Two Injective Proofs of a Conjecture of Simion

Miklós Bóna
Department of Mathematics
University of Florida
Gainesville, FL 32611
USA
bona@math.ufl.edu

and

Bruce E. Sagan
Department of Mathematics
Michigan State University
East Lansing, MI 48824-1027
USA
sagan@math.msu.edu

October 31, 2002

Abstract

Simion [9] conjectured the unimodality of a sequence counting lattice paths in a grid with a Ferrers diagram removed from the northwest corner. Recently, Hildebrand [5] and then Wang [11] proved the stronger result that this sequence is actually log concave. Both proofs were mainly algebraic in nature. We give two combinatorial proofs of this theorem.

1 Introduction

In this note we present two injective proofs of a strengthening of a conjecture of Simion [9]. To describe the result, let $\lambda = (\lambda_1, \lambda_2, \cdots, \lambda_k)$ be the Ferrers diagram of a partition viewed as a set of squares in English notation. (See any of the texts [1, 8, 10] for definitions of terms that we do not define here.) The shape λ will be fixed for the rest of this paper.

Consider a grid with the vertices labeled (i, j) for $i, j \geq 0$ as in Figure 1. Place λ in the northwest corner of this array so that its squares coincide with those of the grid.

A *northeastern lattice path* is a lattice path on the grid in which each step goes one unit to the north or one unit to the east. Let $N_\lambda(m, n) = N(m, n)$ be the number of northeastern lattice paths from $(m, 0)$

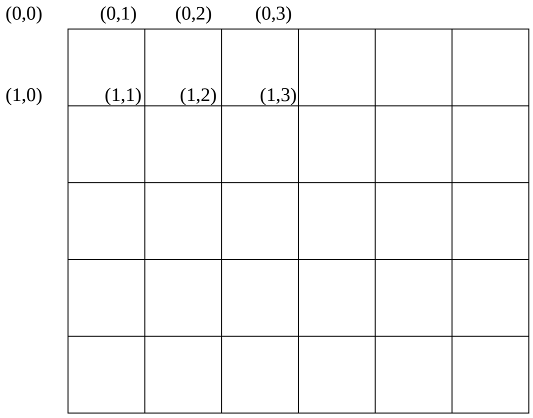

Figure 1: Labeling our points in the grid.

to $(0,n)$ that *do not go inside* λ (although they may touch its southeastern boundary), and let $\mathcal{N}(m,n)$ be the set of such paths. In particular, $N(m,n) = 0$ if either the starting or ending point is inside λ.

Simion [9] conjectured that for all $m, n \geq 0$ the sequence

$$N(0, m+n), N(1, m+n-1), \ldots, N(m+n, 0)$$

is unimodal. Lattice path techniques for proving unimodality were investigated by Sagan [7], but the conjecture remained open at that point. Recently, Hildebrand [5] proved the stronger result that this sequence is actually log concave by mostly algebraic means. Shortly thereafter, Wang [11] simplified Hildebrand's proof using results about Polya frequency sequences. In the present work, we will give two injective proofs of the strong version of Simion's conjecture. The one in Sections 3 and 4 employs ideas from Hildebrand's proof while the one in Section 5 is more direct. Our injections come from a method of Lindström [6], later popularized by Gessel and Viennot [3, 4], that can be used to prove total positivity results for matrices. For an exposition, see Sagan's book [8, pp. 158–163]. Bóna [2] has used related ideas to prove the log concavity of a sequence counting t-stack sortable permutations.

We end this section by reiterating the statement of the main theorem for easy reference. Notice that when $\lambda = \emptyset$ it specializes to the well-known result that the rows of Pascal's triangle are log concave.

Theorem 1 (The Strong Simion Conjecture) *Let λ be the Ferrers diagram of a partition and let $N(m,n)$ be the number of northeastern lattice paths in the grid from $(m,0)$ to $(0,n)$ which do not intersect the interior of λ. Then for all $m, n \geq 0$ the sequence*

$$N(0, m+n), N(1, m+n-1), \ldots, N(m+n, 0)$$

is log concave.

2 A decomposition of the problem

This preliminary part of the first proof is from [5]. We include it so that our exposition will be self contained. We need to prove that for all $m, n > 0$ we have

$$N(m-1, n+1)N(m+1, n-1) \leq N(m, n)^2.$$

To prove this, it suffices to show that for all λ and all $m \geq \lambda_1'$, $n \geq \lambda_1$ (where λ' is the conjugate or transpose of λ) we have

$$N(m-1, n+1)N(m+1, n) \leq N(m, n)N(m, n+1).$$

Then, using the fact that the previous inequality holds for all partitions and that $N_\lambda(m, n) = N_{\lambda'}(n, m)$, we also have

$$N(m+1, n-1)N(m, n+1) \leq N(m, n)N(m+1, n).$$

Now multiplying the last two inequalities together and simplifying gives the first.

The second inequality can be proved by demonstrating another pair of inequalities, namely

$$N(m, n+1)N(m+1, n) \leq N(m, n)N(m+1, n+1), \tag{1}$$

and

$$N(m-1, n+1)N(m+1, n+1) \leq N(m, n+1)^2. \tag{2}$$

Multiplying these two inequalities together and cancelling gives the desired result.

3 The proof of (1)

In this section we prove that (1) holds by constructing an injection

$$\Psi : \mathcal{N}(m, n+1) \times \mathcal{N}(m+1, n) \to \mathcal{N}(m, n) \times \mathcal{N}(m+1, n+1).$$

Consider a path pair $(p, q) \in \mathcal{N}(m, n+1) \times \mathcal{N}(m+1, n)$. Then p and q must intersect. Let C be their first (most southwestern) intersection point. Say that C splits p into parts p_1 and p_2, and splits q into parts q_1 and q_2. Then the concatenation of p_1 and q_2 is a path in $\mathcal{N}(m, n)$, and the concatenation of q_1 and p_2 is a path in $\mathcal{N}(m+1, n+1)$. So define $\Psi(p, q) = (p_1q_2, q_1p_2) = (p', q')$. It is easy to see that the image of Ψ is exactly all $(p', q') \in \mathcal{N}(m, n) \times \mathcal{N}(m+1, n+1)$ such that p' and q' intersect. It is also simple to verify that if $\Psi(p, q) = (p', q')$, then applying the same algorithm to (p', q') recovers (p, q). So Ψ is injective. See Figure 2 for an example.

Figure 2: The action of Ψ.

4 The proof of (2)

In this section we construct an injection

$$\Phi : \mathcal{N}(m-1,n) \times \mathcal{N}(m+1,n) \rightarrow \mathcal{N}(m,n)^2,$$

thus proving (2).

Let $(p,q) \in \mathcal{N}(m-1,n) \times \mathcal{N}(m+1,n)$. If $P=(i,j)$ and $Q=(k,j)$ are vertices with the same second coordinate, then define the *vertical distance from P to Q* to be $d_v(P,Q)=k-i$. Now the vertical distance from a point of p to a point of q starts at 2 for their initial vertices and ends at 0 for their final ones. Since vertical distance can change by at most one with a step of a path, there must be some vertical distance equal to 1. Let $P \in p$ and $Q \in q$ be the first (most southwest) pair of points with $d_v(P,Q)=1$.

Let p_1 and p_2 be the portions of p before and after P, respectively, and similarly for q. Now let

$$\Phi(p,q) = (p_1' q_2, q_1' p_2)$$

where p_1' is p_1 moved south one unit and q_1' is q_1 moved north one unit. Since P and Q are the first pair of points at vertical distance one, q_1' will not go inside λ and the concatenations are valid paths in $\mathcal{N}(m,n)$. This proves that Φ is well defined.

We now prove that Φ is injective by showing that given any (p',q') in the image of Φ, there is a unique pair (p,q) mapping onto it. Since (p',q') is in Φ's image, there must be a pair of points $P' \in p'$ and $Q' \in q'$ with $d_v(P',Q')=-1$. Furthermore, if we pick P',Q' to be the first such pair, then by the definition of Φ we must have $P'=Q$ and $Q'=P$. It follows that p,q are uniquely determined by moving the portion of p' up to P' north one unit and the portion of q' up to Q' south one unit. This proves injectivity and completes the first proof of the Strong Simion Conjecture. For an example with $\lambda=(2,1)$ and $(m,n)=(4,3)$, see Figure 3. $\diamond$

Figure 3: The action of Φ.

5 A more direct proof

The reader may wonder if we can do away with splitting our problem into two parts, that is, inequalities (1) and (2). The answer is yes, and the necessary injection is just a modification $\overline{\Phi}$ of the map Φ. This will give us a second, completely combinatorial, proof of our main theorem

Take a path pair $(p, q) \in \mathcal{N}(m-1, n+1) \times \mathcal{N}(m+1, n-1)$. Notice that p and q must intersect. So before the first intersection there must be a first pair of points $P \in p$ and $Q \in q$ with $d_v(P, Q) = 1$. Similarly, after the last intersection there must be a last pair of points $\overline{P} \in p$ and $\overline{Q} \in q$ with $d_h(\overline{P}, \overline{Q}) = -1$ where d_h is horizontal distance which is defined analogously. Let P and $\overline{P}$ divide p into subpaths p_1, p_2, p_3 and use the same notation for q. Then define

$$\overline{\Phi}(p, q) = (p_1' q_2 p_3'', q_1' p_2 q_3'')$$

where p_1' is p_1 moved south one unit, p_3'' is p_3 moved west one unit, and q_1', q_3'' are defined in the analogous way but moving in the opposite directions. It is a simple job to verify that $\overline{\Phi}$ is well defined and injective just as we did with Φ.

This completes the second proof of Theorem 1. $\diamond$

We have two final remarks. First of all, it is clear from the geometry of the situation that if λ is self-conjugate then the sequence in Theorem 1 is also symmetric, but this does not hold in general. One might also wonder if this sequence has the stronger property that the associated polynomial generating function has only real zeros. This is not always true as can be seen by taking $\lambda = (1)$ and $m + n = 4$. In this case the associated polynomial is $x(3x^2 + 5x + 3)$ which has two complex roots. It might be interesting to determine for which shapes the real zero property holds.

Acknowledgment. We are indebted to the two anonymous referees for suggestions which improved the exposition.

References

[1] M. Bóna, "A Walk Through Combinatorics," World Scientific, 2002.

[2] M. Bóna, Log-concavity for t-stack sortable permutations, *J. Comb. Theory, Series A*, submitted.

[3] I. Gessel and G. Viennot, Binomial determinants, paths, and hook length formulae, *Advances in Math.* **58** (1985), 300–321.

[4] I. Gessel and G. Viennot, Determinants, paths, and plane partitions, in preparation.

[5] M. Hildebrand, Log Concavity of a Sequence in a Conjecture of Simion, *J. Comb. Theory, Series A*, **97** (2002) 108-116.

[6] B. Lindström, On the vector representation of induced matroids, *Bull. London Math. Soc.* **5** (1973), 85–90.

[7] B. Sagan, Unimodality and the Reflection Principle, *Ars Combin.*, **48** (1998), 65-72.

[8] B. Sagan, "The Symmetric Group: Representations, Combinatorial Algorithms, and Symmetric Functions," 2nd edition, Springer-Verlag, New York, 2001.

[9] R. Simion, Combinatorial Statistics and Noncrossing partitions, *J. Comb. Theory, Series A*, **94** (1994) 270-301.

[10] R. P. Stanley, "Enumerative Combinatorics, Volume 2," Cambridge University Press, Cambridge, 1999.

[11] Y. Wang, A simple proof of a conjecture of Simion, preprint.