Combinatorial and colorful proofs of cyclic sieving phenomena

Bruce Sagan
Michigan State University
and the National Science Foundation
www.math.msu.edu/~sagan

and

Yuval Roichman
Bar Ilan University

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Suppose S is a set and let C be a finite cyclic group acting on S. If $g \in C$, we let

$$S^g = \{t \in S \ : \ gt = t\} \text{ and } o(g) = \text{order of } g \text{ in } C.$$

We also let

$$\omega_d = \text{primitive } d\text{th root of unity.}$$

Finally, suppose we are given $f(q) \in \mathbb{R}[q]$, a polynomial in q.

Definition (Reiner-Stanton-White, 2004)

The triple $(S, C, f(q))$ exhibits the cyclic sieving phenomenon (c.s.p.) if, for all $g \in C$, we have

$$\#S^g = f(\omega_{o(g)}).$$

Notes. 1. The case $\#C = 2$ was first studied by Stembridge [1994] and called "the $q = -1$ phenomenon."
2. Recent work by: Bessis, Eu, Fu, Petersen, Pylyavskyy, Rhoades, Serrano, Shareshian, Wachs.

Let $[n] = \{1, 2, \dots, n\}$ and

$$S = \binom{[n]}{k} = \{T \subseteq [n] \ : \ \#T = k\}.$$

Let $C_n = \langle (1, 2, \dots, n) \rangle$. Now $g \in C_n$ acts on $T = \{t_1, \dots, t_k\}$ by

$$gT = \{g(t_1), \dots, g(t_k)\}.$$

Ex. Suppose $n = 4$ and $k = 2$. We have

$$S = \{12,\ 13,\ 14,\ 23,\ 24,\ 34\}.$$

Also

$$C_4 = \langle (1, 2, 3, 4) \rangle = \{e,\ (1, 2, 3, 4),\ (1, 3)(2, 4),\ (1, 4, 3, 2)\}.$$

For $g = (1, 3)(2, 4)$ we have

$$(1,3)(2,4)12 = 34, \quad (1,3)(2,4)13 = 13, \quad (1,3)(2,4)14 = 23,$$
$$(1,3)(2,4)23 = 14, \quad (1,3)(2,4)24 = 24, \quad (1,3)(2,4)34 = 12.$$

Let $[n]_q = 1 + q + q^2 + \cdots + q^{n-1}$ and $[n]_q! = [1]_q[2]_q \cdots [n]_q$. Define the *Gaussian polynomials* or *q-binomial coeffiecients* by

$$\left[\begin{array}{c} n \\ k \end{array} \right]_q = \frac{[n]_q!}{[k]_q![n-k]_q!}.$$

Theorem (Reiner-Stanton-White)

The c.s.p. is exhibited by

$$\left(\binom{[n]}{k}, C_n, \left[\begin{array}{c} n \\ k \end{array} \right]_q \right).$$

Ex. Consider $n = 4$, $k = 2$. So

$$\left[\begin{array}{c} 4 \\ 2 \end{array} \right]_q = \frac{[4]_q!}{[2]_q![2]_q!} = 1 + q + 2q^2 + q^3 + q^4.$$

For $g = (1,3)(2,4)$ we have $o(g) = 2$ and $\omega = -1$ so

$$\left[\begin{array}{c} 4 \\ 2 \end{array} \right]_{-1} = 1 - 1 + 2 - 1 + 1 = 2 = \#S^{(1,3)(2,4)}.$$

Most proofs of the c.s.p. involve either explicitly evaluating polynomials at roots of unity or representation theory. We have given the first purely combinatorial proof. To combinatorially prove $(S, C, f(q))$ exhibits the c.s.p., first find a weight function $\mathrm{wt} : S \to \mathbb{R}[q]$ such that

$$f(q) = \sum_{T \in S} \mathrm{wt}\, T. \tag{1}$$

If $B \subseteq S$ we let $\mathrm{wt}\, B = \sum_{T \in B} \mathrm{wt}\, T$. For each $g \in C$ we then find a partition of S

$$\pi = \pi_g = \{B_1, B_2, \ldots\}$$

satisfying, the following two criteria where $\omega = \omega_{o(g)}$:

(I) For $1 \leq i \leq \#S^g$ we have $\#B_i = 1$ and $\mathrm{wt}\, B_i|_\omega = 1$.

(II) For $i > \#S^g$ we have $\#B_i > 1$ and $\mathrm{wt}\, B_i|_\omega = 0$.

We then have the c.s.p. since for each $g \in C$

$$f(\omega) = \sum_{T \in S} \mathrm{wt}\, T|_\omega = \sum_i \mathrm{wt}\, B_i|_\omega = \overbrace{1 + \cdots + 1}^{\#S^g} + 0 + 0 + \cdots = \#S^g.$$

Theorem (Reiner, Stanton, White)

The c.s.p. is exhibited by the triple $\left(\binom{[n]}{k}, C_n, \begin{bmatrix} n \\ k \end{bmatrix}_q \right).$

Combinatorial Proof. For $T \in \binom{[n]}{k}$ let wt $T = q^{\sum_{t \in T} t - \binom{k+1}{2}}$.

$$\therefore \sum_{T \in \binom{[n]}{k}} \text{wt}\, T = \begin{bmatrix} n \\ k \end{bmatrix}_q .$$

Suppose $g \in C_n$ with $o(g) = d$, say $g = (1, \ldots, n)^{n/d}$ so

$$g = (1, 1 + n/d, 1 + 2n/d, \ldots)(2, 2 + n/d, 2 + 2n/d, \ldots) \cdots$$

Let $g_i = (i, i + n/d, i + 2n/d, \ldots)$ for $1 \le i \le n/d$. So $T \in S^g$ iff T can be written as $T = g_{i_1} \uplus g_{i_2} \uplus \cdots$

Ex. If $n = 4$ and $k = 2$ then wt$\{t_1, t_2\} = q^{t_1 + t_2 - 3}$. So

T	: 12	13	14	23	24	34,
$\sum_T$ wt T	$= q^0$	$+ \; q^1$	$+ \; q^2$	$+ \; q^2$	$+ \; q^3$	$+ \; q^4$

$$= \begin{bmatrix} 4 \\ 2 \end{bmatrix}_q .$$

If $g = (1, 3)(2, 4)$ then $S^g = \{13, 24\}$.

Let $h = (1, 2, \ldots, d)(d+1, d+2, \ldots, 2d) \cdots$ and $h_i = (id+1, id+2, \ldots, (i+1)d)$ for $0 \le i < n/d$. Since g and h have the same cycle type, $\#\binom{[n]}{k}^g = \#\binom{[n]}{k}^h$. For any $T \in \binom{[n]}{k}$ define the block B of π containing T by as follows. If $hT = T$ then $B = \{T\}$. If $hT \neq T$, then find the smallest index i such that $0 < \#(T \cap h_i) < d$ and let

$$B = \{T, h_i T, h_i^2 T, \ldots, h_i^{d-1} T\}.$$

Proof of (II): If $\omega = \omega_d$, wt $T = q^j$, $\ell = |T \cap h_i|$ then $0 < \ell < d$.

$$\therefore \text{wt}\, B = \text{wt}\, T + \text{wt}\, h_i T + \cdots + \text{wt}\, h_i^{d-1} T$$

$$\therefore \text{wt}\, B|_\omega = \omega^j + \omega^{j+\ell} + \cdots + \omega^{j+(d-1)\ell} = \omega^j \frac{1 - \omega^{d\ell}}{1 - \omega^\ell} = 0$$

since $\omega^d = 1$ and $\omega^\ell \neq 1$. ■

Ex: $n = 4$, $k = 2$, $g = (1,3)(2,4)$. So $h = (1,2)(3,4)$, and π:

$\{12\}$, $\{34\}$, $\{13, (1,2)13\} = \{13, 23\}$, $\{14, (1,2)14\} = \{14, 24\}$.

$\text{wt}\{12\}|_{-1} = (-1)^0 = 1$, $\quad \text{wt}\{13, 23\}|_{-1} = (-1)^1 + (-1)^2 = 0$,
$\text{wt}\{34\}|_{-1} = (-1)^4 = 1$, $\quad \text{wt}\{14, 24\}|_{-1} = (-1)^2 + (-1)^3 = 0$.

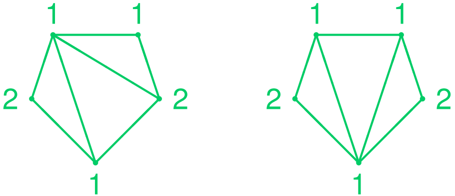

Figure: Two triangulations: proper (left) and improper (right)

A *triangulation*, T, is a subdivision of a regular polygon P into triangles using noncrossing diagonals. Let $\mathcal{T}_n$ be the set of all triangulations of an n-gon. Then

$$\#\mathcal{T}_{n+2} = \frac{1}{n+1}\binom{2n}{n}.$$

Let C_n be the group of rotations of a regular n-gon.

Theorem (Reiner-Stanton-White)

The c.s.p. is exhibited by the triple

$$\left(\mathcal{T}_{n+2},\ C_{n+2},\ \frac{1}{[n+1]_q}\begin{bmatrix} 2n \\ n \end{bmatrix}_q \right). \qquad \blacksquare$$

Label (color) the vertices of P cyclically $1, 2, 1, 2, \ldots$ Call a triangulation *proper* if it contains no monochromatic triangle.

Let $\mathcal{P}_n$ be the set of proper triangulations of a regular n-gon.

Theorem (S)

We have

$$\#\mathcal{P}_{n+2} = \begin{cases} \dfrac{2^m}{2m+1}\dbinom{3m}{m} & \text{if } n = 2m, \\[2ex] \dfrac{2^{m+1}}{2m+2}\dbinom{3m+1}{m} & \text{if } n = 2m+1. \end{cases}$$

■

Note that for n odd, rotation does not preserve properness. If $n = 2m$ then let

$$p_n(q) = \frac{(1+q^2)\left([2]_q^{m-1} - [2]_q^{\lceil m/2\rceil - 1} + 2^{\lceil m/2\rceil - 1}\right)}{[2m+1]_q} \begin{bmatrix} 3m \\ m \end{bmatrix}_q .$$

Theorem (Roichman-S)

If $n = 2m$ then $(\mathcal{P}_{n+2}, C_{n+2}, p_n(q))$ exhibits the c.s.p. ■

I. Is there a combinatorial proof of the Reiner-Stanton-White theorem about (uncolored) triangulations? The first difficulty is to find a weight function wt : $\mathcal{T}_n \to \mathbb{R}[q]$ such that

(a) we have

$$\sum_{T \in \mathcal{T}_{n+2}} \mathrm{wt}\, T = \frac{1}{[n+1]_q} \left[\begin{array}{c} 2n \\ n \end{array} \right]_q ,$$

(b) and wt T is well behaved with respect to rotation.

Note that there are various other families of combinatorial objects (Dyck paths, 2-rowed standard Young tableaux) with a weighting giving the q-Catalan numbers. The hope is that one of these can be reformulated in terms of triangulations in a way that (b) above will be satisfied.

II. Let $\mathcal{D}_{n,k}$ be the set of all dissections of a regular n-gon using k noncrossing diagonals. So if $k = n - 3$ then we have a triangulation. We have

$$\#\mathcal{D}_{n,k} = \frac{1}{n+k}\binom{n+k}{k+1}\binom{n-3}{k}.$$

There is an action of C_n on dissections just as on triangulations.

Theorem (Reiner-Stanton-White)

The c.s.p. is exhibited by the triple

$$\left(\mathcal{D}_{n,k}, C_n, \frac{1}{[n+k]_q} \begin{bmatrix} n+k \\ k+1 \end{bmatrix}_q \begin{bmatrix} n-3 \\ k \end{bmatrix}_q \right). \qquad \blacksquare$$

Burstein-Roichman-S are investigating proper dissections (no monochromatic sub-polygon) even for $q = 1$. So far we have proved a formula for triangulations with a different coloring scheme which involves a new basis for the algebra of symmetric functions.

MERCI BEAUCOUP!