Partially Ordered Sets and their Möbius Functions II: Graph Coloring

Bruce Sagan
Department of Mathematics
Michigan State University
East Lansing, MI 48824-1027
sagan@math.msu.edu
www.math.msu.edu/~sagan

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The Chromatic Polynomial of a Graph

The Characteristic Polynomial of a Poset

Lattices

The Bond Lattice of a Graph

Chromatic Symmetric and Quasisymmetric Functions

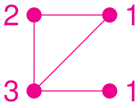
Let $G = (V, E)$ be a finite graph with vertices V and edges E. If S is a set (the color set), then a *coloring* of G is a function

$$c : V \to S.$$

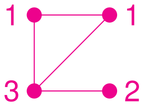
The coloring is *proper* if

$$uv \in E \implies c(u) \neq c(v).$$

Ex. Let $S = [3] = \{1, 2, 3\}$.



is proper,



is not proper, $\mathrm{chr}(G) = 3$.

The *chromatic number* of G is

$$\mathrm{chr}(G) = \text{smallest } \#S \text{ such that there is a proper } c : V \to S.$$

Theorem (Four Color Theorem, Appel-Haken, 1976)

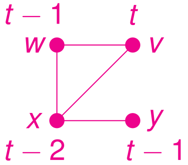
If G is planar (can be drawn in the plane with no edge crossings), then

$$\mathrm{chr}(G) \leq 4. \qquad \square$$

For a positive integer t, the *chromatic polynomial* of G is

$$p(G) = p(G; t) = \# \text{ of proper colorings } c : V \to [t].$$

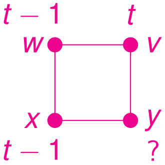
Ex. Coloring vertices in the order v, w, x, y gives choices



$$p(G; t) = t(t-1)(t-2)(t-1)$$
$$= t^4 - 4t^3 + 5t^2 - 2t$$

Note 1. This is a polynomial in t.
2. $\mathrm{chr}(G)$ is the smallest positive integer with $p(G; \mathrm{chr}(G)) > 0$.
3. $p(G; t)$ need not be a product of linear factors.

Ex. Coloring vertices in the order v, w, x, y gives choices



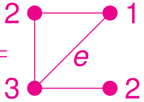
If $G = (V, E)$ is a graph and $e \in E$ then let

$G - e = G$ with e deleted.

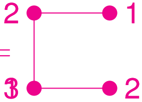
$G/e = G$ with e contracted to a vertex v_e.

Any multiple edge in G/e is replaced by a single edge.

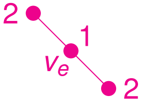
Ex. $G =$



$G - e =$



$G/e =$



Lemma (Deletion-Contraction, DC)

If $G = (V, E)$ is any graph and $e \in E$ then

$$p(G; t) = p(G - e; t) - p(G/e; t).$$

Proof.

Let $e = uv$. It suffices to show $p(G - e) = p(G) + p(G/e)$.

$$\begin{aligned} p(G - e) &= (\#\text{ of proper } c : G - e \to [t] \text{ with } c(u) \neq c(v)) \\ &\quad + (\#\text{ of proper } c : G - e \to [t] \text{ with } c(u) = c(v)) \\ &= p(G) + p(G/e) \end{aligned}$$

as desired. □

$$p(G; t) = p(G - e; t) - p(G/e; t).$$

Corollary (Birkhoff-Lewis, 1946)

For any graph $G = (V, E)$, $p(G; t)$ is a polynomial in t.

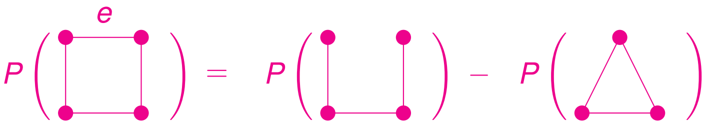
Proof.

Let $|V| = n, |E| = m$. Induct on m. If $m = 0$ then $p(G) = t^n$.
If $m > 0$, then pick $e \in E$. Both $G - e$ and G/e have fewer edges than G. So by DC and induction

$$p(G) = p(G-e) - p(G/e) = \text{polynomial} - \text{polynomial} = \text{polynomial}$$

as desired. □

Ex.



$$= t(t-1)^3 - t(t-1)(t-2).$$

If P is a poset and $x, y \in P$ then an *x–y chain of length r* is

$$C : x = x_0 < x_1 < x_2 < \cdots < x_r = y.$$

So $C \cong C_r$. We say C is *saturated* if it is of the form

$$C : x = x_0 \lhd x_1 \lhd x_2 \lhd \ldots \lhd x_r = y.$$

Call P *ranked* if P has a $\hat{0}$ and, for any $x \in P$, all saturated $\hat{0}$–x chains have the same length. In this case, the *rank of x*, $\rho(x)$, is this common length and

$$\rho(P) = \max_{x \in P} \rho(x).$$

Ex. Posets C_n, B_n, D_n are all ranked.

$$i \in C_n \implies \rho(i) = i.$$

$$S \in B_n \implies \rho(S) = |S|.$$

$$d = \prod_i p_i^{m_i} \in D_n \implies \rho(d) = \sum_i m_i.$$

The characteristic polynomial of a ranked poset P is

$$\chi(P) = \chi(P;t) = \sum_{x \in P} \mu(x) t^{\rho(P)-\rho(x)}.$$

Ex. We have the following characteristic polynomials.

$$\chi(C_n) = \sum_{i=0}^{n} \mu(i) t^{n-i} = t^n - t^{n-1} = t^{n-1}(t-1).$$

$$\chi(B_n) = \sum_{S \in B_n} \mu(S) t^{n-|S|} = \sum_{k=0}^{n} (-1)^k \binom{n}{k} t^{n-k} = (t-1)^n.$$

Note 1. $\chi(C_n)$ and $\chi(B_n)$ factor with nonnegative integer roots.
2. The corank, $\rho(P) - \rho(x)$, is used to make $\chi(P;t)$ monic: the element with the largest corank is $x = \hat{0}$ and $\mu(\hat{0}) = 1$.

Proposition

Let P, Q be ranked posets.
1. $P \cong Q \implies \chi(P;t) = \chi(Q;t)$.
2. $P \times Q$ is ranked and $\chi(P \times Q;t) = \chi(P;t)\chi(Q;t)$. □

If P is a poset then $x, y \in P$ have a *greatest lower bound* or *meet* if there is an element $x \wedge y$ in P such that

1. $x \wedge y \leq x$ and $x \wedge y \leq y$,
2. if $z \leq x$ and $z \leq y$ then $z \leq x \wedge y$.

Also $x, y \in P$ have a *least upper bound* or *join* if there is an element $x \vee y$ in P such that

1. $x \vee y \geq x$ and $x \vee y \geq y$,
2. if $z \geq x$ and $z \geq y$ then $z \geq x \wedge y$.

Call P a *lattice* if every $x, y \in P$ have both a meet and a join.

Ex. 1. C_n is a lattice with $i \wedge j = \min\{i, j\}$ and $i \vee j = \max\{i, j\}$.
2. B_n is a lattice with $S \wedge T = S \cap T$ and $S \vee T = S \cup T$.
3. D_n is a lattice with $c \wedge d = \gcd\{c, d\}$ and $c \vee d = \mathrm{lcm}\{c, d\}$.

Note 1. Any finite lattice L always has a $\hat{0}$, namely $\hat{0} = \bigwedge_{x \in L} x$, and a $\hat{1}$, namely $\hat{1} = \bigvee_{x \in L} x$.
2. If P is a finite poset with a $\hat{1}$ and every pair of element has a meet, then P is a lattice with join

$$x \vee y = \bigwedge_{z \geq x, y} z.$$

If P is a poset with $\hat{0}$ then then *atom set* of P is

$$\mathcal{A}(P) = \{a \in P \ : \ a \rhd \hat{0}\}.$$

Lattice L is *atomic* if every $x \in L$ is a join of atoms.

Ex. $\mathcal{A}(B_n) = \{S \subseteq [n] \ : \ |S| = 1\}$ and B_n is atomic for all n.

A ranked lattice is *semimodular* if, for all $x, y \in L$,

$$\rho(x \wedge y) + \rho(x \vee y) \leq \rho(x) + \rho(y).$$

Ex. C_n, B_n, and D_n are all semimodular. For example, in B_n,

$$\rho(S \wedge T) + \rho(S \vee T) = |S \cap T| + |S \cup T| = |S| + |T| = \rho(S) + \rho(T).$$

Proposition

Lattice L is semimodular $\iff$ for all $x, y \in L$: if x, y cover $x \wedge y$ then $x \vee y$ covers x, y.

Proof. "$\Longrightarrow$" $x, y \rhd x \wedge y$ implies $\rho(x) = \rho(y) = r$ and $\rho(x \wedge y) = r - 1$ for some r. So $x \parallel y$ and $\rho(x \vee y) \geq r + 1$. But

$$\rho(x \vee y) \leq \rho(x) + \rho(y) - \rho(x \wedge y) = r + r - (r - 1) = r + 1.$$

Thus $\rho(x \vee y) = r + 1$ and $x \vee y$ covers x, y. □

A *geometric lattice* is both atomic and semimodular.

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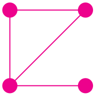
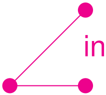
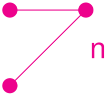
Graph H is a *subgraph* of G, $H \subseteq G$, if $V(H) \subseteq V(G)$ and $E(H) \subseteq E(G)$. Call H *spanning* if $V(H) = V(G)$. Call H *induced* if, for all $v, w \in V(H)$,

$$vw \in E(G) \implies vw \in E(H).$$

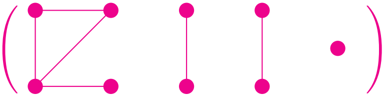
Ex. $G =$  $H =$  induced, $H =$  not induced.

Given $v, w \in V(G)$, a *v–w walk* is

$$W : v = v_0, v_1, \ldots, v_t = w$$

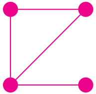
where $v_i v_{i+1} \in E(G)$ for all i. Call G *connected* if there is a v–w walk for all $v, w \in V(G)$. A *component* of G is $K \subseteq G$ which is connected and contained in no larger connected subgraph. Let

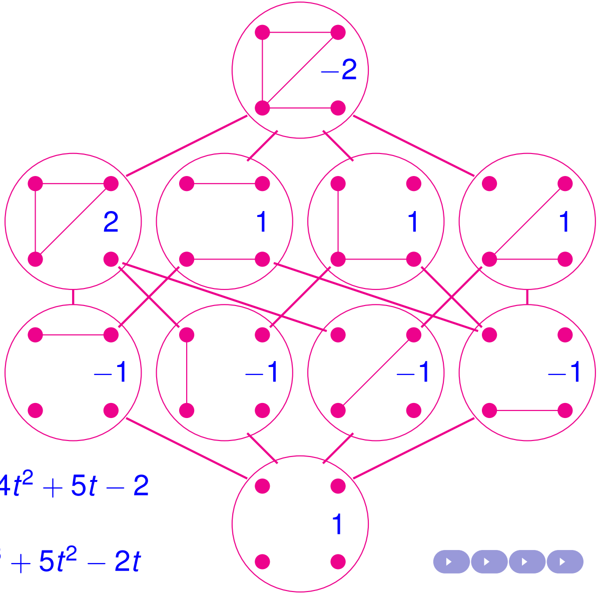
$$k(G) = \text{\# of components of } G.$$

Ex. k  $= 4.$

A *bond* of graph G is a spanning $H \subseteq G$ such that each component of H is induced. The *bond lattice* of G, $L(G)$, is the set of bonds partially ordered by the subgraph relation.

Ex.

$G =$ 

$L(G) =$ 

$\chi(L(G;t)) = t^3 - 4t^2 + 5t - 2$

$p(G;t) = t^4 - 4t^3 + 5t^2 - 2t$

Theorem

For any graph G, the poset $L(G)$ is a geometric lattice.

Proof.

$L(G)$ is finite and has a $\hat{1}$, namely G. So to show it is a lattice, it suffices to show if $H, K \in L(G)$ then $H \wedge K$ exists. Let $J \subseteq G$ be the spanning graph with $E(J) = E(H) \cap E(K)$. Then J is a bond and is the meet of H and K.

To show $L(G)$ is geometric, we first need to prove it is atomic. But $A \in \mathcal{A}(L(G))$ iff $A = A_e$ is a spanning subgraph of G with exactly one edge $e \in E(G)$. Thus for any $H \in L(G)$ we have $H = \vee_{e \in E(H)} H_e$.

To show $L(G)$ is semimodular, suppose $H, K \rhd H \wedge K$ and let the components of $H \wedge K$ have vertices $V_1, \ldots, V_r$. Then the vertices of the components of H are obtained by taking the union of some V_i and V_j and leaving the rest alone, and similarly for the vertices of components of K some V_k asnd V_l. So the vertices of the components of $H \vee K$ are obtained by doing both unions so $H \vee K \rhd H, K$. □

Theorem

For any graph G we have $p(G;t) = t^{k(G)}\chi(L(G);t)$.

Proof. A coloring $c : V(G) \to [t]$, defines a spanning $H_c \subseteq G$ by

$$vw \in E(H_c) \iff vw \in E(G) \text{ and } c(v) = c(w).$$

Then H_c is a bond: If v, w are in the same component of H_c then $c(u) = c(v)$. So if $vw \in E(G)$ then $vw \in E(H_c)$.
Define $f, g : L(G) \to \mathbb{R}$ by

$$\begin{aligned} f(H) &= (\# \text{ of } c : V(G) \to [t] \text{ such that } H_c \supseteq H) &= t^{k(H)}, \\ g(H) &= (\# \text{ of } c : V(G) \to [t] \text{ such that } H_c = H). \end{aligned}$$

Now $f(H) = \sum_{K \geq H} g(K)$. By MIT and $\rho(K) = |V(G)| - k(K)$,

$$p(G) = g(\hat{0}) = \sum_{K \geq \hat{0}} \mu(K) f(K) = \sum_{K \in L(G)} \mu(K) t^{k(K)}$$

$$= t^{k(G)} \sum_K \mu(K) t^{k(K)-k(G)} = t^{k(G)} \sum_K \mu(K) t^{\rho(G)-\rho(K)}$$

$$= t^{k(G)} \chi(L(G)) \qquad \square$$

Let $\mathbf{x} = \{x_1, x_2, \dots\}$. Coloring $c : V(G) \to \mathbb{P}$ has *monomial*

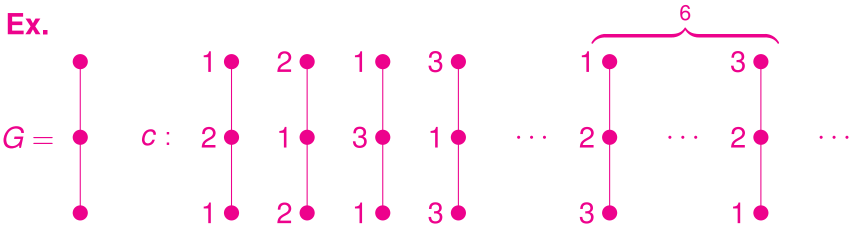
$$\mathbf{x}^c = \prod_{v \in V(G)} x_{c(v)}.$$

The *chromatic symmetric function of G* (Stanley, 1995) is

$$X(G) = X(G; \mathbf{x}) = \sum_{c\,:\,V(G)\to\mathbb{P}\ \text{proper}} \mathbf{x}^c.$$

Note 1. Permuting colors in a proper coloring gives a proper coloring, so $X(G; \mathbf{x})$ is a symmetric function.
2. If $x_i = 1$ for $i \le t$ and $x_i = 0$ for $i > t$ then $X(G; \mathbf{x}) = p(G; t)$.

Ex.



$$X(G; \mathbf{x}) = x_1^2 x_2 + x_1 x_2^2 + x_1^2 x_3 + x_1 x_3^2 + \cdots + 6 x_1 x_2 x_3 + \cdots$$

Bases for the algebra of symmetric functions are indexed by *integer partitions* $\lambda = (\lambda_1, \ldots, \lambda_k)$ where $\lambda_1 \geq \cdots \geq \lambda_k$ are in $\mathbb{P}$. For example, the *power sum* basis is defined by

$$\begin{aligned} p_n(\mathbf{x}) &= x_1^n + x_2^n + x_3^n + \ldots, \\ p_\lambda(\mathbf{x}) &= p_{\lambda_1} p_{\lambda_2} \ldots p_{\lambda_k}. \end{aligned}$$

If G has components $G_1, G_2, \ldots, G_k$ then let

$$\lambda(G) = (|V(G_1)|, |V(G_2)|, \ldots, |V(G_k)|).$$

Theorem (Stanley)

For any graph G we have

$$X(G; \mathbf{x}) = \sum_{K \in L(G)} \mu(K) p_{\lambda(K)}. \qquad \square$$

If $x_i = 1$ for $i \leq t$ and $x_i = 0$ for $i > t$ then $p_n(\mathbf{x}) = t$ and $p_\lambda(\mathbf{x}) = t^k$ where $\lambda = (\lambda_1, \ldots, \lambda_k)$. So the above theorem gives

$$P(G; t) = \sum_{K \in L(G)} \mu(K) t^{k(G)}.$$

Let $V(G) = [n]$. Coloring $c : V(G) \to \mathbb{P}$ has *ascent number*

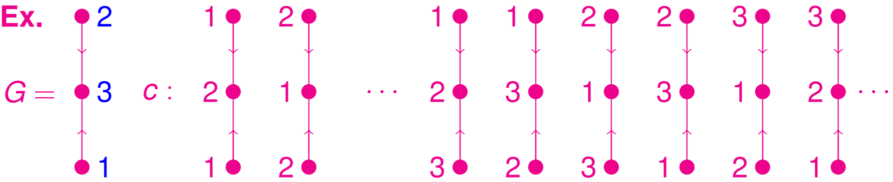
$$\operatorname{asc} G = \#\{vw \in E(G)) : \ v < w \text{ and } c(v) < c(w)\}.$$

Replacing $vw \in E(G)$ with $v < w$ by an arc $\vec{vw}$, the arc of an ascent points from a smaller vertex to a larger. The *chromatic quasisymmetric function of G* (Shareshian-Wachs, 2014) is

$$X(G; \mathbf{x}, t) = \sum_{c : V(G) \to \mathbb{P} \ \text{ proper}} t^{\operatorname{asc} G} \mathbf{x}^c.$$

Note 1. An order-preserving permutation of colors preserves ascents, so the coefficient of t^k in $X(G; \mathbf{x}, t)$ is quasisymmetric.
2. $X(G, \mathbf{x}, 1) = X(G; \mathbf{x})$.

Ex.



$$X(G; \mathbf{x}, t) = t^2 x_1^2 x_2 + x_1 x_2^2 + \cdots + (t + t^2 + 1 + t^2 + 1 + t) x_1 x_2 x_3 + \cdots$$