Quasisymmetric Macdonald Polynomials

Sarah K Mason
Wake forest University

MSU Combinatorics and Graph Theory Seminar
February 24, 2021

Outline

GOAL: Introduce a quasisymmetric analogue of Macdonald Polynomials

Sym, QSym and Polynomials

Polynomials Sums of monomials (no restrictions)

⋆ Bases indexed by weak compositions

$x^{102} = x_1 x_3^2$

QSym Shifting nonzero exponents fixes polynomial

⋆ Bases indexed by strong compositions

$M_{12} = x_1 x_2^2 + x_1 x_3^2 + x_2 x_3^2$

Sym Permuting indices fixes polynomial ($s_i(f) = f$)

⋆ Bases indexed by partitions

$m_{21} = x_1^2 x_2 + x_1^2 x_3 + x_2^2 x_3 + x_1 x_2^2 + x_1 x_3^2 + x_2 x_3^2$

Sym, QSym and Polynomials

Polynomials Sums of monomials (no restrictions)

⋆ Bases indexed by weak compositions

$x^{102} = x_1 x_3^2$

QSym Shifting nonzero exponents fixes polynomial

⋆ Bases indexed by strong compositions

$M_{12} = x_1 x_2^2 + x_1 x_3^2 + x_2 x_3^2 = x_1^1 x_2^2 x_3^0 + x_1^1 x_2^0 x_3^2 + x_1^0 x_2^1 x_3^2$

Sym Permuting indices fixes polynomial $(s_i(f) = f)$

⋆ Bases indexed by partitions

$m_{21} = x_1^2 x_2 + x_1^2 x_3 + x_2^2 x_3 + x_1 x_2^2 + x_1 x_3^2 + x_2 x_3^2$

Sym, QSym and Polynomials

Polynomials Sums of monomials (no restrictions)

⋆ Bases indexed by weak compositions

$x^{102} = x_1 x_3^2$

QSym Shifting nonzero exponents fixes polynomial

⋆ Bases indexed by strong compositions

$$M_{12} = x_1x_2^2 + x_1x_3^2 + x_2x_3^2 = x_1^1x_2^2x_3^0 + x_1^1x_2^0x_3^2 + x_1^0x_2^1x_3^2$$

Sym Permuting indices fixes polynomial ($s_i(f) = f$)

⋆ Bases indexed by partitions

$$m_{21} = x_1^2x_2 + x_1^2x_3 + x_2^2x_3 + x_1x_2^2 + x_1x_3^2 + x_2x_3^2$$
$$= x_1^2x_2^1x_3^0 + x_1^2x_2^0x_3^1 + x_1^0x_2^2x_3^1 + x_1^1x_2^2x_3^0 + x_1^1x_2^0x_3^2 + x_1^0x_2^1x_3^2$$

Bases for Symmetric Functions

- **Monomial:** $$m_\lambda(x) = \sum_{\mathrm{dec}(\gamma)=\lambda} x^\gamma$$

$$m_{21}(x_1,x_2,x_3) = x_1^2x_2 + x_1^2x_3 + x_2^2x_3 + x_1x_2^2 + x_1x_3^2 + x_2x_3^2$$

- **Elementary:** $$e_r(x) = m_{(1^r)}(x), \qquad e_\lambda = e_{\lambda_1}e_{\lambda_2}\cdots e_{\lambda_k}$$

$$e_{21}(x_1,x_2,x_3) = (x_1x_2 + x_1x_3 + x_2x_3)(x_1+x_2+x_3)$$

- **Homogeneous:** $$h_r(x) = \sum_{|\lambda|=r} m_\lambda, \qquad h_\lambda = h_{\lambda_1}h_{\lambda_2}\cdots h_{\lambda_k}$$

$$h_{21}(x_1,x_2,x_3) = (m_2 + m_{11})(m_1)$$

- **Power Sum:** $$p_r = m_r, \qquad p_\lambda = p_{\lambda_1}p_{\lambda_2}\cdots p_{\lambda_k}$$

$$p_{21}(x_1,x_2,x_3) = (x_1^2+x_2^2+x_3^2)(x_1+x_2+x_3)$$

Schur functions

Definition

$$s_\lambda(x_1, \dots, x_n) = \sum_{T \in SSYT(\lambda)} x^T,$$

where $SSYT(\lambda)$ is the set of all SSYT of shape λ, meaning row entries weakly increase left to right and column entries strictly increase bottom to top (French notation).

Example

$$s_{2,1}(x_1, x_2, x_3) =$$

$$\begin{array}{|c|c} \hline 2 & \\ \hline 1 & \multicolumn{1}{c|}{1} \\ \hline \end{array} \quad \begin{array}{|c|c} \hline 3 & \\ \hline 1 & \multicolumn{1}{c|}{1} \\ \hline \end{array} \quad \begin{array}{|c|c} \hline 2 & \\ \hline 1 & \multicolumn{1}{c|}{2} \\ \hline \end{array} \quad \begin{array}{|c|c} \hline 3 & \\ \hline 1 & \multicolumn{1}{c|}{2} \\ \hline \end{array} \quad \begin{array}{|c|c} \hline 2 & \\ \hline 1 & \multicolumn{1}{c|}{3} \\ \hline \end{array} \quad \begin{array}{|c|c} \hline 3 & \\ \hline 1 & \multicolumn{1}{c|}{3} \\ \hline \end{array} \quad \begin{array}{|c|c} \hline 3 & \\ \hline 2 & \multicolumn{1}{c|}{2} \\ \hline \end{array} \quad \begin{array}{|c|c} \hline 3 & \\ \hline 2 & \multicolumn{1}{c|}{3} \\ \hline \end{array}$$

$$x_1^2 x_2 + x_1^2 x_3 + x_1 x_2^2 \ + \ 2x_1 x_2 x_3 + x_1 x_3^2 + x_2^2 x_3 + x_2 x_3^2$$

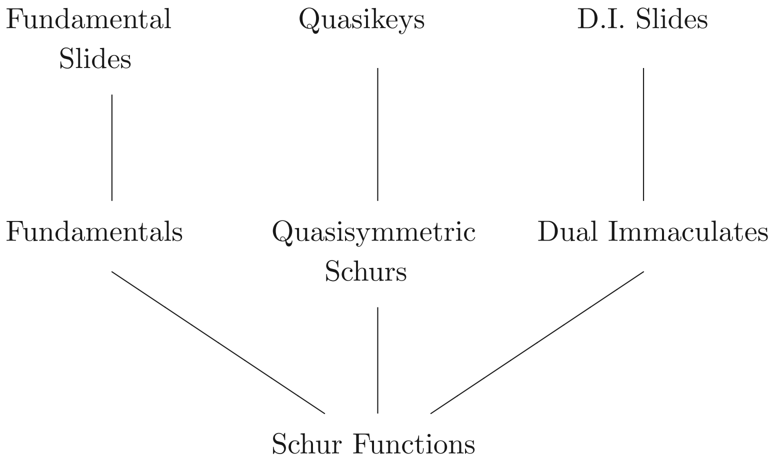
Schur functions...

- form a basis for all symmetric functions.
- are closely related to other symmetric function bases.

$$e_r = s_{(1^r)}, \qquad h_r = s_r$$

- correspond to characters of irr reps of GL_n.
- describe the cohomology of the Grassmannian.
- have many nice combinatorial properties.
- generalize to Macdonald polynomials $(P_\lambda(X; q, t))$.

Analogues of Schur functions in QSym and the Polynomial Ring



Macdonald Polynomials (I.G. Macdonald, 1988)

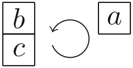
$$\underline{\text{Triangularity}}\text{: } P_\lambda(X;q,t) = m_\lambda + \sum_{\mu<\lambda} c_{\lambda,\mu}(q,t)m_\mu$$

$$\underline{\text{Orthogonality}}\text{: } \langle P_\lambda, P_\mu \rangle_{q,t} = 0 \quad \Longleftrightarrow \quad \lambda \neq \mu$$

- Unique eigenfunction of a certain divided difference operator
- Contain other symmetric function bases as special cases
- Generalize Jack symmetric functions, Hall-Littlewood polynomials

HHL Formula for Modified Macdonald Polynomials

Inversion Triples



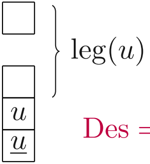
$a < b \leq c$

$c < a < b$

$b \leq c < a$

$\mathrm{inv}(\sigma) = \#$ inversion triples

Descents



$\mathrm{Des} = \{u | \sigma(u) > \sigma(\underline{u})\}$

$$\mathrm{maj}(\sigma) = \sum_{u \in \mathrm{Des}(\sigma)} (\mathrm{leg}(u) + 1)$$

Example

$\sigma =$

5	4	6	
2	1	6	
1	4	5	4
∞	∞	∞	∞

$\mathrm{inv}(\sigma) = 3$

$\mathrm{maj}(\sigma) = 6$

$x^\sigma = x_1^2 x_2 x_4^3 x_5^2 x_6^2$

HHL Formula for Modified Macdonald Polynomials

Theorem (Haglund-Haiman-Loehr, 2004)

$$\tilde{H}_\lambda(X;q,t) = \sum_{\sigma:\mathrm{dg}(\lambda)\to\mathbb{Z}^+} x^\sigma q^{\mathrm{inv}(\sigma)} t^{\mathrm{maj}(\sigma)}$$

Example: $\tilde{H}_{21}(X;q,t)$

$$m_3 : \begin{array}{cc} 1 & \\ 1 & 1 \end{array} \qquad m_{21} : \begin{array}{cc} 2 & \\ 1 & 1 \end{array}, \quad \begin{array}{cc} 1 & \\ 1 & 2 \end{array}, \quad \begin{array}{cc} 1 & \\ 2 & 1 \end{array}$$

$$1 \qquad\qquad t \qquad 1 \qquad q$$

$$m_{111} : \begin{array}{cc} 3 & \\ 1 & 2 \end{array}, \quad \begin{array}{cc} 2 & \\ 1 & 3 \end{array}, \quad \begin{array}{cc} 3 & \\ 2 & 1 \end{array}, \quad \begin{array}{cc} 1 & \\ 2 & 3 \end{array}, \quad \begin{array}{cc} 2 & \\ 3 & 1 \end{array}, \quad \begin{array}{cc} 1 & \\ 3 & 2 \end{array}$$

$$t \qquad t \qquad qt \qquad 1 \qquad q \qquad q$$

$$H_{21}(X;q,t) = m_3 + (1+t+q)m_{21} + (1+2t+2q+qt)m_{111}$$

Nonsymmetric Macdonald Polynomials ($E_\gamma(X;q,t)$)

Macdonald polynomials break down into non-symmetric components (Cherednik, Macdonald, Marshall, Opdam,...):

$$P_\mu(X;q,t) = \prod_{s\in\mu}(1-q^{leg(s)+1}t^{arm(s)}) \sum_{\mathrm{dec}(\gamma)=\mu} \frac{E_\gamma(X;q,t)}{\prod_{s\in\gamma}(1-q^{leg(s)+1}t^{arm(s)})}$$

which can then be specialized to $q=t=0$ to obtain a decomposition of the Schur functions:

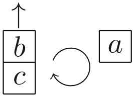
$$P_\mu(X_n;0,0) = s_\mu(X_n) = \sum_{\mathrm{dec}(\gamma)=\mu} E_\gamma(X_n;0,0),$$

$\mathrm{dec}(\gamma)$ = rearrangement of γ into weakly decreasing order.

$s_{21} = E_{210}+E_{201}+E_{021}+E_{120}+E_{102}+E_{012}$ (specialized to $q=t=0$)

Coinversions and arms

Type A Coinv Triples

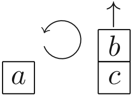


$b < a \leq c$

$c < b \leq a$

$a \leq c < b$

Type B Coinv Triples



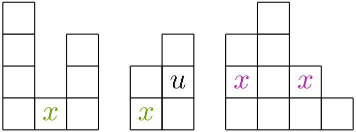
$b < a \leq c$

$a < c \leq b$

$c < b \leq a$

$$\mathrm{coinv}(\sigma) = \#\text{coinversion triples}$$

$\mathrm{arm}(u) =$
cells to R in same row
(weakly shorter column),
cells to L in row below
(strictly shorter column)



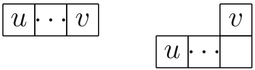
Permuted Basement Macdonald Polynomials

Definition (Ferreira (2011), Alexandersson (2016))

Let γ be a composition with n parts and $\sigma \in S_n$.

$$E_\gamma^\sigma(X;q,t) = \sum_{F \in \mathrm{NAF}} x^F q^{\mathrm{maj}(F)} t^{\mathrm{coinv}(F)} \prod_{\substack{u \in F \\ F(\underline{u}) \neq F(u)}} \frac{1-t}{1-q^{1+\mathrm{leg}(u)} t^{1+\mathrm{arm}(u)}}$$

NAF : non-attacking filling ($u \neq v$)



Example: E_{110}^{312}

$$\begin{array}{|c|c|c}\hline 1 & 2 & \\ \hline 3 & 1 & \multicolumn{1}{c|}{2} \\ \hline\end{array} \qquad \frac{qt(1-t)^2 x_1 x_2}{(1-qt)(1-qt^2)}$$

$$\begin{array}{|c|c|c}\hline 2 & 1 & \\ \hline 3 & 1 & \multicolumn{1}{c|}{2} \\ \hline\end{array} \qquad \frac{(1-t) x_1 x_2}{(1-qt^2)}$$

$$\begin{array}{|c|c|c}\hline 3 & 1 & \\ \hline 3 & 1 & \multicolumn{1}{c|}{2} \\ \hline\end{array} \qquad x_1 x_3$$

$$\begin{array}{|c|c|c}\hline 3 & 2 & \\ \hline 3 & 1 & \multicolumn{1}{c|}{2} \\ \hline\end{array} \qquad \frac{q(1-t) x_2 x_3}{(1-qt)}$$

Quasisymmetric Macdonald Polynomials $G_\gamma(X;q,t)$

Desirable properties for quasisymmetric Macdonald polynomials:

- $G_\gamma(X;q,t)$ is quasisymmetric.
- $G_\gamma(X;0,0)$ is the quasisymmetric Schur function.
- $P_\lambda(X;q,t)$ is a positive sum of quasisymmetric Macdonald polynomials.
- $G_\gamma(X;q,t)$ has a combinatorial formula.
- $G_\gamma(X;q,t)$ satisfies triangularity and orthogonality properties.

Previous approaches:

Weighted Sums of Nonsymmetric Macdonald Polynomials

$$\sum_{\gamma^+=\alpha} c(q,t) E_\gamma(X;q,t)$$

CANNOT be quasisymmetric for certain choices of α.

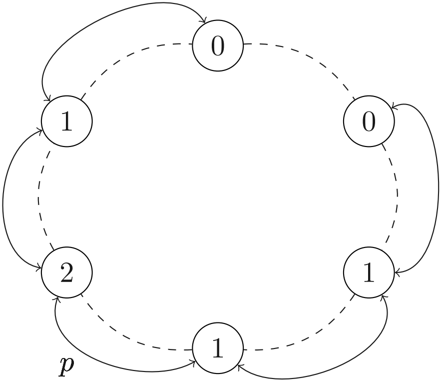
Quasisymmetric Power Sums (Ballantine, Daugherty, Hicks, M, Niese)

$$\Psi_\alpha = z_\alpha \sum_{\beta \succeq \alpha} \frac{1}{\pi(\alpha,\beta)} M_\beta$$

- product formula (shuffles)
- expansion into fundamentals
- positivity results (Alexandersson and Sulzgruber)

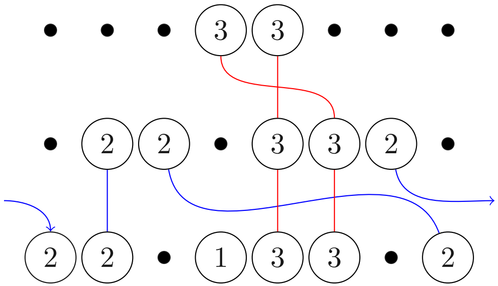
Asymmetric Simple Exclusion Process (ASEP)

- 1960s (Macdonald-Gibbs-Pipkin, Spitzer)
- weighted particles hopping on a 1-dimensional lattice
- multispecies ASEP (different "weights")
- probability of place-swapping 1 or t



Multiline Queues

- stationary distribution of multispecies ASEP on a ring (James Martin 2018)
- Weighted sums of MLQs (Corteel, Mandelshtam, Williams)



type: $(2,2,0,1,3,3,0,2)$

$$wt = x_1 x_2^2 x_3 x_4^2 x_5^3 x_6^2 x_7 x_8 \cdot \frac{(1-t)^3 t^2 q}{(1-qt^4)(1-qt^3)(1-qt^2)}$$

Macdonald Polynomials from Multiline Queues

$$F_\gamma(X;q,t) = \sum_Q wt(Q)$$

Theorem (Corteel, Mandelshtam, Williams)

$$P_\lambda(X;q,t) = \sum_{dec(\gamma)=\lambda} F_\gamma(X;q,t)$$

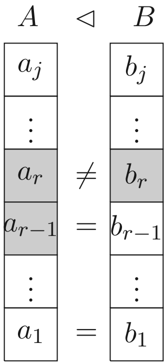
Example: $P_{21} = F_{210} + F_{201} + F_{120} + F_{102} + F_{021} + F_{012}$

Theorem (Corteel, Mandelshtam, Williams)

$$F_\gamma(X;q,t) = E^\sigma_{\mathrm{inc}(\gamma)},$$

where σ is the longest permutation s.t.
$\gamma_{\sigma(1)} \leq \gamma_{\sigma(2)} \leq \cdots \leq \gamma_{\sigma(n)}$.

Sorted tableaux



grey cells:
non-inversion triple
(clockwise)

A filling σ of the diagram of a partition λ is a sorted tableau if the columns of a fixed height appear in weakly increasing order with respect to $\lhd$.

$$\boxed{\mathrm{ST}(\lambda) = \text{all sorted tableaux of } \mathrm{dg}(\lambda)}$$

A sorted tableau of dg(5, 5, 3, 3, 3, 3)

$$F = \begin{array}{|c|c|cccc}\hline 4 & 5 & & & & \\ \hline 3 & 1 & & & & \\ \hline 5 & 2 & \multicolumn{1}{c|}{1} & \multicolumn{1}{c|}{3} & \multicolumn{1}{c|}{2} & \multicolumn{1}{c|}{2} \\ \hline 4 & 4 & \multicolumn{1}{c|}{5} & \multicolumn{1}{c|}{4} & \multicolumn{1}{c|}{1} & \multicolumn{1}{c|}{1} \\ \hline 5 & 5 & \multicolumn{1}{c|}{2} & \multicolumn{1}{c|}{3} & \multicolumn{1}{c|}{3} & \multicolumn{1}{c|}{3} \\ \hline \end{array}$$

For rectangles, set $\mathrm{perm}_t(\sigma) = \binom{n}{u_1,\ldots,u_j}_t$, where u_i = number of copies of i^{th} distinct column. Then for σ a concatenation of rectangular sorted tableaux $\sigma_1, \ldots, \sigma_\ell$ define

$$\mathrm{perm}_t(\sigma) = \prod_{i=1}^{\ell} \mathrm{perm}_t(\sigma_i).$$

$$\mathrm{perm}_t(F) = \binom{2}{1,1}_t \binom{4}{1,1,2}_t = 1 + 2t + 3t^2 + 3t^3 + 2t^4 + t^5$$

Macdonald Polynomials from Sorted Tableaux

Theorem (Corteel-Haglund-Mandelshtam-M-Williams)

$$\tilde{H}_\lambda(X;q,t) = \sum_{\sigma \in \mathrm{ST}(\lambda)} x^\sigma t^{\mathrm{inv}(\sigma)} q^{\mathrm{maj}(\sigma)} \,\mathrm{perm}_t(\sigma)$$

- More compact than the HHL formula
- Origin: multiline queues
- Polyqueue tableau formulation (Ayyer-Mandelshtam-Martin)

$\tilde{H}_{22}(X;q,t)$

$$m_4 : \underset{1}{\begin{array}{|c|c|}\hline 1&1\\\hline 1&1\\\hline\end{array}} \qquad m_{31} : \underset{q[2]_t}{\begin{array}{|c|c|}\hline 2&1\\\hline 1&1\\\hline\end{array}},\ \underset{[2]_t}{\begin{array}{|c|c|}\hline 1&1\\\hline 1&2\\\hline\end{array}} \qquad m_{22} : \underset{q^2}{\begin{array}{|c|c|}\hline 2&2\\\hline 1&1\\\hline\end{array}},\ \underset{q[2]_t}{\begin{array}{|c|c|}\hline 1&2\\\hline 1&2\\\hline\end{array}},\ \underset{[2]_t}{\begin{array}{|c|c|}\hline 2&1\\\hline 1&2\\\hline\end{array}},\ \underset{q^2}{\begin{array}{|c|c|}\hline 1&1\\\hline 2&2\\\hline\end{array}}$$

$$m_{211} : \underset{q^2[2]_t}{\begin{array}{|c|c|}\hline 2&3\\\hline 1&1\\\hline\end{array}},\ \underset{tq[2]_t}{\begin{array}{|c|c|}\hline 1&3\\\hline 1&2\\\hline\end{array}},\ \underset{q[2]_t}{\begin{array}{|c|c|}\hline 3&1\\\hline 1&2\\\hline\end{array}},\ \underset{t[2]_t}{\begin{array}{|c|c|}\hline 1&2\\\hline 1&3\\\hline\end{array}},\ \underset{q[2]_t}{\begin{array}{|c|c|}\hline 2&1\\\hline 1&3\\\hline\end{array}},\ \underset{[2]_t}{\begin{array}{|c|c|}\hline 1&1\\\hline 2&3\\\hline\end{array}}$$

$$m_{1111} : \underset{q^2[2]_t}{\begin{array}{|c|c|}\hline 3&4\\\hline 1&2\\\hline\end{array}},\ \underset{tq^2[2]_t}{\begin{array}{|c|c|}\hline 4&3\\\hline 1&2\\\hline\end{array}},\ \underset{q^2[2]_t}{\begin{array}{|c|c|}\hline 2&4\\\hline 1&3\\\hline\end{array}},\ \underset{tq[2]_t}{\begin{array}{|c|c|}\hline 4&2\\\hline 1&3\\\hline\end{array}},\ \underset{q[2]_t}{\begin{array}{|c|c|}\hline 2&3\\\hline 1&4\\\hline\end{array}},\ \underset{tq[2]_t}{\begin{array}{|c|c|}\hline 3&2\\\hline 1&4\\\hline\end{array}}$$

$$\underset{tq[2]_t}{\begin{array}{|c|c|}\hline 1&4\\\hline 2&3\\\hline\end{array}},\ \underset{q[2]_t}{\begin{array}{|c|c|}\hline 4&1\\\hline 2&3\\\hline\end{array}},\ \underset{t[2]_t}{\begin{array}{|c|c|}\hline 1&3\\\hline 2&4\\\hline\end{array}},\ \underset{q[2]_t}{\begin{array}{|c|c|}\hline 3&1\\\hline 2&4\\\hline\end{array}},\ \underset{[2]_t}{\begin{array}{|c|c|}\hline 1&2\\\hline 3&4\\\hline\end{array}},\ \underset{t[2]_t}{\begin{array}{|c|c|}\hline 2&1\\\hline 3&4\\\hline\end{array}}$$

Quasisymmetric Macdonald polynomials

Definition (Corteel-Haglund-M-Mandelshtam-Williams)

$$G_\alpha(X;q,t) = \sum_{\gamma^+=\alpha} F_\gamma(X;q,t) = \sum_{\gamma^+=\alpha} E^{\beta(\gamma)}_{\mathrm{inc}(\gamma)}(X;q,t)$$

Example

$$G_{12}(X;q,t) = F_{120}(X;q,t) + F_{102}(X;q,t) + F_{012}(X;q,t)$$

$$= E^{312}_{012}(X;q,t) + E^{213}_{012}(X;q,t) + E^{123}_{012}(X;q,t)$$

Proposition

$$P_\lambda(X;q,t) = \sum_{\mathrm{dec}(\alpha)=\lambda} G_\alpha(X;q,t),$$

since $\displaystyle\sum_{\mathrm{dec}(\alpha)=\lambda} G_\alpha(X;q,t) = \sum_{\mathrm{dec}(\alpha)=\lambda}\sum_{\gamma^+=\alpha} F_\gamma(X;q,t).$

Properties of Quasisymmetric Macdonald Polynomials

- $G_\alpha(X;q,t)$ is quasisymmetric for all α.
- $G_\alpha(X;0,0) = QS_\alpha(X)$
- Triangularity but not (naive) orthogonality

Theorem (Corteel, Mandelshtam, Roberts)

$$G_\alpha(X;q,t) = \sum_{\substack{T \in \mathrm{NAT}(\alpha) \\ T \text{ packed}}} q^{\mathrm{maj}(T)} t^{\mathrm{coinv}(T)} a(q,t) M_{\mathrm{content}(T)},$$

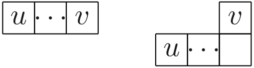
$$G_\alpha(X;q,t) = \sum_{\tau \in ST(\alpha)} b(q,t) \sum_{U \subseteq W(\tau)} (-t)^{|U|} c(q,t) F_{V(\tau) \cup U},$$

where $a(q,t), b(q,t)$, and $c(q,t)$ are nonzero rational functions in q and t.

A few more definitions...

$T \in \mathrm{NAT}(\alpha)$

- Non-attacking filling of $\mathrm{inc}(\alpha)$



- Let σ be the longest permutation s.t. $\sigma \circ \alpha = \mathrm{inc}(\alpha)$
- First row has pattern σ

Packed

- Uses every integer from $[m]$ (and no others) for some m

Example

			3
		7	2
	6	1	7
4	2	1	6
5	7	2	4

$\in \mathrm{NAT}((4, 5, 2, 3))$

Triangularity

Corollary (to CMR expansion)

$$G_\alpha(X;q,t) = M_\alpha + \sum_{\beta<\alpha} c_{\alpha,\beta}(q,t) M_\beta,$$

where $<$ is any extension of the dominance order on the underlying partition.

Sketch of Proof

- $G_\alpha(X;q,t) = \sum_{\substack{T\in \mathrm{NAT}(\alpha) \\ T \text{ packed}}} q^{\mathrm{maj}(T)} t^{\mathrm{coinv}(T)} a(q,t) M_{\mathrm{content}(T)}$
- Let $\gamma = \lambda(\mathrm{content}(T))$. Then $\gamma_1 \leq$ (largest part of α).
- Similarly, $\gamma_1 + \gamma_2 \leq$ (sum of largest two parts of α), etc...
- The non-attacking condition implies that $\mathrm{content}(T) = \alpha$ if $\mathrm{content}(T)$ is a rearrangement of α

References

- Corteel, Haglund, Mandelshtam, Mason, and Williams. *Compact formulas for Macdonald polynomials and quasisymmetric Macdonald polynomials*, Sem. Lothar. Combin., 2020.
- Corteel, Mandelshtam, and Roberts. *Expanding the quasisymmetric Macdonald polynomials in the fundamental basis*, pre-print, 2020.
- Corteel, Mandelshtam, and Williams. *From multiline queues to Macdonald polynomials via the exclusion process*, Sem. Lothar. Combin., 2020.
- Haglund, Haiman, and Loehr. *Combinatorial theory of Macdonald polynomials I: Proof of Haglund's formula*, Proc. Nat. Acad. Sci., 2005.
- Mandelshtam, Ayyer, and Martin. *Stationary probabilities of the multispecies TAZRP and modified Macdonald polynomials: I*, pre-print, 2020.