Counting neighborhood-restricted graphs

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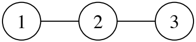

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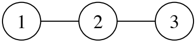
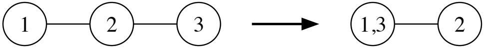


Here vertices 1 and 3 have the same neighborhood, $\{2\}$, so this graph is not point-determining. Note that if $N(u) = N(v)$ then u and v are not adjacent.

If G is an arbitrary graph then we can construct a point-determining graph from G by identifying vertices with the same neighborhood.



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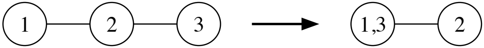

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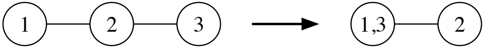

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This decomposition yields an identity of generating functions.

Exponential generating functions

Let $P(x) = \sum_{n=0}^{\infty} p_n \frac{x^n}{n!}$ be the exponential generating function for point-determining graphs.

Let $G(x) = \sum_{n=0}^{\infty} 2^{\binom{n}{2}} \frac{x^n}{n!}$ be the exponential generating function for all graphs.

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By the theory of exponential generating functions, the decomposition just described implies that $G(x) = P(e^x - 1)$ so

$$\begin{aligned} P(x) &= G\big(\log(1+x)\big) \\ &= 1 + x + \frac{x^2}{2!} + 4\frac{x^3}{3!} + 32\frac{x^4}{4!} + 588\frac{x^5}{5!} + 21476\frac{x^6}{6!} + \cdots \end{aligned}$$

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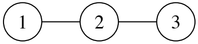

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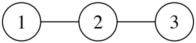

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The decomposition approach does not work for counting noncomplementary neighborhood graphs. Instead we will use inclusion-exclusion.

Inclusion-exclusion

We have a set S of objects and a set $\mathscr{P}$ of conditions. For each condition p there is a subset $S_p \subseteq S$ of objects that satisfy condition p. We would like to find the number of objects satisfying none of the conditions in $\mathscr{P}$, i.e., the size of $\bigcap_{p \in \mathscr{P}} \overline{S_p} = S - \bigcup_{p \in \mathscr{P}} S_p$.

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For inclusion-exclusion to be useful, we must be able to compute $|S_A|$ for each $A \subseteq \mathscr{P}$.

Partitions

Let's apply inclusion-exclusion to count certain partitions. (This is closely related to counting point-determining graphs.)

Let S be the set of partitions Π of $[n]$, so $|S|$ is the nth Bell number B_n. We take conditions of the form

$$c_{i,j} : i \text{ and } j \text{ are in the same block of } \Pi.$$

The number of partitions of $[n]$ in which 1 and 2 are in the same block is B_{n-1} since to construct such a partition, we can join 1 and 2 together, and then take a partition of the $(n-1)$-element set $\{\boxed{12}, 3, 4, \ldots, n\}$.

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The set of conditions $\{c_{1,2}, c_{2,3}, c_{1,3}\}$ is equivalent to the set $\{c_{1,2}, c_{1,3}\}$, so the number of partitions satisfying these three conditions is also B_{n-2}.

Let's use inclusion-exclusion to count partitions of $[n]$ in which 1, 2, and 3 are all in separate blocks. We sum over subsets of the conditions $\{c_{1,2}, c_{1,3}, c_{2,3}\}$. We obtain

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However, for our purposes, the full inclusion-exclusion formula is easier to work with, even though there will be cancellation.

Now let's count partitions of $[n]$ that satisfy none of the conditions $c_{i,j}, 1 \leq i < j \leq n$, using exponential generating functions. Of course it's easy to count them directly: there is only one, the partition $\{\{1\}, \{2\}, \ldots, \{n\}\}$. But we want to count them by inclusion-exclusion.

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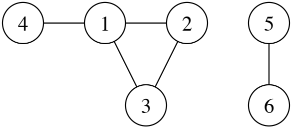
To every set A of conditions, we associate a condition graph G_A. For example, if $A = \{c_{1,2}, c_{1,3}, c_{1,4}, c_{2,3}, c_{5,6}\}$ then G_A is



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If G_A has k components, then the number of partitions satisfying all the conditions in A is the Bell number B_k.

So the exponential generating function for partitions in which no two elements are in the same block is

$$\sum_{n=0}^{\infty} \frac{x^n}{n!} \sum_G (-1)^{\text{\#edges of } G} B_{\text{\#components of } G}$$

where the sum is over all graphs G on $[n]$, and this may be written

$$\sum_{n=0}^{\infty} \frac{x^n}{n!} \sum_{k,i} (-1)^i h_{n,k,i} B_k,$$

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If we start with $g = e^f$ then we can recover f as $\log g$ and then $f^k/k! = (\log g)^k/k!$.

Now let's take the case in which f counts connected graphs by edges and g counts all graphs by edges, where edges are weighted y. (Soon we will set $y = -1$.)

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Setting $y = -1$ gives $g = 1 + x$ and $f = \log(1 + x)$.

So $\log(1 + x)$ counts connected graphs where edges are weighted -1.

Recall that we showed earlier that the exponential generating function for partitions in which no two elements are in the same block is

$$\sum_{k=0}^{\infty} B_k \sum_{n,i} (-1)^i h_{n,k,i} \frac{x^n}{n!}$$

where B_k is the Bell number and $h_{n,k,i}$ is the number of graphs on $[n]$ with k components and i edges.

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The inner sum is just $f^k/k! = \log(1+x)^k/k!$. So the sum is just

$$R(x) := \sum_{k=0}^{\infty} B_k \frac{\log(1+x)^k}{k!} = B\big(\log(1+x)\big),$$

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So $R(x) = e^x = \sum_{n=0}^{\infty} x^n/n!$.

Point-determining graphs

Now let's return to point-determining graphs. We want to count them by inclusion-exclusion. (We call the graphs to be counted object graphs to distinguish them from condition graphs.) We start with the set of graphs with vertex set $[n]$. There are $2^{\binom{n}{2}}$ of them. We want to count the graphs satisfying none of the conditions

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The number of object graphs satisfying any single condition $p_{i,j}$ is $2^{\binom{n-1}{2}}$ because a graph satisfying it can be constructed by contracting i and j to a single vertex, picking a graph on these $n-1$ vertices, and then replacing the contracted vertex with vertices i and j, with the same neighborhood as the contracted vertex.

An arbitrary set of conditions works exactly like our earlier example of partitions. To every set A of conditions, we associate a condition graph G_A whose edges correspond to the conditions in A. The object graphs on $[n]$ satisfying all the conditions in A are those in which the vertices of each connected component of the condition graph G_A all have the same neighborhood.

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So by the same reasoning as before, the generating function for graphs with all neighborhoods distinct is $G\big(\log(1+x)\big)$ where $G(x) = \sum_{n=0}^{\infty} 2^{\binom{n}{2}} \frac{x^n}{n!}$ is the exponential generating function for all graphs.

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(Recall that $\log(1+x)$ is the exponential generating function for connected graphs with edges weighted -1.)

Noncomplementary neighborhood graphs

Now we count noncomplementary neighborhood graphs. We consider conditions

$$q_{i,j} : i \text{ and } j \text{ have complementary neighborhoods.}$$

The number of graphs on $[n]$ satisfying one of these conditions is $2^{\binom{n-1}{2}}$ by the same kind of contraction argument as before. (Note that if i and j have complementary neighborhoods then i and j must be adjacent.) If there is more than one condition the situation is similar but some sets of conditions are inconsistent.

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For example, suppose conditions $q_{1,2}$ and $q_{2,3}$ are satisfied. Then 1 and 3 must have the same neighborhood, so they cannot have complementary neighborhoods. So the set $\{q_{1,2}, q_{2,3}, q_{1,3}\}$ is inconsistent.

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So by the same argument as before, the exponential generating function for noncomplementary neighborhood graphs is $G(b(x))$ where $G(x) = \sum_n 2^{\binom{n}{2}} x^n/n!$ and $b(x)$ is the exponential generating function for connected bipartite graphs in which edges are weighted -1.

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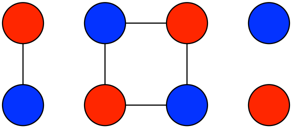
How do we count connected bipartite graphs?

Counting bipartite graphs

To count bipartite graphs, we start by counting bicolored graphs. These are graphs in which the vertices are colored red or blue, and every edge joins a red vertex and a blue vertex:



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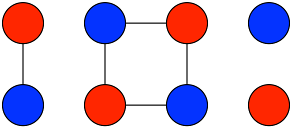

We can count bicolored graphs on $[n]$ directly. If we weight edges by y, the contribution from graphs with i red and $n-i$ blue vertices is $\binom{n}{i}(1+y)^{i(n-i)}$ so for all bicolored graphs on $[n]$ we have

$$b_n(y) := \sum_{i=0}^{n} \binom{n}{i} (1+y)^{i(n-i)}$$

So the exponential generating function for all bicolored graphs is

$$B(x) := \sum_{n=0}^{\infty} b_n(y) \frac{x^n}{n!}$$

and the exponential generating function for connected bicolored graphs is $\log B(x)$.

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Now we set $y = -1$. We have $b_0(-1) = 1$ and $b_n(-1) = 2$ for $n > 0$. So $B(x)$ reduces to $2e^x - 1$ and the exponential generating function for connected bipartite graphs reduces to $b(x) = \frac{1}{2} \log(2e^x - 1)$.

Therefore, the exponential generating function for noncomplementary neighborhood graphs is

$$G\left(\tfrac{1}{2}\log(2e^x - 1)\right)$$
$$= 1 + x + \frac{x^2}{2!} + 5\frac{x^3}{3!} + 33\frac{x^4}{4!} + 629\frac{x^5}{5!} + 21937\frac{x^6}{6!} + 1570213\frac{x^7}{7!} + \cdots$$

Can we say anything about the coefficients of

$$J(x) := \tfrac{1}{2}\log(2e^x - 1) = x - \frac{x^2}{2!} + 3\frac{x^3}{3!} - 13\frac{x^4}{4!} + 75\frac{x^5}{5!} - \cdots?$$

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These are (up to sign) the Fubini or ordered Bell numbers that count ordered partitions of a set:

$$J'(x) = \frac{1}{2 - e^{-x}}.$$

Could we count noncomplementary neighborhood graphs using a decomposition?

Could we count noncomplementary neighborhood graphs using a decomposition? Probably not.

Could we count noncomplementary neighborhood graphs using a decomposition?

Let $K(x) = J(x)^{\langle -1 \rangle}$ be the compositional inverse of $J(x)$, so if $N(x)$ is the exponential generating function for noncomplementary neighborhood graphs then

$$G(x) = N\big(K(x)\big).$$

The coefficients of $K(x)$ aren't positive, though they do have some nice properties.

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We have

$$\begin{aligned} K(x) &= \log\big(\tfrac{1}{2}(e^{2x}+1)\big) \\ &= x + \frac{x^2}{2!} - 2\frac{x^4}{4!} + 16\frac{x^6}{6!} - 272\frac{x^8}{8!} + \cdots \end{aligned}$$

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These are (signed) tangent numbers: $K'(x) = 1 + \tanh x$.

Point-determining and noncomplementary neighborhood graphs

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We can count them by using inclusion-exclusion, but there is a shortcut. Once we know the exponential generating function $N(x)$ for noncomplementary neighborhood graphs, we can get the exponential generating function $M(x)$ for graphs that are both point-determining and noncomplementary neighborhood by the decomposition method:

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$$\begin{aligned} M(x) &= N\bigl(\log(1+x)\bigr) = G\bigl(\tfrac{1}{2}\log(1+2x)\bigr) \\ &= 1 + x + 4\frac{x^3}{3!} + 8\frac{x^4}{4!} + 448\frac{x^5}{5!} + 14336\frac{x^6}{6!} + 1202432\frac{x^7}{7!} + \cdots \end{aligned}$$

We can invert this to get

$$G(x) = M\Big(\tfrac{1}{2}(e^{2x} - 1)\Big)$$

but I don't know of a combinatorial interpretation to this formula. If we count these graphs directly, the condition graphs are balanced signed graphs.

Möbius inversion

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We can eliminate some of the cancellation that occurs in inclusion-exclusion by using Möbius inversion. This doesn't really help with the derivation, but it shows an interesting example of Möbius inversion. First we look at the point-determining case.

Let us say that two condition graphs (which are arbitrary graphs in this case) are equivalent if they yield equivalent sets of conditions. Then two condition graphs are equivalent if the have the same connected components (as sets of vertices). So the equivalence classes are partitions of $[n]$, and we have Möbius inversion in the lattice of partitions of a set.

The exponential generating function $\log(1+x)^k/k!$ that we computed earlier, that counts graphs with k components, weighted by $(-1)^{\text{\# edges}}$, is the sum of the Möbius functions of all partitions with k blocks (Whitney number of the second kind).

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So the exponential generating function for the characteristic polynomials of the lattice of partitions is

$$e^{t\log(1+x)} = (1+x)^t = \sum_{n=0}^{\infty} t(t-1)\cdots(t-n+1)\frac{x^n}{n!}$$

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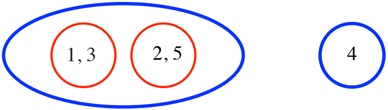

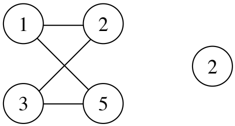

Here the generating function for the characteristic polynomials is

$$
\begin{aligned}
\exp\left(\tfrac{t}{2}\log(2e^x - 1)\right) &= (2e^x - 1)^{t/2} \\
&= 1 + tx + (t^2 - t)\frac{x^2}{2!} + (t^3 - 3t^2 + 3t)\frac{x^3}{3!} \\
&\qquad + (t^4 - 6t^3 + 15t^2 - 13t)\frac{x^4}{4!} + \cdots
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$$\begin{aligned}\exp\left(\tfrac{t}{2}\log(2e^x-1)\right) &= (2e^x-1)^{t/2}\\ &= 1+tx+(t^2-t)\frac{x^2}{2!}+(t^3-3t^2+3t)\frac{x^3}{3!}\\ &\qquad +(t^4-6t^3+15t^2-13t)\frac{x^4}{4!}+\cdots\end{aligned}$$

For graphs that are both point-determining and noncomplementary neighborhood, the exponential generating function for the characteristic polynomials is

$$\begin{aligned}e^{\frac{t}{2}\log(1+2x)} &= (1+2x)^{t/2}\\ &= 1+tx+t(t-2)\frac{x^2}{2!}+t(t-2)(t-4)\frac{x^3}{3!}\\ &\qquad +t(t-2)(t-4)(t-6)\frac{x^6}{6!}+\cdots\end{aligned}$$

Closed neighborhoods

Instead of neighborhoods we might consider closed neighborhoods which are defined by

$$\overline{N}(v) = N(v) \cup \{v\}.$$

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Thus a graph is co-point-determining if and only if its complement is point-determining. So co-point-determining graphs are easy to count.

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What about graphs that are both point-determining and co-point-determining? (Bi-point-determining)

We can count them using the decomposition approach but it is fairly complicated. They can be counted more easily by inclusion-exclusion. The generating function is $G\big(2\log(1+x)-x\big)$.

More combinations

There are four restrictions on neighborhoods of graphs that we can work with using inclusion-exclusion:

- distinct neighborhoods
- noncomplementary neighboroods
- distinct closed neighborhoods
- noncomplementary closed neighborhoods

There are 2^4 subsets of these conditions, but by considering graph complements, they reduce (e.g., by Burnside's Lemma) to only 10 inequivalent subsets, of which one is the empty set. So there are 9 nontrival problems that can be solved by the inclusion-exclusion method.

Unlabeled graphs

The decomposition method also enables us to count unlabeled point-determining and bi-point-determining graphs. (This was done by Ronald Read for point-determining graphs by Ji Li and me for bi-point-determining graphs.)

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Open question:

Is there any way to count unlabeled noncomplementary neighborhood graphs?