known, but no complete characterization.

Lecture II: Graph Coloring

1. Let $G = (V, E)$ be a graph with chromatic polynomial

$$p(G; t) = a_0 t^n + a_1 t^{n-1} + \cdots + a_n$$

   (a) Prove that $n = |V|$.

   (b) Prove that $a_0 = 1$ and $a_1 = -|E|$.

   (c) Prove that the coefficients alternate in sign with $a_{2i} \geq 0$ and $a_{2i+1} \leq 0$ for all i.

   (d) Prove that if G has components $G_1, \ldots, G_k$ then $P(G) = P(G_1) \ldots P(G_k)$.

2. A *cycle* in a graph G is a sequence of distinct vertices $C : v_1, \ldots, v_k$ with $v_i v_{i+1} \in E(G)$ for all i modulo k.

   (a) A *tree* is a graph T which is connected and has no cycles. Show that if $|V(T) = n$ then

$$p(T; t) = t(t-1)^{n-1}.$$

   (b) A *forest* is a graph F all of whose components are trees. Show that if $|V(F)| = n$ and F has k components then

$$p(F; t) = t^k (t-1)^{n-k}.$$

   (c) Show that if T is a tree with $|V(T)| = n$ then the bond lattice of T satisfies

$$L(T) \cong B_{n-1}.$$

   You may assume that all trees on n vertices have $n - 1$ edges.

3. The *complete graph* on n vertices, K_n, has $|V(K_n)| = n$ and all possible $\binom{n}{2}$ edges.

(a) Show that the bond lattice of K_n satisfies

$$L(K_n) \cong \Pi_n.$$

where Π_n is the partition lattice of Exercise 3 from Lecture I.

(b) Use part (a) to determine $\chi(\Pi_n; t)$.

(c) Use part (b) to determine $\mu(\hat{1})$ in Π_n.

(d) Use part (c) and Exercise 3 from Lecture I to determine $\mu(\pi, \sigma)$ for any $\pi \leq \sigma$ in Π_n.

4. Let P, Q be ranked posets.
   (a) Prove that $P \cong Q \implies \chi(P; t) = \chi(Q; t)$.
   (b) Prove that $P \times Q$ is ranked and $\chi(P \times Q; t) = \chi(P; t)\chi(Q; t)$.

5. (a) Prove that C_n is semimodular.
   (b) Prove that D_n is semimodular.

6. Show that for any graph G we have

$$X(G; \mathbf{x}) = \sum_{K \in L(G)} \mu(K) p_{\lambda(K)}.$$

7. Call graphs G and H *isomorphic*, $G \cong H$, if there is a bijection $f : V(G) \to V(H)$ such that

$$vw \in E(G) \iff f(v)f(w) \in E(H).$$

   (a) Show that if $G \cong H$ then $X(G; \mathbf{x}, t) = X(H; \mathbf{x}, t)$.
   (b) (Research Problem) We saw in Exercise 2 above that any two trees on n vertices have the same chromatic polynomial. Stanley conjectured that the opposite is true for the chromatic generating function. Precisely, if T, T' are trees with $T \ncong T'$ then $X(T; \mathbf{x}, t) \neq X(T'; \mathbf{x}, n)$.

Lecture III: Topology of Posets

1. (a) The *n-dimensional tetrahedron (simplex)*, T^n, is the simplicial complex which consists of all subsets of the set $\{0, 1, \ldots, n\}$. Show that

$$\Delta(C_n) = T^{n-2} \cong B^{n-2}.$$

   (b) Given a geometric simplicial complex Δ its *barycentric subdivision*, Δ^*, is the simplicial complex whose simplices are all those of the form $v_1, v_2, \ldots, v_k$ where $F_1 \subset F_2 \subset \ldots \subset F_k$ is a sequence of faces of Δ and v_i is the barycenter (centroid/center of mass) of F_i for all i. Show that

$$\Delta(B_n) = (\partial T^{n-1})^* \cong S^{n-2}.$$

2. Show that the labelings given for the four examples in the lecture are indeed EL-labelings of C_n, B_n, D_n and Π_n by checking the two conditions in the definition for every interval $[x, y]$ (not just the interval $[\hat{0}, \hat{1}]$).

3. Suppose that P is a graded poset with an EL-labeling and $F_1, \ldots, F_k$ is the list of saturated $\hat{0}$–$\hat{1}$ chains in lexicographic order.

   (a) Show that $\overline{F}_1, \ldots, \overline{F}_k$ is a shelling of $\Delta(P)$.

   (b) Show, for all j, that if $r(\overline{F}_j) = \overline{F}_j$ then $\ell(F_j)$ is strictly decreasing

4. Fill in the details of the shelling proof that

$$\mu(D_n) = \begin{cases} (-1)^k & \text{if } n = p_1 \ldots p_k \text{ distinct primes,} \\ 0 & \text{else.} \end{cases}$$

5. If P is a poset, then a *multichain* with n elements in P is

$$M : x_1 \leq x_2 \leq \cdots \leq x_n.$$

   Define the *zeta polynomial* of P to be, for $n \geq 2$,

$$Z(P; n) = \text{the number of multichains in } P \text{ with } n-1 \text{ elements.}$$

   (a) Show that if P is bounded then

$$Z(P; n) = \zeta^n(\hat{0}, \hat{1})$$

   (b) For any poset P, show that

$$Z(P; n) = \sum_{i \geq 2} b_i \binom{n-2}{i-2}$$

   where

$$b_i = \text{the number of chains in } P \text{ with } i-1 \text{ elements.}$$

   (c) Use part (b) to show that $Z(P; n)$ is a polynomial in n with degree

$$d = \text{length of the longest chain in } P,$$

   and leading coefficient $b_{d+1}/d!$

   (d) Use part (c) to extend the definition of $Z(P; n)$ to $n = 1$. Show that if P is bounded then

$$Z(P; 1) = \tilde{\chi}(P) + 1.$$

6. (Research Problem) Consider the poset $\mathfrak{S}$ defined in Exercise 8 from Lecture 1. Characterize which intervals of $\mathfrak{S}$ are shellable. Characterize which intervals of $\mathfrak{S}$ are EL-shellable.

Lecture IV: Factoring the Characteristic Polynomial

1. Prove that if $P/\sim$ is a homogeneous quotient then the relation $X \le Y$ is reflexive and transitive.

2. Prove that the three conditions of the Main Theorem hold for any $\pi \in \Pi_n$ (not just $\pi = \hat{1}$).

3. Prove that for any poset P and any $x, y, z \in P$ with $y \le z$ we have
$$y \vee (x \wedge z) \le (y \vee x) \wedge z.$$

4. Give a direct proof (without using the increasing forest generating function) that if a graph has a perfect elimnation order then its chromatic polynomial has nonnegative integral roots.

5. If P is a poset and $X \subseteq P$ then the *lower order ideal generated by* X is
$$L(X) = \{y \in P \ : \ y \le x \text{ for some } x \in X\}.$$
Also, if P has a $\hat{1}$ then a *coatom* of P is $d \in P$ with $d \lhd \hat{1}$.

   (a) Let $P/\sim$ be a homogeneous quotient. Suppose that for all $X \in T/\sim$ we have
   $$\sum_{y \in L(X)} \mu(y) = \delta_{\hat{0},X}.$$
   Prove that
   $$\mu(X) = \sum_{x \in X} \mu(x).$$

   (b) Let P be bounded and d a coatom of P. Let $\sim$ be the equivalence relation with all equivalence class $\{d, \hat{1}\}$ and all other classes being singletons. Show that $P/\sim$ is homogeneous and
   $$\mu([\hat{1}]) = \mu(d) + \mu(\hat{1})$$
   where $[x] \in P/\sim$ is the equivalence class of $x \in P$.

   (c) Use part (b) to prove Phillip Hall's Theorem: for any poset P and $x, y \in P$ we have
   $$\mu(x, y) = \sum_{i \ge 0} (-1)^i c_i$$
   where
   $$c_i = \text{number of } x\text{–}y \text{ chains of length } i.$$

   (d) Show that under the hypotheses of part (b), if P is a lattice then so is $P/\sim$ where
   $$[x] \vee [y] = [x \vee y]$$
   for all $x, y \in P$, and
   $$[x] \wedge [y] = [x \wedge y]$$
   for all $x, y \in P$ such that $[x], [y] \ne [\hat{1}]$.

(e) Use parts (b) and (d) to prove Weisner's Theorem: Let L be a lattice with $|L| \geq 2$ and $a \in L - \{\hat{0}\}$. Then

$$\mu(\hat{1}) = - \sum_{x \neq \hat{1},\ x \vee a = \hat{1}} \mu(x).$$

(f) Use part (e) to rederive the fact that in Π_n we have

$$\mu(\hat{1}) = (-1)^{n-1}(n-1)!$$

6. (Research Problem) A *weighted set partition of* $[n]$ is a set

$$\{(B_1, w_1), (B_2, w_2), \ldots, (B_k, w_k)\}$$

where $B_1/B_2/\ldots/B_k \vdash [n]$ and the $w_i \in \{0, 1, \ldots, |B_i| - 1\}, 1 \leq i \leq k$, are called *weights*. Put a partial order on the set W_n of such partitions by starting with $\hat{0} = \{(1, 0), (2, 0), \ldots, (n, 0)\}$ and then defining the covering relation by

$$\{(B_1, w_1), (B_2, w_2), \ldots, (B_k, w_k)\} \lhd \{(C_1, x_1), (C_2, x_2), \ldots, (C_{k-1}, x_{k-1})\}$$

where

$$B_1/B_2/\ldots/B_k \lhd C_1/C_2/\ldots/C_{k-1}$$

and if B_i and B_j were merged to form C_l then

$$x_l = w_i + w_j \text{ or } w_i + w_j + 1$$

with all other weights being unchanged. González D'León and Wachs have shown that

$$\chi(W_n; t) = (t - n)^{n-1}.$$

Can this result be derived using quotients?