A poset of lattice path matroids

Carolina Benedetti Velásquez

Universidad de los Andes

with K. Knauer (≥ 20)

Combinatorics Seminar
MSU

Outline

Matroids and Grassmannian

The (real) **grassmannian** $Gr_{k,n}$ consists of all the k-dimensional vector spaces V in $\mathbb{R}^n$. Every $V \in Gr_{k,n}$ can be represented as a full rank matrix $A_{k \times n}$.

Matroids and Grassmannian

The (real) **grassmannian** $Gr_{k,n}$ consists of all the k-dimensional vector spaces V in $\mathbb{R}^n$. Every $V \in Gr_{k,n}$ can be represented as a full rank matrix $A_{k \times n}$.

For instance,

$$V = \langle (2,0,0,1),(1,1,0,2) \rangle \in Gr_{2,4}$$

Matroids and Grassmannian

The (real) **grassmannian** $Gr_{k,n}$ consists of all the k-dimensional vector spaces V in $\mathbb{R}^n$. Every $V \in Gr_{k,n}$ can be represented as a full rank matrix $A_{k\times n}$.

For instance,

$$V = \langle (2,0,0,1),(1,1,0,2)\rangle \in Gr_{2,4} \rightsquigarrow A = \begin{pmatrix} 2 & 0 & 0 & 1 \\ 1 & 1 & 0 & 2 \end{pmatrix}.$$

Matroids and Grassmannian

The (real) **grassmannian** $Gr_{k,n}$ consists of all the k-dimensional vector spaces V in $\mathbb{R}^n$. Every $V \in Gr_{k,n}$ can be represented as a full rank matrix $A_{k\times n}$.

For instance,

$$V = \langle (2,0,0,1),(1,1,0,2)\rangle \in Gr_{2,4} \rightsquigarrow A = \begin{pmatrix} 2 & 0 & 0 & 1 \\ 1 & 1 & 0 & 2 \end{pmatrix}.$$

- $Gr_{k,n}$ can be thought of as $M_{k,n}/\sim$

Matroids and Grassmannian

The (real) **grassmannian** $Gr_{k,n}$ consists of all the k-dimensional vector spaces V in $\mathbb{R}^n$. Every $V \in Gr_{k,n}$ can be represented as a full rank matrix $A_{k\times n}$.

For instance,

$$V = \langle (2,0,0,1),(1,1,0,2)\rangle \in Gr_{2,4} \rightsquigarrow A = \begin{pmatrix} 2 & 0 & 0 & 1 \\ 1 & 1 & 0 & 2 \end{pmatrix}.$$

- $Gr_{k,n}$ can be thought of as $M_{k,n}/\sim$

Every $V \in Gr_{k,n}$ gives rise to a **linear matroid** $M = ([n],\mathcal{B})$ of rank k where $B \in \mathcal{B}$ if and only if $p_B \neq 0$. Here p_B is the $k \times k$ determinant of the matrix whose columns are those indexed by B.

Matroids and Grassmannian

The (real) **grassmannian** $Gr_{k,n}$ consists of all the k-dimensional vector spaces V in $\mathbb{R}^n$. Every $V \in Gr_{k,n}$ can be represented as a full rank matrix $A_{k\times n}$.

For instance,

$$V = \langle (2,0,0,1),(1,1,0,2)\rangle \in Gr_{2,4} \rightsquigarrow A = \begin{pmatrix} 2 & 0 & 0 & 1 \\ 1 & 1 & 0 & 2 \end{pmatrix}.$$

- $Gr_{k,n}$ can be thought of as $M_{k,n}/\sim$

Every $V \in Gr_{k,n}$ gives rise to a **linear matroid** $M = ([n], \mathcal{B})$ of rank k where $B \in \mathcal{B}$ if and only if $p_B \neq 0$. Here p_B is the $k \times k$ determinant of the matrix whose columns are those indexed by B.

$$V : \begin{pmatrix} 2 & 0 & 0 & 1 \\ 1 & 1 & 0 & 2 \end{pmatrix} \rightsquigarrow$$

Matroids and Grassmannian

The (real) **grassmannian** $Gr_{k,n}$ consists of all the k-dimensional vector spaces V in $\mathbb{R}^n$. Every $V \in Gr_{k,n}$ can be represented as a full rank matrix $A_{k \times n}$.

For instance,

$$V = \langle (2,0,0,1),(1,1,0,2) \rangle \in Gr_{2,4} \rightsquigarrow A = \begin{pmatrix} 2 & 0 & 0 & 1 \\ 1 & 1 & 0 & 2 \end{pmatrix}.$$

- $Gr_{k,n}$ can be thought of as $M_{k,n}/\sim$

Every $V \in Gr_{k,n}$ gives rise to a **linear matroid** $M = ([n], \mathcal{B})$ of rank k where $B \in \mathcal{B}$ if and only if $p_B \neq 0$. Here p_B is the $k \times k$ determinant of the matrix whose columns are those indexed by B.

$$V : \begin{pmatrix} 2 & 0 & 0 & 1 \\ 1 & 1 & 0 & 2 \end{pmatrix} \rightsquigarrow M_V = ([4], \{12, 14, 24\}).$$

Matroids and Grassmannian

The (real) **grassmannian** $Gr_{k,n}$ consists of all the k-dimensional vector spaces V in $\mathbb{R}^n$. Every $V \in Gr_{k,n}$ can be represented as a full rank matrix $A_{k \times n}$.

For instance,

$$V = \langle (2,0,0,1),(1,1,0,2) \rangle \in Gr_{2,4} \rightsquigarrow A = \begin{pmatrix} 2 & 0 & 0 & 1 \\ 1 & 1 & 0 & 2 \end{pmatrix}.$$

- $Gr_{k,n}$ can be thought of as $M_{k,n}/\sim$

Every $V \in Gr_{k,n}$ gives rise to a **linear matroid** $M = ([n], \mathcal{B})$ of rank k where $B \in \mathcal{B}$ if and only if $p_B \neq 0$. Here p_B is the $k \times k$ determinant of the matrix whose columns are those indexed by B.

$$V : \begin{pmatrix} 2 & 0 & 0 & 1 \\ 1 & 1 & 0 & 2 \end{pmatrix} \rightsquigarrow M_V = ([4], \{12, 14, 24\}).$$

- Every linear matroid arises this way.

Positroids - Postnikov '05-

The **totally nonnegative Grassmannian** $Gr_{k,n}^{\geq 0}$ is the subset of $Gr_{k,n}$ consisting of those $A_{k \times n}$ s.t. all its maximal minors are ≥ 0.

Positroids - Postnikov '05-

The **totally nonnegative Grassmannian** $Gr_{k,n}^{\geq 0}$ is the subset of $Gr_{k,n}$ consisting of those $A_{k \times n}$ s.t. all its maximal minors are ≥ 0.

A **positroid** of rank k is a matroid $P = ([n], \mathcal{B})$ such that P can be represented by some $A_{k \times n} \in Gr_{k,n}^{\geq 0}$.

Positroids - Postnikov '05-

The **totally nonnegative Grassmannian** $Gr_{k,n}^{\geq 0}$ is the subset of $Gr_{k,n}$ consisting of those $A_{k\times n}$ s.t. all its maximal minors are ≥ 0.

A **positroid** of rank k is a matroid $P = ([n], \mathcal{B})$ such that P can be represented by some $A_{k\times n} \in Gr_{k,n}^{\geq 0}$.

- $P = ([5], \mathcal{B})$ where $\mathcal{B} = \{13, 14, 15, 34, 35, 45\}$:

Positroids - Postnikov '05-

The **totally nonnegative Grassmannian** $Gr_{k,n}^{\geq 0}$ is the subset of $Gr_{k,n}$ consisting of those $A_{k\times n}$ s.t. all its maximal minors are ≥ 0.

A **positroid** of rank k is a matroid $P = ([n], \mathcal{B})$ such that P can be represented by some $A_{k\times n} \in Gr_{k,n}^{\geq 0}$.

- $P = ([5], \mathcal{B})$ where $\mathcal{B} = \{13, 14, 15, 34, 35, 45\}$: $\begin{pmatrix} 1 & 0 & 1 & 0 & -2 \\ 0 & 0 & 1 & 1 & 2 \end{pmatrix}$ ✓

Positroids - Postnikov '05-

The **totally nonnegative Grassmannian** $Gr_{k,n}^{\geq 0}$ is the subset of $Gr_{k,n}$ consisting of those $A_{k\times n}$ s.t. all its maximal minors are ≥ 0.

A **positroid** of rank k is a matroid $P = ([n], \mathcal{B})$ such that P can be represented by some $A_{k\times n} \in Gr_{k,n}^{\geq 0}$.

- $P = ([5], \mathcal{B})$ where $\mathcal{B} = \{13, 14, 15, 34, 35, 45\}$: $\begin{pmatrix} 1 & 0 & 1 & 0 & -2 \\ 0 & 0 & 1 & 1 & 2 \end{pmatrix}$ ✓
- $M = ([4], \mathcal{B})$ where $\mathcal{B} = \{12, 14, 23, 34\}$ is linear but *is not* a positroid.

Positroids - Postnikov '05-

The **totally nonnegative Grassmannian** $Gr_{k,n}^{\geq 0}$ is the subset of $Gr_{k,n}$ consisting of those $A_{k\times n}$ s.t. all its maximal minors are ≥ 0.

A **positroid** of rank k is a matroid $P = ([n], \mathcal{B})$ such that P can be represented by some $A_{k\times n} \in Gr_{k,n}^{\geq 0}$.

- $P = ([5], \mathcal{B})$ where $\mathcal{B} = \{13, 14, 15, 34, 35, 45\}$: $\begin{pmatrix} 1 & 0 & 1 & 0 & -2 \\ 0 & 0 & 1 & 1 & 2 \end{pmatrix}$ ✓
- $M = ([4], \mathcal{B})$ where $\mathcal{B} = \{12, 14, 23, 34\}$ is linear but *is not* a positroid.
- Positroids care about the labelling of the ground set.

Positroids - Postnikov '05-

The **totally nonnegative Grassmannian** $Gr_{k,n}^{\geq 0}$ is the subset of $Gr_{k,n}$ consisting of those $A_{k\times n}$ s.t. all its maximal minors are ≥ 0.

A **positroid** of rank k is a matroid $P = ([n], \mathcal{B})$ such that P can be represented by some $A_{k\times n} \in Gr_{k,n}^{\geq 0}$.

- $P = ([5], \mathcal{B})$ where $\mathcal{B} = \{13, 14, 15, 34, 35, 45\}$: $\begin{pmatrix} 1 & 0 & 1 & 0 & -2 \\ 0 & 0 & 1 & 1 & 2 \end{pmatrix}$ ✓
- $M = ([4], \mathcal{B})$ where $\mathcal{B} = \{12, 14, 23, 34\}$ is linear but *is not* a positroid.
- Positroids care about the labelling of the ground set. $M = ([4], \mathcal{B})$ where

$\mathcal{B} = \{13, 14, 23, 24\}$ *is* a positroid:

Positroids - Postnikov '05-

The **totally nonnegative Grassmannian** $Gr_{k,n}^{\geq 0}$ is the subset of $Gr_{k,n}$ consisting of those $A_{k \times n}$ s.t. all its maximal minors are ≥ 0.

A **positroid** of rank k is a matroid $P = ([n], \mathcal{B})$ such that P can be represented by some $A_{k \times n} \in Gr_{k,n}^{\geq 0}$.

- $P = ([5], \mathcal{B})$ where $\mathcal{B} = \{13, 14, 15, 34, 35, 45\}$: $\begin{pmatrix} 1 & 0 & 1 & 0 & -2 \\ 0 & 0 & 1 & 1 & 2 \end{pmatrix}$ ✓
- $M = ([4], \mathcal{B})$ where $\mathcal{B} = \{12, 14, 23, 34\}$ is linear but *is not* a positroid.
- Positroids care about the labelling of the ground set. $M = ([4], \mathcal{B})$ where $\mathcal{B} = \{13, 14, 23, 24\}$ *is* a positroid: $\begin{pmatrix} 1 & 1 & 0 & 0 \\ 0 & 0 & 1 & 1 \end{pmatrix}$

Positroids - Postnikov '05-

The **totally nonnegative Grassmannian** $Gr_{k,n}^{\geq 0}$ is the subset of $Gr_{k,n}$ consisting of those $A_{k\times n}$ s.t. all its maximal minors are ≥ 0.

A **positroid** of rank k is a matroid $P = ([n], \mathcal{B})$ such that P can be represented by some $A_{k\times n} \in Gr_{k,n}^{\geq 0}$.

The positroid $P = ([5], \mathcal{B})$ where $\mathcal{B} = \{13, 14, 15, 34, 35, 45\}$ can be encoded by its

Positroids - Postnikov '05-

The **totally nonnegative Grassmannian** $Gr_{k,n}^{\geq 0}$ is the subset of $Gr_{k,n}$ consisting of those $A_{k\times n}$ s.t. all its maximal minors are ≥ 0.

A **positroid** of rank k is a matroid $P = ([n], \mathcal{B})$ such that P can be represented by some $A_{k\times n} \in Gr_{k,n}^{\geq 0}$.

The positroid $P = ([5], \mathcal{B})$ where $\mathcal{B} = \{13, 14, 15, 34, 35, 45\}$ can be encoded by its

- Grassmann necklace $I_P = (13, 34, 34, 45, 51)$

Positroids - Postnikov '05-

The **totally nonnegative Grassmannian** $Gr_{k,n}^{\geq 0}$ is the subset of $Gr_{k,n}$ consisting of those $A_{k\times n}$ s.t. all its maximal minors are ≥ 0.

A **positroid** of rank k is a matroid $P = ([n], \mathcal{B})$ such that P can be represented by some $A_{k\times n} \in Gr_{k,n}^{\geq 0}$.

The positroid $P = ([5], \mathcal{B})$ where $\mathcal{B} = \{13, 14, 15, 34, 35, 45\}$ can be encoded by its

- Grassmann necklace $I_P = (13, 34, 34, 45, 51)$
- Decorated permutation $\pi = 4\underline{2}513$

Positroids - Postnikov '05-

The **totally nonnegative Grassmannian** $Gr_{k,n}^{\geq 0}$ is the subset of $Gr_{k,n}$ consisting of those $A_{k\times n}$ s.t. all its maximal minors are ≥ 0.

A **positroid** of rank k is a matroid $P = ([n], \mathcal{B})$ such that P can be represented by some $A_{k\times n} \in Gr_{k,n}^{\geq 0}$.

The positroid $P = ([5], \mathcal{B})$ where $\mathcal{B} = \{13, 14, 15, 34, 35, 45\}$ can be encoded by its

- Grassmann necklace $I_P = (13, 34, 34, 45, 51)$
- Decorated permutation $\pi = 4\underline{2}513$
- and many more combinatorial objects...

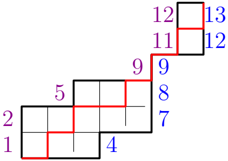
Lattice path matroids LPMs

Fix $0 \leq k \leq n$ and let $U, L \in \binom{[n]}{k}$.
The **lattice path matroid** $M[U, L]$ is the matroid on $[n]$ whose bases are those $B \in \binom{[n]}{k}$ such that $U \leq B \leq L$.

Lattice path matroids LPMs

Fix $0 \leq k \leq n$ and let $U, L \in \binom{[n]}{k}$.
The **lattice path matroid** $M[U, L]$ is the matroid on $[n]$ whose bases are those $B \in \binom{[n]}{k}$ such that $U \leq B \leq L$.

For instance, let $k = 6, n = 13$, $U = \{1, 2, 5, 9, 11, 12\}$, $L = \{4, 7, 8, 9, 12, 13\}$.

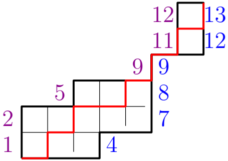


then $B = \{2, 4, 7, 9, 11, 13\}$ is a basis of $M[U, L]$.

Lattice path matroids LPMs

Fix $0 \leq k \leq n$ and let $U, L \in \binom{[n]}{k}$.
The **lattice path matroid** $M[U, L]$ is the matroid on $[n]$ whose bases are those $B \in \binom{[n]}{k}$ such that $U \leq B \leq L$.

For instance, let $k = 6, n = 13$, $U = \{1, 2, 5, 9, 11, 12\}$, $L = \{4, 7, 8, 9, 12, 13\}$.



then $B = \{2, 4, 7, 9, 11, 13\}$ is a basis of $M[U, L]$.

◦ Every LPM is a positroid.

Flags of matroids a.k.a. quotients of matroids

A point in the **(full) flag variety** $\mathcal{F}\ell_n$ is a flag $F: V_1 \subset V_2 \subset \cdots \subset V_n = \mathbb{R}^n$ of subspaces with $\dim V_i = i$. Every $F \in \mathcal{F}\ell_n$ can be thought of as a full rank $n \times n$ matrix A.

Flags of matroids a.k.a. quotients of matroids

A point in the **(full) flag variety** $\mathcal{F}\ell_n$ is a flag $F: V_1 \subset V_2 \subset \cdots \subset V_n = \mathbb{R}^n$ of subspaces with $\dim V_i = i$. Every $F \in \mathcal{F}\ell_n$ can be thought of as a full rank $n \times n$ matrix A.

$$\underbrace{\begin{pmatrix} 0 & 1 & 0 \end{pmatrix}}_{M_1} \subset \underbrace{\begin{pmatrix} 0 & 1 & 0 \\ 1 & 2 & -1 \end{pmatrix}}_{M_2} \subset \underbrace{\begin{pmatrix} 0 & 1 & 0 \\ 1 & 2 & -1 \\ 2 & 1 & 1 \end{pmatrix}}_{M_3}$$

Flags of matroids a.k.a. quotients of matroids

A point in the **(full) flag variety** $\mathcal{F}\ell_n$ is a flag $F: V_1 \subset V_2 \subset \cdots \subset V_n = \mathbb{R}^n$ of subspaces with $\dim V_i = i$. Every $F \in \mathcal{F}\ell_n$ can be thought of as a full rank $n \times n$ matrix A.

$$\underbrace{\begin{pmatrix} 0 & 1 & 0 \end{pmatrix}}_{M_1} \subset \underbrace{\begin{pmatrix} 0 & 1 & 0 \\ 1 & 2 & -1 \end{pmatrix}}_{M_2} \subset \underbrace{\begin{pmatrix} 0 & 1 & 0 \\ 1 & 2 & -1 \\ 2 & 1 & 1 \end{pmatrix}}_{M_3}$$

◦ A *circuit* of M is a minimal linearly dependent subset of $[n]$.
◦ A matroid M is a **quotient** of N if every circuit of N is union of circuits of M.
◦ A collection of matroids $\{M_1, \ldots, M_n\}$ on the set $[n]$ are a **(full) flag matroid** F if M_{i-1} is a quotient of M_i for $1 < i < n$.

Flags of matroids a.k.a. quotients of matroids

A point in the **(full) flag variety** $\mathcal{F}\ell_n$ is a flag $F: V_1 \subset V_2 \subset \cdots \subset V_n = \mathbb{R}^n$ of subspaces with $\dim V_i = i$. Every $F \in \mathcal{F}\ell_n$ can be thought of as a full rank $n \times n$ matrix A.

$$\underbrace{\begin{pmatrix} 0 & 1 & 0 \end{pmatrix}}_{M_1} \subset \underbrace{\begin{pmatrix} 0 & 1 & 0 \\ 1 & 2 & -1 \end{pmatrix}}_{M_2} \subset \underbrace{\begin{pmatrix} 0 & 1 & 0 \\ 1 & 2 & -1 \\ 2 & 1 & 1 \end{pmatrix}}_{M_3}$$

- A *circuit* of M is a minimal linearly dependent subset of $[n]$.
- A matroid M is a **quotient** of N if every circuit of N is union of circuits of M.
- A collection of matroids $\{M_1, \ldots, M_n\}$ on the set $[n]$ are a **(full) flag matroid** F if M_{i-1} is a quotient of M_i for $1 < i < n$.

$\mathcal{C}(M_2)$

Flags of matroids a.k.a. quotients of matroids

A point in the **(full) flag variety** $\mathcal{F}\ell_n$ is a flag $F: V_1 \subset V_2 \subset \cdots \subset V_n = \mathbb{R}^n$ of subspaces with $\dim V_i = i$. Every $F \in \mathcal{F}\ell_n$ can be thought of as a full rank $n \times n$ matrix A.

$$\underbrace{\begin{pmatrix} 0 & 1 & 0 \end{pmatrix}}_{M_1} \subset \underbrace{\begin{pmatrix} 0 & 1 & 0 \\ 1 & 2 & -1 \end{pmatrix}}_{M_2} \subset \underbrace{\begin{pmatrix} 0 & 1 & 0 \\ 1 & 2 & -1 \\ 2 & 1 & 1 \end{pmatrix}}_{M_3}$$

◦ A *circuit* of M is a minimal linearly dependent subset of $[n]$.
◦ A matroid M is a **quotient** of N if every circuit of N is union of circuits of M.
◦ A collection of matroids $\{M_1, \ldots, M_n\}$ on the set $[n]$ are a **(full) flag matroid** F if M_{i-1} is a quotient of M_i for $1 < i < n$.

$\mathcal{C}(M_2) = \{13\}$,

Flags of matroids a.k.a. quotients of matroids

A point in the **(full) flag variety** $\mathcal{F}\ell_n$ is a flag $F: V_1 \subset V_2 \subset \cdots \subset V_n = \mathbb{R}^n$ of subspaces with $\dim V_i = i$. Every $F \in \mathcal{F}\ell_n$ can be thought of as a full rank $n \times n$ matrix A.

$$\underbrace{\begin{pmatrix} 0 & 1 & 0 \end{pmatrix}}_{M_1} \subset \underbrace{\begin{pmatrix} 0 & 1 & 0 \\ 1 & 2 & -1 \end{pmatrix}}_{M_2} \subset \underbrace{\begin{pmatrix} 0 & 1 & 0 \\ 1 & 2 & -1 \\ 2 & 1 & 1 \end{pmatrix}}_{M_3}$$

∘ A *circuit* of M is a minimal linearly dependent subset of $[n]$.
∘ A matroid M is a **quotient** of N if every circuit of N is union of circuits of M.
∘ A collection of matroids $\{M_1, \ldots, M_n\}$ on the set $[n]$ are a **(full) flag matroid** F if M_{i-1} is a quotient of M_i for $1 < i < n$.

$\mathcal{C}(M_2) = \{13\}$, $\mathcal{C}(M_1)$

Flags of matroids a.k.a. quotients of matroids

A point in the **(full) flag variety** $\mathcal{F}\ell_n$ is a flag $F: V_1 \subset V_2 \subset \cdots \subset V_n = \mathbb{R}^n$ of subspaces with $\dim V_i = i$. Every $F \in \mathcal{F}\ell_n$ can be thought of as a full rank $n \times n$ matrix A.

$$\underbrace{\begin{pmatrix} 0 & 1 & 0 \end{pmatrix}}_{M_1} \subset \underbrace{\begin{pmatrix} 0 & 1 & 0 \\ 1 & 2 & -1 \end{pmatrix}}_{M_2} \subset \underbrace{\begin{pmatrix} 0 & 1 & 0 \\ 1 & 2 & -1 \\ 2 & 1 & 1 \end{pmatrix}}_{M_3}$$

◦ A *circuit* of M is a minimal linearly dependent subset of $[n]$.
◦ A matroid M is a **quotient** of N if every circuit of N is union of circuits of M.
◦ A collection of matroids $\{M_1, \ldots, M_n\}$ on the set $[n]$ are a **(full) flag matroid** F if M_{i-1} is a quotient of M_i for $1 < i < n$.

$\mathcal{C}(M_2) = \{13\}$, $\mathcal{C}(M_1) = \{1, 3\}$.

Let's recap

$V \in Gr_{k,n}$	Linear $M = ([n], \mathcal{B})$ of rank k

Let's recap

$V \in Gr_{k,n}$	Linear $M = ([n], \mathcal{B})$ of rank k
Richardson cell X_U^L	LPM $M(U, L)$

Let's recap

$V \in Gr_{k,n}$	Linear $M = ([n], \mathcal{B})$ of rank k
Richardson cell X_U^L	LPM $M(U, L)$
$A \in Gr_{k,n}^{\geq 0}$	Positroids of rank k

Let's recap

$V \in Gr_{k,n}$	Linear $M = ([n], \mathcal{B})$ of rank k
Richardson cell X_U^L	LPM $M(U, L)$
$A \in Gr_{k,n}^{\geq 0}$	Positroids of rank k
$F \in \mathcal{F}\ell_n$	Linear flag matroid $M_1 \subset \cdots \subset M_n$

Let's recap

$V \in Gr_{k,n}$	Linear $M = ([n], \mathcal{B})$ of rank k
Richardson cell X_U^L	LPM $M(U, L)$
$A \in Gr_{k,n}^{\geq 0}$	Positroids of rank k
$F \in \mathcal{F}\ell_n$	Linear flag matroid $M_1 \subset \cdots \subset M_n$
$F \in \mathcal{F}\ell_n^{\geq 0}$	?

- $F\ell_n^{\geq 0}$: $A_{n \times n}$ whose top i rows give a point in $Gr_{i,n}^{\geq 0}$, for $i \in [n]$.

Let's recap

$V \in Gr_{k,n}$	Linear $M = ([n], \mathcal{B})$ of rank k
Richardson cell X_U^L	LPM $M(U, L)$
$A \in Gr_{k,n}^{\geq 0}$	Positroids of rank k
$F \in \mathcal{F}\ell_n$	Linear flag matroid $M_1 \subset \cdots \subset M_n$
$F \in \mathcal{F}\ell_n^{\geq 0}$	?

- $F\ell_n^{\geq 0}$: $A_{n \times n}$ whose top i rows give a point in $Gr_{i,n}^{\geq 0}$, for $i \in [n]$.

Let's recap

$V \in Gr_{k,n}$	Linear $M = ([n], \mathcal{B})$ of rank k
Richardson cell X_U^L	LPM $M(U, L)$
$A \in Gr_{k,n}^{\geq 0}$	Positroids of rank k
$F \in \mathcal{F}\ell_n$	Linear flag matroid $M_1 \subset \cdots \subset M_n$
$F \in \mathcal{F}\ell_n^{\geq 0}$	?

- $F\ell_n^{\geq 0}$: $A_{n \times n}$ whose top i rows give a point in $Gr_{i,n}^{\geq 0}$, for $i \in [n]$.

Problems:

(1) Given two positroids P, Q on $[n]$, can you tell combinatorially if P is a quotient of Q, or viceversa?

(2) Is every flag $P_1 \subset \cdots \subset P_n$ of positroids a point in $\mathcal{F}\ell_n^{\geq 0}$?

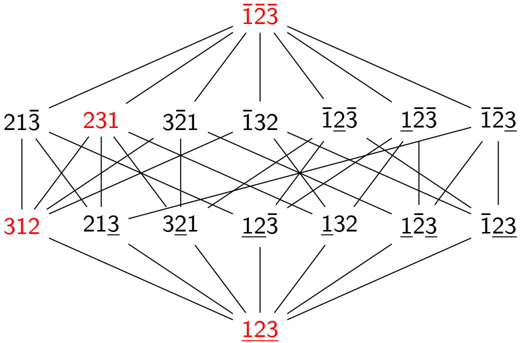
(3) What can we say about flags $L_1 \subset \cdots \subset L_n$ of LPMs?

(1) Given two positroids P, Q on $[n]$, how to tell (combinatorially) if P is a quotient of Q, or viceversa?

(1) Given two positroids P, Q on $[n]$, how to tell (combinatorially) if P is a quotient of Q, or viceversa?

A. Chavez
UC Davis

D. Tamayo
U. Paris-Saclay



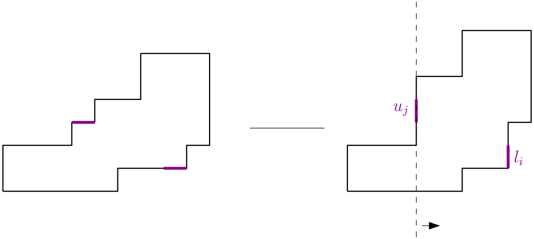
Quotients of uniform positroids. arXiv:1912.06873

(1) Given two positroids P, Q on $[n]$, how to tell (combinatorially) if P is a quotient of Q, or viceversa?

(1) Given two positroids P, Q on $[n]$, how to tell (combinatorially) if P is a quotient of Q, or viceversa?

K. Knauer
U. of Barcelona

Theorem [B-Knauer'20]: Let $M = M[U, L]$ be an LPM of rank k on $[n]$ and let $i, j \in [n]$. Then $M[U/j, L/i]$ is a quotient of M if and only if $\max(0, u_j - \ell_i) \leq j - i$.

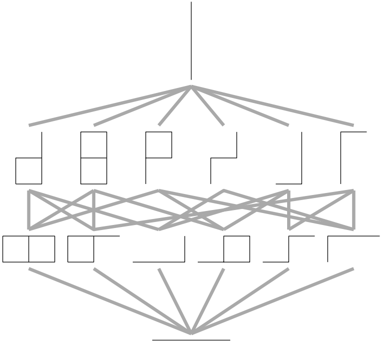


A poset of LPMs

◦ Given two LPMs P, Q on $[n]$ let $P \leq Q$ if and only if P is a quotient of Q.

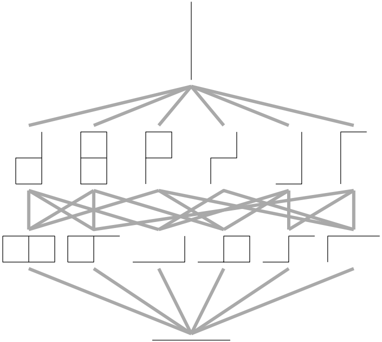
A poset of LPMs

◦ Given two LPMs P, Q on $[n]$ let $P \leq Q$ if and only if P is a quotient of Q.



A poset of LPMs

∘ Given two LPMs P, Q on $[n]$ let $P \leq Q$ if and only if P is a quotient of Q.

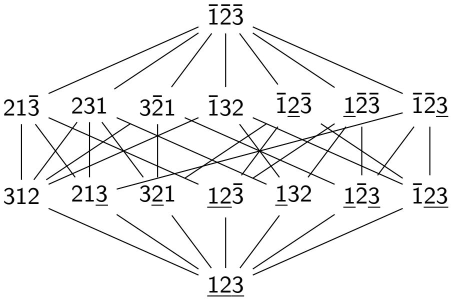


∘ We know:
$\#\{P \in LPM_n \ : \ r(P) = n - k\} = a(n+1, k+1)$ [Narayana numbers]

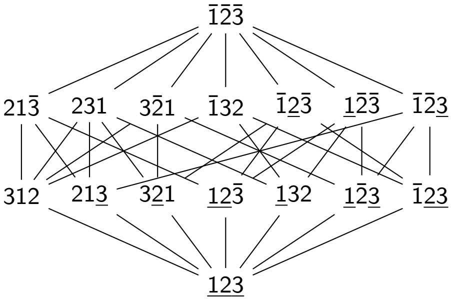
∘ We don't know: Möbius function of this poset.

(2) Is every flag $P_1 \subset \cdots \subset P_n$ of positroids a point in $\mathcal{F}\ell_n^{\geq 0}$?

(2) Is every flag $P_1 \subset \cdots \subset P_n$ of positroids a point in $\mathcal{F}\ell_n^{\geq 0}$?



(2) Is every flag $P_1 \subset \cdots \subset P_n$ of positroids a point in $\mathcal{F}\ell_n^{\geq 0}$?



NO.
The flag $3\underline{2}1 \lessdot 231$ is not a point in $\mathcal{F}\ell_n^{\geq 0}$:

$3\underline{2}1 : \{1, 3\}$

$231 : \{12, 23, 13\}$

$$\begin{pmatrix} a & 0 & b \\ c & d & e \end{pmatrix}$$

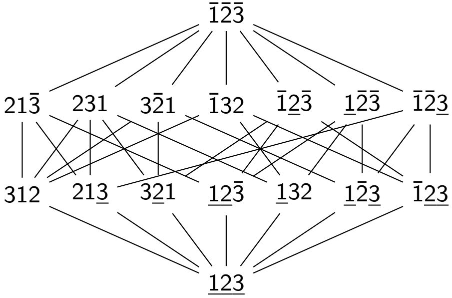
(3) Is every flag $L_1 \subset \cdots \subset L_n$ of LPMs a point in $\mathcal{F}\ell_n^{\geq 0}$?

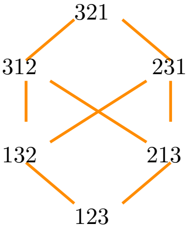
(3) Is every flag $L_1 \subset \cdots \subset L_n$ of LPMs a point in $\mathcal{F}\ell_n^{\geq 0}$?

[Tsukerman, Williams'15] If $F \in \mathcal{F}\ell_n^{\geq 0}$ then $F : P_1 \subset \cdots \subset P_n$ is a flag positroid and is flag positroid polytope $\Delta_F = \Delta_{P_1} + \cdots + \Delta_{P_n}$ is a Bruhat interval polytope.

> (3) Is every flag $L_1 \subset \cdots \subset L_n$ of LPMs a point in $\mathcal{F}\ell_n^{\geq 0}$?

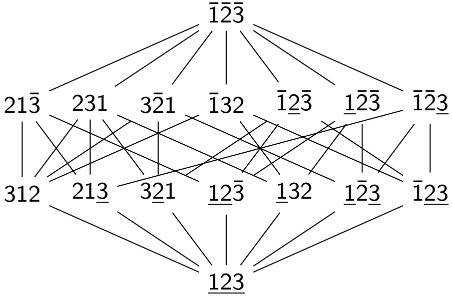
[Tsukerman, Williams'15] If $F \in \mathcal{F}\ell_n^{\geq 0}$ then $F : P_1 \subset \cdots \subset P_n$ is a flag positroid and is flag positroid polytope $\Delta_F = \Delta_{P_1} + \cdots + \Delta_{P_n}$is a Bruhat interval polytope.

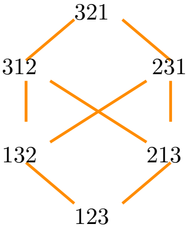




(3) Is every flag $L_1 \subset \cdots \subset L_n$ of LPMs a point in $\mathcal{F}\ell_n^{\geq 0}$?

[Tsukerman, Williams'15] If $F \in \mathcal{F}\ell_n^{\geq 0}$ then $F : P_1 \subset \cdots \subset P_n$ is a flag positroid and is flag positroid polytope $\Delta_F = \Delta_{P_1} + \cdots + \Delta_{P_n}$ is a Bruhat interval polytope.

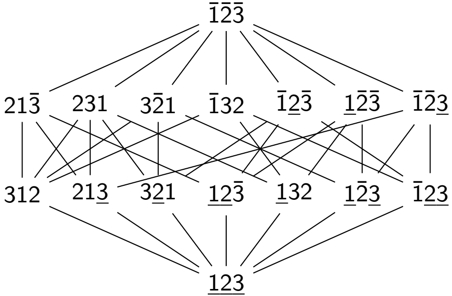


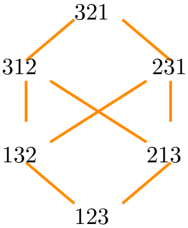


- Out of the 22 flags of positroids on [3], only 19 correspond to points in $\mathcal{F}\ell_n^{\geq 0}$.

(3) Is every flag $L_1 \subset \cdots \subset L_n$ of LPMs a point in $\mathcal{F}\ell_n^{\geq 0}$?

[Tsukerman, Williams'15] If $F \in \mathcal{F}\ell_n^{\geq 0}$ then $F : P_1 \subset \cdots \subset P_n$ is a flag positroid and is flag positroid polytope $\Delta_F = \Delta_{P_1} + \cdots + \Delta_{P_n}$ is a Bruhat interval polytope.

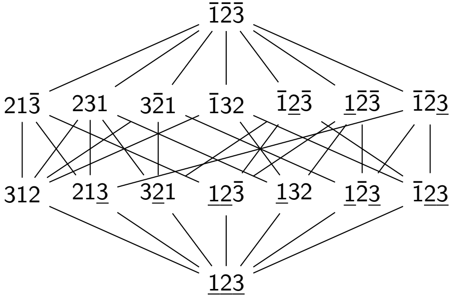


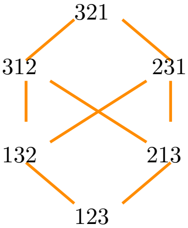


- Out of the 22 flags of positroids on [3], only 19 correspond to points in $\mathcal{F}\ell_n^{\geq 0}$.
- Out of these 19 flags in $\mathcal{F}\ell_n^{\geq 0}$, 17 are flags of LPMs.

(3) Is every flag $L_1 \subset \cdots \subset L_n$ of LPMs a point in $\mathcal{F}\ell_n^{\geq 0}$?

[Tsukerman, Williams'15] If $F \in \mathcal{F}\ell_n^{\geq 0}$ then $F : P_1 \subset \cdots \subset P_n$ is a flag positroid and is flag positroid polytope $\Delta_F = \Delta_{P_1} + \cdots + \Delta_{P_n}$ is a Bruhat interval polytope.

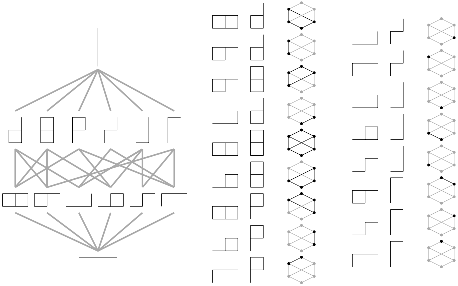




- Out of the 22 flags of positroids on [3], only 19 correspond to points in $\mathcal{F}\ell_n^{\geq 0}$.
- Out of these 19 flags in $\mathcal{F}\ell_n^{\geq 0}$, 17 are flags of LPMs.

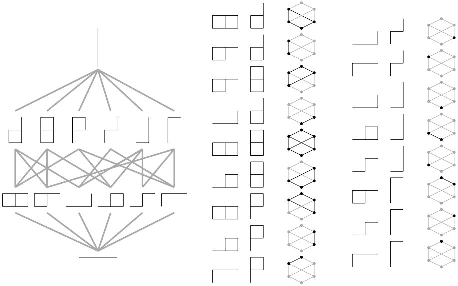
Theorem: [B-Knauer'20]
Every flag $L_1 \subset \cdots \subset L_n$ of LPMs is an interval in the Bruhat order.

Theorem: [B-Knauer'20]
Every flag $L_1 \subset \cdots \subset L_n$ of LPMs is an interval in the Bruhat order.



(3') What intervals in the Bruhat order correspond to flags $L_1 \subset \cdots \subset L_n$ of LPMs?

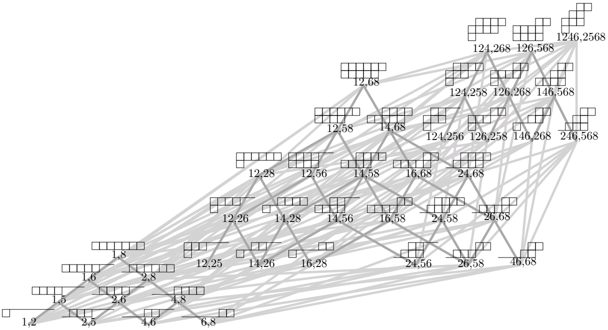
Theorem: [B-Knauer'20]
Every flag $L_1 \subset \cdots \subset L_n$ of LPMs is an interval in the Bruhat order.



(3') What intervals in the Bruhat order correspond to flags $L_1 \subset \cdots \subset L_n$ of LPMs?

Proposition: [B-Knauer'20]
If an interval $[u, v]$ in the (right weak) Bruhat order is a hypercube then it is a flag of LPMs.

Thank you!


1246,2568
124,268
126,568
12,68
124,258
126,268
146,568
12,58
14,68
124,256
126,258
146,268
246,568
12,28
12,56
14,58
16,68
24,68
12,26
14,28
14,56
16,58
24,58
26,68
1,8
12,25
14,26
16,28
24,56
26,58
46,68
1,6
2,8
1,5
2,6
4,8
1,2
2,5
4,6
6,8