Partially Ordered Sets and their Möbius Functions III: Topology of Posets

Bruce Sagan
Department of Mathematics
Michigan State University
East Lansing, MI 48824-1027
sagan@math.msu.edu
www.math.msu.edu/~sagan

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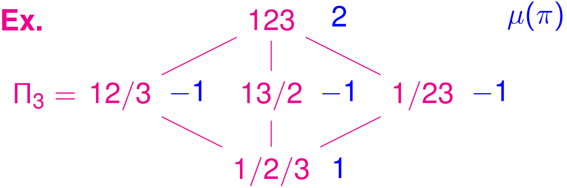
A *partition* of a set S is a family π of nonempty sets $B_1, \ldots, B_k$ called *blocks* such that $\uplus_i B_i = S$ (disjoint union). We write $\pi = B_1/\ldots/B_k \vdash S$ omitting braces and commas in blocks.

Ex. $\pi = acf/bg/de \vdash \{a, b, c, d, e, f, g\}$.

The *partition lattice* is $\Pi_n = \{\pi \ : \ \pi \vdash [n]\}$ ordered by $B_1/\ldots/B_k \leq C_1/\ldots/C_l$ if for each B_i there is a C_j with $B_i \subseteq C_j$.

If P has a $\hat{0}$ and a $\hat{1}$ we write $\mu(P) = \mu_P(\hat{0}, \hat{1})$ and similarly for other elements of $I(P)$.

Ex.



n	1	2	3	4	5	6
$\mu(\Pi_n)$	1	-1	2	-6	24	-120

Theorem

We have: $\mu(\Pi_n) = (-1)^{n-1}(n-1)!$

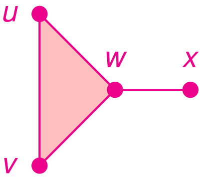
An *(abstract) simplicial complex* is a finite nonempty family Δ of finite sets called *faces* such that

$$F \in \Delta \quad \text{and} \quad F' \subseteq F \quad \Longrightarrow \quad F' \in \Delta.$$

A *geometric realization* of Δ has a $(d-1)$-dimensional simplex (tetrahedron) for each d-element set in Δ. The *dimension* of $F \in \Delta$ is $\dim F = \#F - 1$. Face F is a *vertex or edge* if $\dim F = 0$ or 1, respectively.

Ex. $\Delta = \{\emptyset, u, v, w, x, uv, uw, vw, wx, uvw\}$

$\dim u = 0$ a vertex,
$\dim uv = 1$, an edge
$\dim uvw = 2$.
uvw and wx are facets.
Not pure.

$\Delta =$ 

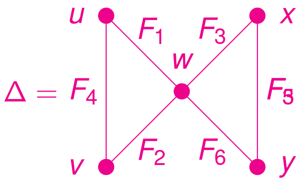
Face F is a *facet* if it is containment-maximal in Δ. We say Δ is *pure of dimension d*, and write $\dim \Delta = d$, if $\dim F = d$ for all facets F of Δ.

Note. A simplicial complex pure of dimension 1 is just a graph.

Let Δ be pure of dimension d. We say Δ is *shellable* if there is an ordering of its facets (a *shelling*) $F_1, \ldots, F_k$ such that for each $j \leq k$:

$$F_j \bigcap \left(\cup_{i<j} F_i\right) \text{ is a union of } (d-1)\text{-dimensional faces of } F_j.$$

Ex. For the graph at right
$uw,\ vw,\ wx,\ uv,\ xy,\ wy$ is a shelling.
So Δ is shellable.
Any sequence beginning $uw,\ vw,\ xy$
is not a shelling since $xy \cap (uw \cup vw) = \emptyset$.
In the original shelling:
$r(uw) = \emptyset$, $r(vw) = v$, $r(wx) = x$,
$r(uv) = uv$, $r(xy) = y$, $r(wy) = wy$.



Note. A graph is shellable iff it is connected.

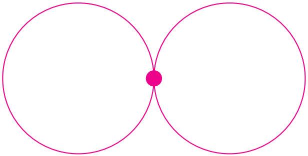
Given a shelling $F_1, \ldots, F_k$, the *restriction of F_j* is

$$r(F_j) = \{v \text{ a vertex of } F_j \ :\ F_j - v \subseteq \left(\cup_{i<j} F_i\right)\}.$$

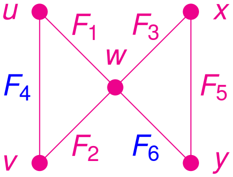
Let S^d denote the d-sphere (sphere of dimension d). To form the *bouquet* or *wedge* of k spheres of dimension d, $\vee^k S^d$, take a point of each sphere and identify the points.

Ex.

$\vee^2 S^1 =$



$\simeq$



$r(uw) = \emptyset, \quad r(vw) = v, \quad r(wx) = x, \quad r(uv) = uv,$
$r(xy) = y, \quad r(wy) = wy.$

If topological spaces X and Y are *homotopic*, write $X \simeq Y$.

Theorem

If Δ is a shellable simplicial complex pure of dimension d, then

$$\Delta \simeq \vee^k S^d$$

where k is the number of facets satisfying $r(F) = F$ in a shelling of Δ. □

Let X be a toplogical space, say $X \subseteq \mathbb{R}^n$ for some n. If X has dimension d then we write $X = X^d$.

Ex. 1. S^d, the d-sphere. For example S^1 is a circle.
2. B^d, the closed d-ball. For example, B^2 is a closed disc.

The *boundary* of $X = X^d$, ∂X, is the set of $p \in X$ such that any (deformed) open d-ball centerd at p contains points both in and out of X.

Ex. 1. $\partial B^d = S^{d-1}$. 2. $\partial S^d = \emptyset$.

Call $C = C^i \subseteq X$ an *i-cycle* if $\partial C = \emptyset$. Call two cycles *equivalent* if they form the boundary of a subset of X.

Ex. If X is a hollow cylinder, then the two copies of S^1 at either end are equivalent.

The *ith reduced Betti number* of X is

$$\tilde{\beta}_i(X) = \text{minimum number of inequivalent } i\text{- cycles which are not boundaries of some subset of } X \text{ and generate all } i\text{-cycles.}$$

If $X \simeq Y$ then $\tilde{\beta}_i(X) = \tilde{\beta}_i(Y)$ for all i. We use reduced Betti numbers since then $\tilde{\beta}_0(X) = 0$ for a connected X.

Proposition

We have

$$\tilde{\beta}_i(S^d) = \begin{cases} 1 & \text{if } i = d, \\ 0 & \text{if } i \neq d. \end{cases}$$

Proof.

We will prove this for S^2. First consider $i = 2$. We have already seen that $\partial S^2 = \emptyset$, so S^2 is a cycle. And it can not be a boundary, since if $\partial Y = S^2$ then Y would have dimension 3 and so $Y \not\subseteq S^2$. Thus $\tilde{\beta}_2(S^2) = 1$.
Now consider $i = 1$. If we have a 1-cylce $C \subset S^2$, then $C = \partial D$ where $D \subseteq S^2$ is the disc interior to C. So every 1-cycle is also a boundary and $\tilde{\beta}_1(S^2) = 0$.
Finally, for $i = 0$. S^2 is connected so $\tilde{\beta}_0(S^2) = 0$. □

Taking wedges adds reduced Betti numbers.

Corollary

We have

$$\tilde{\beta}_i(\vee^k S^d) = \begin{cases} k & \text{if } i = d, \\ 0 & \text{if } i \neq d. \end{cases}$$

□

The *reduced Euler characteristic* of X is

$$\tilde{\chi}(X) = \sum_{i \geq -1} (-1)^i \tilde{\beta}_i(X) = -\tilde{\beta}_{-1}(X) + \tilde{\beta}_0(X) - \tilde{\beta}_1(X) + \cdots$$

By the previous proposition $\tilde{\beta}_i(\vee^k S^d) = k$ if $i = d$ and zero else.

Corollary

We have $\tilde{\chi}(\vee^k S^d) = (-1)^d k$. □

The *ith face number* of a simplicial complex Δ is

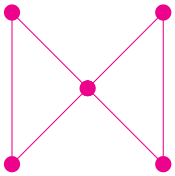
$$f_i(\Delta) = (\text{\# of faces of dimension } i) = (\text{\# of faces of cardinality } i+1.)$$

Theorem

$$\tilde{\chi}(\Delta) = \sum_{i \geq -1} (-1)^i f_i(X) = -f_{-1}(X) + f_0(X) - f_1(X) + \cdots$$ □

Ex. $\Delta \simeq \vee^2 S^1 \overset{\text{Cor}}{\Longrightarrow} \tilde{\chi}(\Delta) = \tilde{\chi}(\vee^2 S^1) = -2.$

$\dim F = -1 \implies F = \emptyset \implies f_{-1}(\Delta) = 1,$

$\dim F = 0 \implies F = \text{vertex} \implies f_0(\Delta) = 5,$ $\Delta =$ 

$\dim F = 1 \implies F = \text{edge} \implies f_1(\Delta) = 6,$

$i \geq 2 \implies f_i(\Delta) = 0, \quad \therefore \tilde{\chi}(\Delta) = -1 + 5 - 6 = -2.$

If $x, y \in P$ (poset) then an *x–y chain of length l in P* is a subposet $C : x = x_0 < x_1 < \ldots < x_l = y$. If P is bounded, let

$$\overline{P} = P - \{\hat{0}, \hat{1}\}.$$

The *order complex* of a bounded P is

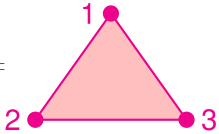
$$\Delta(P) = \text{ set of all chains in } \overline{P}.$$

A subset of a chain is a chain so $\Delta(P)$ is a simplicial complex.

Ex. $P = C_4$,

$\therefore \overline{C}_4 =$

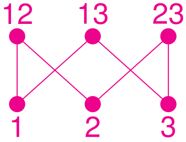
and $\Delta(C_4) =$



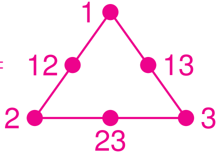
In general $\Delta(C_n) \simeq B^0$, a point.

Ex. $P = B_3$,

$\therefore \overline{B}_3 =$



and $\Delta(B_3) =$



In general $\Delta(B_n) \simeq S^{n-2}$.

Lemma

In $I(P)$: $(\zeta-\delta)^l(x,y) =$ *# of x–y chains of length l.*

Proof. We have $(\zeta-\delta)(x,y) = 1$ if $x < y$ and zero else. So

$$(\zeta-\delta)^l(x,y) = \sum_{x=x_0,x_1,\ldots,x_l=y} (\zeta-\delta)(x_0,x_1)\cdots(\zeta-\delta)(x_{l-1},x_l)$$

$$= \sum_{x=x_0<x_1<\ldots<x_l=y} 1 = \text{ \# of } x\text{–}y \text{ chains of length } l. \quad \square$$

Theorem

In a bounded poset P with $\hat{0} \neq \hat{1}$: $\mu(P) = \tilde{\chi}(\Delta(P))$.

Proof. Using the definition of μ and the lemma,

$$\mu(P) = \zeta^{-1}(P) = (\delta + (\zeta-\delta))^{-1}(P) = \textstyle\sum_{l\geq 0}(-1)^l(\zeta-\delta)^l(P)$$

$$= \textstyle\sum_{l\geq 1}(-1)^l(\text{\# of } \hat{0}\text{–}\hat{1} \text{ chains of length } l \text{ in } P)$$

$$= \textstyle\sum_{l\geq 1}(-1)^{l-2}(\text{\# of chains of length } l-2 \text{ in } \overline{P})$$

$$= \textstyle\sum_{i\geq -1}(-1)^i f_i(\Delta(P)) = \tilde{\chi}(\Delta(P)). \quad \square$$

A poset P is *graded* if it is bounded and ranked.

Ex. Our example posets C_n, B_n, D_n, Π_n are all graded.

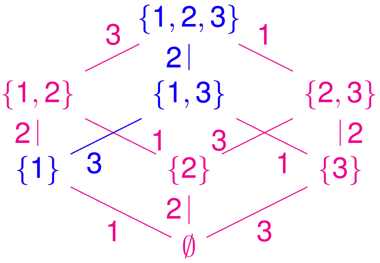
Let $E(P)$ be the edge set of the Hasse diagram of P. A labeling $\ell : E(P) \to \mathbb{R}$ induces a labeling of saturated chains by

$$\ell(x_0 \lhd x_1 \lhd \ldots \lhd x_l) = (\ell(x_0 \lhd x_1), \ldots, \ell(x_{l-1} \lhd x_l)).$$

Ex. For B_n, let

$$\ell(S \lhd T) = T - S.$$

$B_3 =$



$$\ell(\{1\} \lhd \{1,3\} \lhd \{1,2,3\}) = (3,2).$$

Say saturated chain C has a property if $\ell(C)$ has that property. An *EL-labelling* of a graded poset P is $\ell : E \to \mathbb{R}$ such that, for each interval $[x, y] \subseteq P$

1. there is a unique weakly increasing x–y chain C_{xy},
2. C_{xy} is lexicographically least among saturated x–y chains.

All four of our example posets have EL-labelings. We will give the labeling and verify the two conditions for the interval $[\hat{0}, \hat{1}]$.

1. In C_n, let $\ell(i-1 \lhd i) = i$. Then there is only one saturated chain and $\ell(0 \lhd 1 \lhd \ldots \lhd n) = (1, 2, \ldots, n)$.

2. In B_n, let $\ell(S \lhd T) = T - S$. Then ℓ is a bijection between saturated $\hat{0}$–$\hat{1}$ chains and permutations of $[n]$

$$\ell(\hat{0} \lhd \{x_1\} \lhd \{x_1, x_2\} \lhd \ldots \lhd \hat{1}) = (x_1, x_2, \ldots, x_n).$$

There is a unique weakly increasing permutation, $(1, 2, \ldots, n)$, and it is lexicographically smaller than any other permutation.

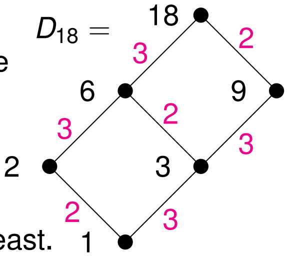
3. In D_n. let $\ell(c \lessdot d) = d/c$.

If $n = \prod_{i=1}^{k} p_i^{m_1}$ then ℓ is a bijection between saturated $\hat{0}$–$\hat{1}$ chains and permutations of the multiset

$$M = \{\{\overbrace{p_1, \ldots, p_1}^{m_1}, \ldots, \overbrace{p_k, \ldots, p_k}^{m_k}\}\}.$$

There is a unique weakly increasing permutation of M and it is lexicographically least.

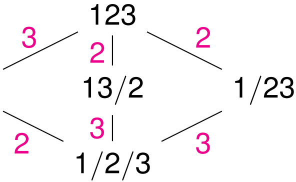
$D_{18} =$



4. In Π_n, if $\pi = B_1/\ldots/B_k$ and merging B_i with B_j forms σ then $\ell(\pi \lessdot \sigma) = \max\{\min B_i, \min B_j\}$.

If C is a saturated $\hat{0}$–$\hat{1}$ chain then $\ell(C)$ is a permutation of $\{2, \ldots, n\}$: for all π, σ we have $2 \le \ell(\pi \lessdot \sigma) \le n$, $\#\ell(C) = n - 1 = \#\{2, \ldots, n\}$,

$\Pi_3 =$



and m appears as a label in C at most once since after merging it is no longer a min. Permutation $(2, \ldots, n)$ only occurs once: $\ell(\hat{0} \lessdot 12/3/\ldots/n \lessdot 123/4/\ldots/n \lessdot \ldots \lessdot \hat{1})$.

Theorem (Björner, 1980)

Let P be a graded poset. If P has an EL-labelling then $\Delta(P)$ is shellable. In fact, if $F_1, \ldots, F_k$ is a list of the saturated $\hat{0} - \hat{1}$ chains in lexicographic order, then $\overline{F}_1, \ldots, \overline{F}_k$ is a shelling of $\Delta(P)$. Furthermore

$$\mu(P) = (-1)^{\rho(P)}(\text{\# of strictly decreasing } F_j). \qquad (1)$$

Proof of (??). Using the first half of the theorem

$$\mu(P) = \tilde{\chi}(\Delta(P)) = (-1)^{\dim \Delta(P)}(\text{\# of } \overline{F}_j \text{ with } r(\overline{F}_j) = \overline{F}_j).$$

The power of -1 is as desired since $\dim \Delta(P) = \rho(P) - 2$. So it suffices to show that $\ell(F_j)$ is strictly decreasing iff $r(\overline{F}_j) = \overline{F}_j$. "$\Longrightarrow$" ("$\Longleftarrow$" is similar) Suppose $F_j : x_0 \lhd \ldots \lhd x_n$ is strictly decreasing. We must show that given any $x_r \in \overline{F}_j$ there is F_i with $i < j$ and $F_i \cap F_j = F_j - \{x_r\}$. Now $x_{r-1} \lhd x_r \lhd x_{r+1}$ is strictly decreasing. Let $x_{r-1} \lhd y_r \lhd x_{r+1}$ be the weakly increasing chain in $[x_{r-1}, x_{r+1}]$. Then $F_i = F_j - \{x_r\} \cup \{y_r\}$ is lexicographically smaller than F_j. So $i < j$ and $F_i \cap F_j = F_j - \{x_r\}$. □

Corollary

(a) $\mu(C_n) = 0$ *if* $n \geq 2$.

(b) $\mu(B_n) = (-1)^n$,

(c) $\mu(D_n) = \begin{cases} (-1)^k & \text{if } n = p_1 \dots p_k \text{ distinct primes,} \\ 0 & \text{else.} \end{cases}$

(d) $\mu(\Pi_n) = (-1)^{n-1}(n-1)!$

Proof. (a) C_n has a single chain which is weakly increasing. So it has no strictly decreasing chain and $\mu(C_n) = (-1)^n \cdot 0 = 0$.
(b) The $\ell(F_i)$ are in bijection with the permutations of $\{1, \dots, n\}$. The unique strictly decreasing permutation is $(n, n-1, \dots, 1)$.
(c) Combine the proofs in (a) and (b).
(d) The $\ell(F_i)$ are permutations of $\{2, \dots, n\}$. Suppose $\ell(F_i) = (n, n-1, \dots, 2)$ where $F_i = \pi_0 \lhd \pi_1 \lhd \dots \lhd \pi_{n-1}$. Then π_1 is obtained from π_0 by merging $\{n\}$ with another block, giving $n-1$ choices. So $n-1$ is still a minimum of some block which must be merged with one of the $n-2$ other blocks to form π_2. Continuing in this manner gives $(n-1)!$ chains. □