Set partition statistics and q-Fibonacci numbers

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Abstract

We consider the set partition statistics ls and rb introduced by Wachs and White and investigate their distribution over set partitions avoiding certain patterns. In particular, we consider those set partitions avoiding the pattern $13/2$, $\Pi_n(13/2)$, and those avoiding both $13/2$ and 123, $\Pi_n(13/2, 123)$. We show that the distribution over $\Pi_n(13/2)$ enumerates certain integer partitions, and the distribution over $\Pi_n(13/2, 123)$ gives q-Fibonacci numbers. These q-Fibonacci numbers are closely related to q-Fibonacci numbers studied by Carlitz and by Cigler. We provide combinatorial proofs that these q-Fibonacci numbers satisfy q-analogues of many Fibonacci identities. Finally, we indicate how p, q-Fibonacci numbers arising from the bistatistic (ls, rb) give rise to p, q-analogues of identities.

1 Introduction and Preliminary Results

Define the Fibonacci numbers F_n as satisfying the initial conditions $F_0 = 1$, $F_1 = 1$, and the recursion $F_n = F_{n-1} + F_{n-2}$ for $n \geq 2$. This paper focuses on q-Fibonacci numbers which arise naturally from the study of set partition statistics on pattern restricted partitions. Wachs and White [19] introduced the statistics we will use and showed how that could be used to define q-Stirling numbers of the second kind. In [16], Simion studied the distribution of these statistics over non-crossing partitions (those avoiding $13/24$) to get q-analogues of the Catalan numbers. Further work on this subject was done by Wachs in [20] and by White in [23]

The q-Fibonacci numbers studied here are closely related to the q-Fibonacci numbers studied by Carlitz [4, 5] and Cigler [6, 7, 8, 9]. In this section we provide the necessary definitions and background for understanding pattern restricted set partitions and the two statistics. Section 2

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contains an exploration of a familiar q-analogue of 2^n and its relationship to integer partitions. In Section 3 we define the q-Fibonacci numbers arising from set partition statistics, introduce the q-Fibonacci numbers of Carlitz and Cigler, and explore their relationships. The next two sections are devoted to showing that known Fibonacci identities and their bijective proofs easily lead to bijective proofs of q-analogues of these identities. Finally, p,q-analogues of Fibonacci identities are discussed in the last section.

Let $[n] = \{1, 2, \ldots, n\}$ and $[k, n] = \{k, k+1, \ldots, n\}$. A *partition* π of $[n]$, written $\pi \vdash [n]$, is a family of nonempty, disjoint subsets $B_1, B_2, \ldots, B_k$ of $[n]$, called *blocks*, such that $\bigcup_{i=1}^{k} B_i = [n]$. If π has k blocks then we say that the *length* of π is k, written $l(\pi) = k$. We write $\pi = B_1/B_2/\ldots/B_k$, omitting set braces and commas, and where we always list the blocks in the *standard order*

$$\min B_1 < \min B_2 < \ldots < \min B_k.$$

Suppose $\pi = A_1/A_2/\ldots/A_k \vdash [m]$ and $\sigma = B_1/B_2/\ldots/B_l \vdash [n]$. We say π *is contained in* σ, written $\pi \subseteq \sigma$, if there are k distinct blocks $B_{i_1}, B_{i_2}, \ldots, B_{i_k}$ of σ such that $A_j \subseteq B_{i_j}$. For example, if $\sigma = 137/25/4/6$ then $\pi = 25/3$ is contained in σ, but $\pi' = 2/5/6$ is not because the 2 and the 5 would have to be contained in separate blocks of σ.

Given a set of integers S with cardinality $\#S = n$, define the *standardization* map $St_S : S \to [n]$ to be the unique order preserving bijection between these sets. For example if $S = \{3, 5, 10\}$ then $St_S : S \to [3]$ and $St_S(3) = 1$, $St_S(5) = 2$, and $St_S(10) = 3$. We drop the subscript S when this will cause no confusion. We let St act element-wise on set partitions.

Let $\pi \vdash [m]$ and $\sigma \vdash [n]$. We say σ contains the *pattern* π if there is a partition π' such that $\pi' \subseteq \sigma$ and $St(\pi') = \pi$, otherwise we say that σ *avoids* π. A copy of $\pi = 12/3$ in $\sigma = 137/25/4/6$ is $25/6$. The partition $12/34$ is not contained in σ since the only two blocks with more than one element are $\{1, 3, 7\}$ and $\{2, 5\}$, and the latter block can not act as either of the two smallest or two largest elements of $12/34$.

Define

$$\Pi_n = \{\pi \vdash [n]\},$$

$$\Pi = \bigcup_{n \geq 0} \Pi_n,$$

and for any set of partitions $P \subseteq \Pi$,

$$\Pi_n(P) = \{\pi \vdash [n] : \pi \text{ avoids every partition in } P\}.$$

A *layered* partition is a partition of the form $\pi = [1, i]/[i+1, j]/\ldots/[k+1, n]$, and a *matching* is a partition $B_1/B_2/\ldots/B_k$ where $\#B_i \leq 2$ for all i. For example $123/4567/8/9$ is a layered partition and $12/34/5/67/8$ is a layered matching.

In [13] Sagan characterized $\Pi_n(\pi)$ for all $\pi \vdash [3]$. In particular, he showed that $\Pi_n(123)$ is the set of matchings of $[n]$ and that $\Pi_n(13/2)$ is the set of layered partitions of $[n]$. It's not hard to see that $\#\Pi_n(13/2) = 2^{n-1}$. Goyt [11] determined $\Pi_n(P)$ for all sets P of partitions of [3]. He noted that $\Pi_n(13/2, 123)$ is the set of layered matchings of $[n]$, from which it follows easily that $\#\Pi_n(13/2, 123) = F_n$.

We are interested in the distributions of set partition statistics on $\Pi_n(13/2)$ and $\Pi_n(13/2, 123)$ and the resulting q-analogues of 2^n and F_n. Of the known set partition statistics, only the left smaller, ls, and right bigger, rb, statistics of Wachs and White [19] seem to give interesting distributions. We now describe these statistics.

Let $\pi = B_1/B_2/\ldots/B_k$ be a partition and $b \in B_i$, then we will say that (b, B_j) is a *left smaller pair* of π if $j < i$ and $\min B_j < b$. So, because of standard ordering, for a given block B_j, the elements in left smaller pairs with B_j are exactly those in $B_{j+1}, \ldots, B_k$. We will say that (b, B_j) is a *right bigger pair* of σ if $j > i$ and $\max B_j > b$. Define $ls(\pi)$ to be the number of left smaller pairs of π and $rb(\pi)$ to be the number of right bigger pairs of π. Wachs and White proved that ls and rb are equidistributed over Π_n. turns out that ls and rb are also equidistributed over $\Pi_n(13/2)$ and $\Pi_n(13/2, 123)$.

Theorem 1.1 *For any n,*

$$\sum_{\pi\in\Pi_n(13/2)} q^{ls(\pi)} = \sum_{\pi\in\Pi_n(13/2)} q^{rb(\pi)},$$

and

$$\sum_{\pi\in\Pi_n(13/2,123)} q^{ls(\pi)} = \sum_{\pi\in\Pi_n(13/2,123)} q^{rb(\pi)}.$$

Proof: Given a set partition $\pi = B_1/B_2/\ldots/B_k \in \Pi_n(13/2)$, let the complement of π be the partition $\pi^c = B_k^c/\ldots/B_2^c/B_1^c$, where $B_i^c = \{n - b + 1 : b \in B_i\}$. Notice that taking the complement reverses the order of the blocks since π is layered. Clearly complementation is an involution, and so bijective. To prove the first equality it suffices to show that it exchanges ls and rb. This follows easily because the block order is reversed and minima are exchanged with maxima. Also, complementation does not alter the block sizes and so restricts to a map on $\Pi_n(13/2, 123)$. $\square$

2 Distribution over $\Pi_n(13/2)$

Define

$$A_n(q) = \sum_{\pi\in\Pi_n(13/2)} q^{rb(\pi)}.$$

It will be useful to think of the rb statistic in the following way. Consider a block B_j. Let $\pi = B_1/B_2/\ldots/B_k$ be a partition. For each element $b \in B_i$ with $i < j$ and $b < \max B_j$, we have that (b, B_j) is a right bigger pair. The number of right bigger pairs of the form (b, B_j) will be the *contribution* of B_j to rb. When restricted to layered partitions, the contribution of B_j is

$$\sum_{i<j} \#B_i = \min B_j - 1.$$

The generating function $A_n(q)$ is closely related to integer partitions. A *partition* $\lambda = (\lambda_1, \lambda_2, \ldots, \lambda_k)$ *of the integer* d is a weakly decreasing sequence of positive integers such that $\sum_{i=1}^{k} \lambda_i = d$; the λ_i are called *parts*. We let $|\lambda| = \sum_{i=1}^{k} \lambda_i$. Denote by D_{n-1} the set of integer partitions with distinct parts of size at most $n-1$. It is well known that

$$\sum_{\lambda \in D_{n-1}} q^{|\lambda|} = \prod_{i=1}^{n-1} (1+q^i).$$

For the rest of this chapter we will refer to a set partition as just a partition and an integer partition by its full name.

For the following proof, it will be more convenient for us to list the parts of an integer partition in weakly increasing order. Let $\phi : \Pi_n(13/2) \to D_{n-1}$ be the map defined by

$$\phi(B_1/B_2/\ldots/B_k) = (\lambda_1, \lambda_2, \ldots, \lambda_{k-1}),$$

where $\lambda_j = \sum_{i=1}^{j} \# B_i$.

Theorem 2.1 *The map ϕ is a bijection, and for $\pi \in \Pi_n(13/2)$, $rb(\pi) = |\phi(\pi)|$. Hence,*

$$A_n(q) = \prod_{i=1}^{n-1} (1+q^i).$$

Proof: Given $\lambda = (\lambda_1, \lambda_2, \ldots, \lambda_{k-1})$ consider, for $1 \le j \le k$, the differences $d_j = \lambda_j - \lambda_{j-1}$, where $\lambda_0 = 0$ and $\lambda_k = n$. If $\lambda \in D_{n-1}$ then we have $d_j > 0$ for all j. And if $\phi(\pi) = \lambda$ then the d_j give the block sizes of π. So a given λ determines a unique sequence of block sizes, and this sequence determines a unique layered π. Thus, ϕ is bijective. Since $\lambda_j = \sum_{i \le j} \# B_i$ is the contribution of B_{j+1} to rb (and B_1 makes no contribution) we have $rb(\pi) = |\phi(\pi)|$ as desired. $\square$

3 q-Fibonacci Numbers Past and Present

We turn our focus to the distribution of rb over $\Pi_n(13/2, 123)$. As remarked in the introduction,

$$\#\Pi_n(13/2, 123) = F_n.$$

The distribution of rb over $\Pi_n(13/2, 123)$ gives a nice q-analogue of the Fibonacci numbers. Let

$$F_n(q) = \sum_{\pi \in \Pi_n(13/2,123)} q^{rb(\pi)}.$$

Proposition 3.1 *The generating function $F_n(q)$ satisfies the boundary conditions $F_0(q) = 1$, $F_1(q) = 1$, and the recursion*

$$F_n(q) = q^{n-1} F_{n-1}(q) + q^{n-2} F_{n-2}(q).$$

Proof: Let $\pi \in \Pi_n(13/2, 123)$. Since π is a matching it must end in a block of size one or of size two. If π ends in a singleton then the singleton is $\{n\}$, which contributes $n-1$ to rb, and the remaining elements form a partition in $\Pi_{n-1}(13/2, 123)$. Similarly the doubleton case is counted by the second term on the right-hand side of the recursion. $\square$

We now introduce the q-Fibonacci numbers of Carlitz and Cigler and explore their relationship to the q-Fibonacci numbers as defined above. Let BS_n be the set of binary sequences $\beta = b_1 \dots b_n$ of length n without consecutive ones. It is well known that $\#BS_n = F_{n+1}$. In [4, 5], Carlitz defined and studied a statistic on BS_n as follows. Let $\rho : BS_n \to \mathbb{N}$ be given by

$$\rho(\beta) = \rho(b_1 \dots b_n) = b_1 + 2b_2 + \dots + nb_n,$$

and define

$$F_n^K(q) = \sum_{\beta \in BS_{n-1}} q^{\rho(\beta)}.$$

Carlitz showed that $F_n^K(q)$ satisfies $F_0^K(q) = 1$, $F_1^K(q) = 1$, and

$$F_n^K(q) = F_{n-1}^K(q) + q^{n-1} F_{n-2}^K(q).$$

Cigler [7] defined his q-Fibonacci polynomials using Morse sequences. A *Morse sequence* of length n is a sequence of dots and dashes, where each dot has length 1 and each dash has length 2. For example, $\nu = \bullet\bullet - - \bullet -$ is a Morse sequence of length 9. Let MS_n be the set of Morse sequences of length n. Each Morse sequence corresponds to a layered matching where a dot is replaced by a singleton block and a dash by a doubleton. So, $\#MS_n = F_n$.

Define the weight of a dot to be x and the weight of a dash to be yq^{a+1} where a is the length of the portion of the sequence appearing before the dash. Also, define a weight on the Morse sequences, $w : MS_n \to \mathbb{Z}[x, y, q]$, by letting $w(\nu)$ be the product of the weights of its dots and dashes. For example, the sequence above has weight $(x)(x)(yq^3)(yq^5)x(yq^8) = x^3y^3q^{16}$. Let

$$F_n^C(x, y, q) = \sum_{\nu \in MS_n} w(\nu).$$

Cigler shows that $F_n^C(x, y, q)$ satisfies $F_0^C(x, y, q) = 1$, $F_1^C(x, y, q) = x$, and

$$F_n^C(x, y, q) = xF_{n-1}^C(x, y, q) + yq^{n-1}F_{n-2}(x, y, q).$$

Note that $F_n^C(1, 1, q) = F_n^K(q)$. In fact, Cigler [6, 7, 8, 9] studied more general q-Fibonacci numbers satisfying the above recursion with yq^{n-1} replaced by $t(yq^{n-1})$, where t is an arbitrary nonzero function. One could apply our method to such q-analogues, but we choose t to be the identity for simplicity.

Proposition 3.2 *For all $n \geq 0$,*

$$F_n(q) = q^{\binom{n}{2}} F_n^K(1/q).$$

Proof: It suffices to construct a bijection $\Pi_n(13/2,123) \leftrightarrow BS_{n-1}$ such that if $\pi \leftrightarrow \beta$ then $rb(\pi) = \binom{n}{2} - \rho(\beta)$.

Let $\pi \in \Pi_n(13/2,123)$ be mapped to the binary sequence $\beta = b_1 \dots b_{n-1}$ where $b_i = 0$ if i and $i+1$ are in separate blocks and $b_i = 1$ otherwise. For example, $1/2/34/56 \leftrightarrow 00101$. We first show that this map is well defined. Suppose $\pi \mapsto b_1 \dots b_{n-1}$, where $b_i = 1$ and $b_{i+1} = 1$ for some i, then i, $i+1$, and $i+2$ must be in a block together. This contradicts the fact that the blocks may only be of size at most 2. Proving that this map is a bijection is straightforward.

Now, suppose that $\pi \leftrightarrow \beta$ and $\beta = b_1 \dots b_{n-1}$. If $\pi = \pi_0 = 1/2/\dots/n$ so that $b_i = 0$ for all i, then

$$rb(\pi) = \sum_{i=1}^{n-1} i = \binom{n}{2} = \binom{n}{2} - \rho(0\dots 0).$$

If $b_i = 1$ for some i then i and $i+1$ are in the same block. In π_0, the contribution of the blocks $\{i\}$ and $\{i+1\}$ to rb was $(i-1)+i = 2i-1$. But in π the contribution of $\{i, i+1\}$ is only $i-1$. Thus, for each $b_i = 1$ we reduce $rb(\pi_0)$ by i and hence,

$$rb(\pi) = \binom{n}{2} - \sum_{i\ :\ b_i=1} i = \binom{n}{2} - \rho(\beta). \ \square$$

In order to describe the relationship between $F_n(q)$ and $F_n^C(x,y,q)$ we will define a weight, ω, on the partitions in $\Pi_n(13/2,123)$. Let

$$\omega : \Pi_n(13/2,123) \to \mathbb{Z}[x,y,q],$$

with $\omega(\pi) = \omega(B_1/B_2/\dots/B_k) = \prod_{i=1}^{k} \omega(B_i)$, where

$$\omega(B_j) = \begin{cases} xq^{\min B_j - 1} & \text{if } \#B_j = 1, \\ yq^{\min B_j - 1} & \text{if } \#B_j = 2. \end{cases}$$

Now, let

$$F_n(x,y,q) = \sum_{\pi \in \Pi_n(13/2,123)} \omega(\pi).$$

Let $s(\pi)$ be the number of singletons of π, $d(\pi)$ be the number of doubletons of π. It is easy to see directly from the definitions that

$$F_n(x,y,q) = \sum_{\pi \in \Pi_n(13/2,123)} x^{s(\pi)} y^{d(\pi)} q^{rb(\pi)}.$$

The proof of Proposition 3.1 also shows that

$$F_n(x,y,q) = xq^{n-1} F_{n-1}(x,y,q) + yq^{n-2} F_{n-2}(x,y,q). \tag{1}$$

The demonstration of the next result is omitted since it parallels that of Proposition 3.2, using the bijection $\Pi_n(13/2,123) \leftrightarrow MS_n$ mentioned above.

Proposition 3.3 *For all $n \geq 0$,*

$$F_n(x,y,q) = q^{\binom{n}{2}} F_n^C(x,y,1/q). \ \square$$

4 q-Fibonacci Identities

We now provide bijective proofs of q-analogues of Fibonacci identities. Many of the proofs in this paper are simply q-analogues of the tiling scheme proofs of Fibonacci identities given in [2, 3]. We should also note that Shattuck and Wagner [14, 15] have used various statitics on domino arrangements to obtain q-identities and parity results for Fibonacci and Lucas numbers.

It is impressive that merely using the rb statistic on $\Pi_n(13/2, 123)$ gives so many identities with relatively little effort. We will state our identities for $F_n(x, y, q)$, but one can translate them in terms of $F_n^K(q)$ or $F_n^C(x, y, q)$ using Propositions 3.2 or 3.3, respectively.

Theorem 4.1 *For all $n \geq 0$,*

$$F_{n+2}(x, y, q) = x^{n+2}q^{\binom{n+2}{2}} + \sum_{j=0}^{n} x^j y q^{\binom{j+1}{2}} F_{n-j}(xq^{j+2}, yq^{j+2}, q).$$

Proof: There is exactly one partition in $\Pi_{n+2}(13/2, 123)$ with all singleton blocks and the weight of this partition is $x^{n+2}q^{\binom{n+2}{2}}$. The remaining partitions have at least one doubleton. Consider all partitions where the first doubleton is $\{j+1, j+2\}$. There are exactly j singletons preceding this doubleton contributing $x^j q^{\binom{j}{2}}$ to the weight of each partition. The doubleton contributes weight yq^j. The remaining blocks of these partitions form layered matchings of $[j+3, n+2]$. We may think of these as being layered matchings of $[n-j]$ where the contribution of each block to the rb statistic is increased by $j+2$. Thus, these contribute weight $F_{n-j}(xq^{j+2}, yq^{j+2}, q)$. Hence, the contributed weight of the partitions whose first doubleton is $\{j+1, j+2\}$, is $x^j y q^{\binom{j+1}{2}} F_{n-j}(xq^{j+2}, yq^{j+2}, q)$. Summing from $j = 0$ to n completes the proof. □

Theorem 4.2 *For all $n \geq 0$,*

$$F_{2n+1}(x, y, q) = \sum_{j=0}^{n} xy^j q^{j(j+1)} F_{2n-2j}(xq^{2j+1}, yq^{2j+1}, q),$$

and

$$F_{2n}(x, y, q) = y^n q^{n(n-1)} + \sum_{j=0}^{n-1} xy^j q^{j(j+1)} F_{2n-2j-1}(xq^{2j+1}, yq^{2j+1}, q).$$

Proof: If $\pi \in \Pi_{2n+1}(13/2, 123)$, then π must have at least one singleton. Consider all partitions with first singleton $\{2j+1\}$. This block must be preceded by j doubletons, which contribute $y^j q^{j(j-1)}$ to the weight. The singleton $\{2j+1\}$ contributes xq^{2j} to the weight. The remaining $2n-2j$ elements form a layered matching of $[2j+2, 2n+1]$. As in the previous proof, we may think of these as being elements of $\Pi_{2n-2j}(13/2, 123)$ where the contribution of each block to rb is increased by $2j+1$. This portion of our partition will contribute $F_{2n-2j}(xq^{2j+1}, yq^{2j+1}, q)$ to the weight. This proves the first identity. The proof of the second identity is similar, and hence omitted. □

Theorem 4.3 *For all $n \geq 0$ and $m \geq 0$,*

$$F_{m+n}(x, y, q) = F_m(x, y, q)F_n(xq^m, yq^m, q) + yq^{m-1}F_{m-1}(x, y, q)F_{n-1}(xq^{m+1}, yq^{m+1}, q).$$

Proof: Every $\pi \in \Pi_{m+n}(13/2, 123)$ does or does not have $\{m, m+1\}$ as a block. If π has $\{m, m+1\}$ as a block then the blocks prior to this block form a partition in $\Pi_{m-1}(13/2, 123)$. This contributes $F_{m-1}(x, y, q)$ to the weight. The doubleton $\{m, m+1\}$ has weight yq^{m-1}. The remaining blocks form a partition in $\Pi_{[m+2,m+n]}(13/2, 123)$. The contribution of each block in this partition to the rb statistic is increased by $m+1$, so this portion of the partition contributes $F_{n-1}(xq^{m+1}, yq^{m+1}, q)$ to the weight. Thus, the sum of $\omega(\pi)$ over all π in $\Pi_{m+n}(13/2, 123)$ with doubleton $\{m, m+1\}$ is the second term in the sum above.

If $\pi \in \Pi_{m+n}(13/2, 123)$ does not have $\{m, m+1\}$ as a block, then we can split π into a partition of $[m]$ and a partition of $[m+1, m+n]$. By a similar argument, the sum of $\omega(\pi)$ over all these π is $F_m(x, y, q)F_n(xq^m, yq^m, q)$. □

For the proof of the next theorem we will need shifted partitions. A partition $\pi \vdash [n]$ *shifted* by k positions, denoted π', consists of a block of k blank positions followed by the partition of $[k+1, k+n]$ obtained by adding k to every element of π. For example, shifting $\pi = 134/25$ by 2 positions gives $\pi' = \_\,\_/356/47$.

Let $\Pi_{n,k}(13/2, 123)$ be the set of partitions $\pi \in \Pi_n(13/2, 123)$ shifted k positions. Notice that the contribution of each block of π' to rb is the contribution of the corresponding block of π increased by k. That is

$$\sum_{\pi \in \Pi_{n,k}(13/2,123)} \omega(\pi) = F_n(xq^k, yq^k, q).$$

Theorem 4.4 *For all $m, n \geq 1$,*

$$F_{m+1}(x, y, q)F_{n+1}(xq^m, yq^m, q) = xq^m F_{m+n+1}(x, y, q) + y^2q^{2m-1}F_{m-1}(x, y, q)F_{n-1}(xq^{m+2}, yq^{m+2}, q).$$

Proof: The left-hand side is the generating function for all pairs $(\pi_1, \pi_2) \in \Pi_{m+1}(13/2, 123) \times \Pi_{n+1,m}(13/2, 123)$. The pair (π_1, π_2) takes one of two forms. Either π_1 ends in a doubleton and π_2 begins with a doubleton or not.

Suppose π_1 ends in a doubleton and π_2 begins with a doubleton. These doubletons contribute y^2q^{2m-1} to $\omega(\pi_1)\omega(\pi_2)$. Dropping these doubletons gives a pair $(\pi_1', \pi_2') \in \Pi_{m-1}(13/2, 123) \times \Pi_{n-1,m+2}(13/2, 123)$. Thus, summing the weights of these pairs gives the second term in the sum above.

Suppose π_1 ends in a singleton, π_2 begins with a singleton, or both. Such a singleton contributes xq^m to $\omega(\pi_1)\omega(\pi_2)$. Removing one singleton (in the case of two singletons, it does not matter which) and concatenating π_1 and π_2 gives a partition $\pi \in \Pi_{m+n+1}(13/2, 123)$. Each $\pi \in \Pi_{m+n+1}(13/2, 123)$ has $m+1$ in a block with $m+2$, in a block with m, or in its own block, corresponding to just π_1 ending in a singleton, just π_2 beginning with a singleton, or both. So every $\pi \in \Pi_{m+n+1}(13/2, 123)$ can be constructed as above. Thus these contribute weight $xq^m F_{m+n+1}(x, y, q)$. □

Theorem 4.5 *For all $n \geq 0$,*

$$F_n(x,y,q)F_{n+1}(x,y,q) = \sum_{j=0}^{n} xy^j q^{\left\lfloor \frac{j^2}{2} \right\rfloor} F_{n-j}(xq^j, yq^j, q) F_{n-j}(xq^{j+1}, yq^{j+1}, q).$$

Proof: Consider a pair

$$(\pi_1, \pi_2) \in \Pi_n(13/2, 123) \times \Pi_{n+1}(13/2, 123)$$

with $\pi_1 = A_1/A_2/\dots/A_\ell$, and $\pi_2 = B_1/B_2/\dots/B_m$. Search through the blocks in the order $B_1, A_1, B_2, A_2, \dots$ and find the first singleton block. Such a block must exists since either n or $n+1$ is odd.

If the first singleton is some $A_i = \{j\}$ then $B_1, \dots, B_i$ are all doubletons, and j is odd. There are $(j-1)/2$ doubletons at the beginning of π_1 and $(j+1)/2$ doubletons at the beginning of π_2 contributing $y^j q^{(j-1)^2/2}$ to the weight. The singleton block A_i has weight xq^{j-1}. The remaining $\ell - i$ blocks of π_1 form a layered matching of $[j+1, n]$ providing a contribution of $F_{n-j}(xq^j, yq^j, q)$. The remaining $m-i$ blocks of π_2 are layered matching of $[j+2, n+1]$ contributing $F_{n-j}(xq^{j+1}, yq^{j+1}, q)$. So, the weight contributed by all pairs (π_1, π_2) with $A_i = \{j\}$ as the first singleton is

$$xy^j q^{\left\lfloor \frac{j^2}{2} \right\rfloor} F_{n-j}(xq^j, yq^j, q) F_{n-j}(xq^{j+1}, yq^{j+1}, q).$$

If the first singleton is some $B_i = \{j+1\}$ then j is even and, by similar arguments, the weight contributed by all such pairs is exactly the same as the one displayed above. Summing over both even and odd j gives the desired identity. $\square$

The identity

$$F_n = \sum_{k \geq 0} \binom{n-k}{k}$$

relates the Fibonacci numbers to the binomial coefficients, where $\binom{n}{k} = 0$ if $k > n$. To state a q-analogue of this identity, we define the *q-binomial coefficients* to be

$$\left[\begin{array}{c} n \\ k \end{array} \right] = \prod_{i=1}^{k} \frac{q^{n-i+1} - 1}{q^i - 1},$$

where, by analogy with binomial coefficients, $\left[\begin{array}{c} n \\ k \end{array} \right] = 0$ if $k > n$.

Carlitz [4] derived the following identity using algebraic and operator methods. We will provide an alternate proof using one of the standard combinatorial interpretations of the q-binomial coefficients. In particular, let $P_{k,l}$ denote the set of all integer partitions with at most l parts, each of size at most k. Then [18, Proposition 1.3.19]

$$\left[\begin{array}{c} n \\ k \end{array} \right] = \sum_{\lambda \in P_{k,n-k}} q^{|\lambda|}. \tag{2}$$

It will be convenient to represent each $\lambda \in P_{k,l}$ as a path p in the integer lattice $\mathbb{Z}^2$ from the origin to (k, l), where each step of p is one unit North (N) or one unit East (E). In this case, the region between p and the y-axis is just the Ferrers diagram of the corresponding λ. And the area of this region is $|\lambda|$.

Theorem 4.6 (Carlitz) *For all $n \geq 0$,*

$$F_n(x, y, q) = \sum_{k \geq 0} x^{n-2k} y^k q^{\binom{k}{2}+\binom{n-k}{2}} \begin{bmatrix} n-k \\ k \end{bmatrix}.$$

Proof: Let $\Pi_n^k(13/2, 123)$ be the set of $\pi \in \Pi_n(13/2, 123)$ with exactly k doubletons. Thus, we have $\Pi_n(13/2, 123) = \biguplus_{2k \leq n} \Pi_n^k(13/2, 123)$, where $\biguplus$ is the disjoint union. This implies that

$$F_n(x, y, q) = \sum_{k \geq 0} x^{n-2k} y^k \left(\sum_{\pi \in \Pi_n^k(13/2,123)} q^{rb(\pi)} \right).$$

So it suffices to show that

$$\sum_{\pi \in \Pi_n^k(13/2,123)} q^{rb(\pi)} = q^{\binom{k}{2}+\binom{n-k}{2}} \begin{bmatrix} n-k \\ k \end{bmatrix}. \tag{3}$$

By equation (2), we will be done if we can find a bijection $\Pi_n^k(13/2, 123) \to P_{k,n-2k}$ such that if $\pi \leftrightarrow \lambda$ then

$$rb(\pi) = |\lambda| + \binom{k}{2} + \binom{n-k}{2}. \tag{4}$$

Map $\pi = B_1/\ldots/B_{n-k}$ to the lattice path $p = s_1, \ldots, s_{n-k}$ where $s_i = N$ or E depending on whether B_i is a singleton or doubleton, respectively. It is easy to see that the corresponding λ is in $P_{k,n-2k}$ and that this is bijective.

As far as the weights, first consider the contribution to $rb(\pi)$ of those pairs (b, B_j) where $b = \min B_i$ for a doubleton B_i. If $|B_j| = 1$, then B_i and B_j contribute an E-step followed later by an N-step in p. Such pairs of steps are in bijection with squares of the Ferrers diagram, and thus such (b, B_j) account for the $|\lambda|$ term of (4). If, on the other hand, $|B_j| = 2$ then there are $\binom{k}{2}$ choices for the pair (b, B_j), giving the next term of our sum.

Finally we need to account for the pairs (b, B_j) where where b is not the minimum of a doubleton. But then b could come from any of the $n-k$ blocks, picking the only element if it is a singleton and the non-minimum if it is a doubleton. So there are $\binom{n-k}{2}$ ways to pick the pair, finishing the proof. $\square$

Theorem 4.7 *For all $n \geq 0$,*

$$F_{2n}(x, y, q) = \sum_{k=0}^{n} x^{n-k} y^k q^{\binom{n+k}{2}-nk} \begin{bmatrix} n \\ k \end{bmatrix} F_{n-k}(xq^{n+k}, yq^{n+k}, q).$$

Proof: Let Δ_k be the set of partitions $\pi \in \Pi_{2n}(13/2, 123)$, which begin with a partition of $[n+k]$ having exactly k doubletons. Then $\Pi_{2n}(13/2, 123)$ is the disjoint union of the Δ_k since the first n blocks of any layered matching of $[2n]$ must form a partition of the desired type. Now the same technique used to prove equation (3) yields

$$\sum_{\pi \in \Delta_k} \omega(\pi) = q^{\binom{n+k}{2} - nk} x^{n-k} y^k \begin{bmatrix} n \\ k \end{bmatrix} F_{n-k}(xq^{n+k}, yq^{n+k}, q).$$

Summing over all k completes the proof. □

We conclude this section by finding a q-analogue of the identity

$$2^n = F_{n+1} + \sum_{k=0}^{n-2} F_k 2^{n-2-k}.$$

We provide a proof for a q-analogue involving only $F_n(q)$, since we will need to consider blocks with more than 2 elements.

Theorem 4.8 *For all $n \geq 0$,*

$$\prod_{i=1}^{n}(1+q^i) = F_{n+1}(q) + \sum_{k=0}^{n-2} q^k F_k(q) \prod_{i=k+3}^{n}(1+q^i). \tag{5}$$

Proof: From Theorem 2.1 we have

$$\prod_{i=1}^{n}(1+q^i) = \sum_{\pi \in \Pi_{n+1}(13/2)} q^{rb(\pi)}.$$

So we need to show that the right-hand side of equation (5) also counts $\Pi_{n+1}(13/2)$ with respect to rb.

The first term on the right-hand side counts those $\pi \in \Pi_{n+1}(13/2)$ that are matchings. For any other π, suppose the first block of size 3 or larger has minimum element $k+1$. The first k elements form a layered matching of $[k]$, and are hence counted by $F_k(q)$. The block containing $k+1$ contributes q^k. And the remaining blocks contribute $\prod_{i=k+3}^{n}(1+q^i)$. □

5 Determinant Identities

In [7], Cigler proved a q-analogue of the Euler-Cassini identity,

$$F_n F_{n+m-1} - F_{n-1} F_{n+m} = (-1)^n F_{m-1}.$$

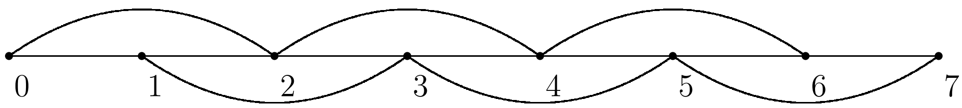
We state his theorem without proof since it will follow from our results later on in this section.

Theorem 5.1 (Cigler) *For all $n, m \geq 1$, the q-Fibonacci polynomials $F_n^C(x, y, q)$ satisfy*

$$F_n^C(x,y,q)F_{n+m-1}^C(x,yq,q) - F_{n-1}^C(x,yq,q)F_{n+m}^C(x,y,q) = (-y)^n q^{\binom{n+1}{2}} F_{m-1}^C(x, yq^{n+1}, q). \square$$

Cigler proves this identity twice, once by using determinants and once by adapting a bijective proof of Zeilberger and Werman [21]. We will prove a q-analogue of the Euler-Cassini identity for $F_n(x, y, q)$ by using weighted lattice paths and their relationship to minors of a Toeplitz-like matrix for the q-Fibonacci sequence. This is a method that appeared in a paper of Lindström [12], and which was later shown to have broad application by Gessel and Viennot [10]. We should note that Benjamin, Cameron, and Quinn [1] have recentlt used this technique to investigate determinants involving ordinary Fibonacci numbers.

Consider the digraph $D = (V, A)$ where the vertices are labeled $0, 1, 2, \ldots$, and the only arcs are from vertex n to vertex $n+1$ and from vertex n to vertex $n+2$ for all nonnegative integers n. The portion of this digraph consisting of the vertices $0, 1, 2, \ldots, 7$ is pictured below. All arcs are directed to the right.



It is easy to see that the number of directed paths from a to b in D is F_{b-a}. Let the arc from n to $n+1$, written $\vec{e}_{n,n+1}$, have weight $\omega(\vec{e}_{n,n+1}) = xq^n$. Let the arc from n to $n+2$ have weight $\omega(\vec{e}_{n,n+2}) = yq^n$. Let p be a directed path from a to b, written $a \xrightarrow{p} b$. We define the weight of p, $\omega(p)$, to be the product of the weights of its arcs. It follows easily from the definitions that

$$\sum_p \omega(p) = F_{b-a}(xq^a, yq^a, q),$$

where the sum is over all paths p from a to b.

Suppose that $\mathbf{u} : u_1 < u_2 < \ldots < u_k$ and $\mathbf{v} : v_1 < v_2 < \ldots < v_k$ are sequences of vertices in D. A *k-tuple* of paths from $\mathbf{u}$ to $\mathbf{v}$ is

$$P = \{u_1 \xrightarrow{p_1} v_{\alpha(1)}, u_2 \xrightarrow{p_2} v_{\alpha(2)}, \ldots, u_k \xrightarrow{p_k} v_{\alpha(k)}\} \tag{6}$$

where $\alpha \in S_k$, the symmetric group on k elements. We will let the weight of such a k-tuple be $\omega(P) = \prod_{i=1}^k \omega(p_i)$. Let $\mathrm{sgn}(P) = \mathrm{sgn}(\alpha)$, where sgn denotes sign.

Now consider the Toeplitz-like matrix

$$F = \begin{bmatrix} F_0(x,y,q) & F_1(x,y,q) & F_2(x,y,q) & F_3(x,y,q) & \cdots \\ 0 & F_0(xq,yq,q) & F_1(xq,yq,q) & F_2(xq,yq,q) & \cdots \\ 0 & 0 & F_0(xq^2,yq^2,q) & F_1(xq^2,yq^2,q) & \cdots \\ \vdots & \vdots & \vdots & \vdots & \ddots \end{bmatrix},$$

where we label the rows and columns starting with 0. Let $F_{\mathbf{u},\mathbf{v}}$ be the submatrix of F with rows and columns indexed by the sequences $\mathbf{u}$ and $\mathbf{v}$, respectively. Directly from the definitions we have that

$$\det F_{\mathbf{u},\mathbf{v}} = \sum_P \operatorname{sgn}(P)\omega(P) \tag{7}$$

where the sum is over all k-tuples of paths of the form (6).

But we can simplify this sum further. We will say that two paths are *noncrossing* if they do not share a vertex.

Theorem 5.2 *Let $\mathbf{u} : u_1 < u_2 < \ldots < u_k$ and $\mathbf{v} : v_1 < v_2 < \ldots < v_k$ be vertices in D. Then*

$$\det F_{\mathbf{u},\mathbf{v}} = \sum_P \operatorname{sgn}(P)\omega(P)$$

where the sum is over all non-crossing k-tuples of paths from $\mathbf{u}$ to $\mathbf{v}$.

Proof: We prove this by giving a weight-preserving, sign-reversing involution on the k-tuples of paths where at least one pair of paths cross. Let k-tuple P have a crossing pair of paths. Let p_i be the path with smallest index of any path which crosses another path. Let w be the first vertex shared by p_i and another path and let p_j be the path of smallest index $j > i$, that goes through w. Exchange the portions of p_i and p_j starting at w. This produces a new k-tuple of paths, Q, and it is easy to check that this is an involution. Since the weight of the k-tuple of paths is just the product of the weights of all arcs appearing in the k-tuple, the weight is preserved. Finally, the permutations for P and Q differ by a transposition, so $\operatorname{sgn}(P) = -\operatorname{sgn}(Q)$. $\Box$

We will now completely characterize the minors of F obtaining, along the way, a q-analogue of the Euler-Cassini identity. Given $\mathbf{u}$ and $\mathbf{v}$ and a vertex c, we call a k-tuple P of paths from $\mathbf{u}$ to $\mathbf{v}$ *reducible at c* if no path in P contains both a vertex less than c and a vertex greater than or equal to c. A family of k-tuples is *reducible at c* if each k-tuple is. If $u_1 < c \leq v_k$, then it is easy to see that the sum of the signed weights for a reducible family can be expressed as a product over two smaller families. So it suffices to consider path families which are not reducible for any c.

Now consider the case when $u_i = v_j$ for some i, j. In any k-tuple P, this forces p_i to be the path of length 0 starting and ending at $c = u_i = v_j$. The case of all k-tuples P which are reducible at c has already been covered. But if P is not reducible at c then, since paths can connect integers at most two apart, P contains exactly one path p beginning before c and ending after c. Furthermore, this path must contain vertices $c-1$ and $c+1$. By adding a new endpoint at $c-1$ and a new initial vertex at $c+1$, one obtains a bijection between all such P and a family of paths which is reducible at c. Thus, taking into account $\omega(\vec{e}_{c-1,c+1})$ and the sign change that occurs, we can determine the signed sum of the weights of the paths in this second case using a reducible family. Thus when $u_i = v_j$ we can compute the determinant using reducible families, and so we will assume from now on that $u_i \neq v_j$ for all i, j.

The next lemma severely limits the number of minors of F which can be nonzero. For a sequence of vertices $\mathbf{u}$ and a nonnegative integer c, define

$$\mathbf{u}(c) = \text{number of } u_i < c.$$

Lemma 5.3 *Suppose the sequences* $\mathbf{u}$ *and* $\mathbf{v}$ *consist of distinct vertices. If* $\det F_{\mathbf{u},\mathbf{v}} \neq 0$ *then we must have*

$$0 \leq \mathbf{v}(c) - \mathbf{u}(c) \leq 2 \tag{8}$$

for all $c \geq 0$.

Proof: We prove both inequalities by contradiction. Suppose first that $\mathbf{v}(c) - \mathbf{u}(c) < 0$. Then, in a corresponding k-tuple P, the number of paths ending before c is greater than the number of paths beginning in that interval. Clearly there is no such k-tuple and so $\det F_{\mathbf{u},\mathbf{v}} = 0$ by equation (7).

On the other hand, suppose $\mathbf{v}(c) - \mathbf{u}(c) \geq 3$. Then there must be at least three paths in P that contain both vertices less than c and vertices greater than or equal to c. Since adjacent vertices on a path are at most two apart as integers, it is impossible for these paths to be nonintersecting. So $\det F_{\mathbf{u},\mathbf{v}} = 0$ by Theorem 5.2. $\square$

The first inequality in the lemma says that the sequence obtained by combining $\mathbf{u}$ and $\mathbf{v}$ is a *ballot sequence*. But the second inequality curtails the number of ballot sequences we need to consider. Also, if $\mathbf{v}(c) - \mathbf{u}(c) = 0$ for some c with $u_1 < c \leq v_k$, then any corresponding noncrossing k-tuple is reducible at c. Thus there is only one sequence satisfying the lemma which is also irreducible, namely

$$u_1 < u_2 < v_1 < u_3 < v_2 < u_4 < v_3 < \ldots < u_k < v_{k-1} < v_k. \tag{9}$$

So to complete our characterization of the minors of F, we need only consider these sequences.

Theorem 5.4 *Let* $\mathbf{u}$ *and* $\mathbf{v}$ *be as in (9). Then*

$$\begin{aligned}\det F_{\mathbf{u},\mathbf{v}} &= (-y)^{\sum_{i=1}^{k-1}[v_i - u_{i+1} + 1]} \; q^{\sum_{i=1}^{k-1}\left[\binom{v_i}{2} - \binom{u_{i+1}-1}{2}\right]} \\ &\cdot F_{u_2-u_1-1}(xq^{u_1}, yq^{u_1}, q)\; F_{v_k - v_{k-1}-1}(xq^{v_{k-1}+1}, yq^{v_{k-1}+1}, q) \prod_{i=1}^{k-2} F_{u_{i+2}-v_i-2}(xq^{v_i+1}, yq^{v_i+1}, q).\end{aligned}$$

Proof: Consider a nonintersecting k-tuple P counted by $\det F_{\mathbf{u},\mathbf{v}}$. Then p_1 starts at u_1 and must contain the point $u_2 - 1$ so as not to intersect p_2. This part of p_1 is counted by the factor $F_{u_2-u_1-1}(xq^{u_1}, yq^{u_1}, q)$. Between u_2 and v_1, the nonintersecting condition forces p_1 to go through exactly the points having the same parity as $u_2 - 1$ and p_2 to go through the others. One of the two paths then terminates at v_1 and the other goes from $v_1 - 1$ to $v_1 + 1$. So the contribution of these steps to the weight is

$$yq^{u_2-1}\; yq^{u_2} \;\cdots\; yq^{v_1-1} = y^{v_1-u_2+1} q^{\binom{v_1}{2} - \binom{u_2-1}{2}}. \tag{10}$$

Whichever path continues on from v_1+1 must then go through $u_3 - 1$ to avoid intersecting p_3, contributing $F_{u_3-v_1-2}(xq^{v_1+1}, yq^{v_1+1}, q)$ to the weight. Next, p_3 and this path alternate vertices between u_3 and v_2, giving a weight which is the same as that in equation (10) but with all the indices increased by one.

It is clear that this pattern continues, giving the rest of the terms of the product. The sign of P is derived in a similar manner, so we omit the proof. $\square$

Taking $k = 2$ and the sequences $\mathbf{u} : 0 < 1$ and $\mathbf{v} : n < n+m$ in the previous theorem immediately gives a q-analogue of the Euler-Cassini Identity.

Corollary 5.5 *For all $n, \geq 1$,*

$$F_n(x,y,q)F_{n+m-1}(xq,yq,q) - F_{n-1}(xq,yq,q)F_{n+m}(x,y,q) = (-y)^n q^{\binom{n}{2}} F_{m-1}(xq^{n+1}, yq^{n+1}, q).$$

Using this corollary and the identity

$$F_n(xq^a, yq^a, q) = q^{\binom{n}{2}+na} F_n^C(x, y/q^a, 1/q)$$

(which is an easy extension of Proposition 3.3) we obtain Ciglers' q-analogue in Theorem 5.1.

6 Other Analogues

The q-Fibonacci numbers that are the focus of this paper come from two statistics, which are equidistributed over the set $\Pi_n(13/2, 123)$. The next natural question is whether the bistatistic (ls, rb) also has nice properties when considered on the set $\Pi_n(13/2, 123)$. The answer is yes. Define

$$F_n(x,y,p,q) = \sum_{\pi \in \Pi_n(13/2,123)} x^{s(\pi)} y^{d(\pi)} p^{ls(\pi)} q^{rb(\pi)}.$$

We will also need the p,q-*binomial coefficient*,

$$\left[\begin{array}{c} n \\ k \end{array} \right]_{p,q} = \prod_{i=1}^{k} \frac{p^{n-i+1} - q^{n-i+1}}{p^i - q^i}.$$

Note that we have $F_0(x,y,p,q) = 1$, $F_1(x,y,p,q) = x$ and for $n \geq 2$

$$F_n(x,y,p,q) = xq^{n-1}F_{n-1}(xp, yp, p, q) + yq^{n-2}F_{n-2}(xp^2, yp^2, p, q).$$

All of the demonstrations for the formulas in Section 4 translate in a straightforward manner to the p,q case. So we will merely list these p,q-identities in the following table and leave the proofs to the reader. In this list, we let $F_n(x,y) = F_n(x,y,p,q)$, and $F_n(xp^a, yp^a)_{1,1}$ be $F_n(xp^a, yp^a, p, q)$ evaluated at $x = y = 1$.

In closing, we should note that there are other ways to obtain q-analogues of Fibonacci numbers which could be studied. Simion and Schmidt [17] discovered a restricted set of permutations which is counted by the Fibonacci numbers. There is also a restricted set of permutations naturally counted by F_{2n}, see the paper of West [22]. Given the plethora of permutation statistics, some of these sets should yield interesting q-analogues.

List of p, q-Fibonacci Identities

$$F_{n+2}(x,y) = x^{n+2}(pq)^{\binom{n+2}{2}} + \sum_{j=0}^{n} x^j y \left(\frac{q}{p}\right)^{\binom{j}{2}} p^{n(j+1)} q^j F_{n-j}(xq^{j+2}, yq^{j+2})$$

$$F_{2n+1}(x,y) = \sum_{j=0}^{n} xy^j p^{(2n-j)(j+1)-j} q^{j(j+1)} F_{2n-2j}(xq^{2j+1}, yq^{2j+1})$$

$$F_{2n}(x,y) = y^n (pq)^{n(n-1)} + \sum_{j=0}^{n-1} xy^j p^{(2n-j-1)(j+1)-j} q^{j(j+1)} F_{2n-2j-1}(xq^{2j+1}, yq^{2j+1})$$

$$F_{m+n}(x,y) = F_m(xp^n, yp^n) F_n(xq^m, yq^m) + \; yp^{n-1} q^{m-1} F_{m-1}(xp^{n+1}, yp^{n+1}) F_{n-1}(xq^{m+1}, yq^{m+1})$$

$$\begin{aligned} F_{m+1}(xp^n, yp^n) F_{n+1}(xq^m, yq^m) &= xp^n q^m F_{m+n+1}(x,y) \\ &+ y^2 p^{2n-1} q^{2m-1} F_{m-1}(xp^{n+2}, yp^{n+2}) F_{n-1}(xq^{m+2}, yq^{m+2}). \end{aligned}$$

$$F_n(x,y) F_{n+1}(x,y) = \sum_{j=0}^{n} xy^j p^{n(j+1)-j(j+3)/2} q^{\left\lfloor \frac{j^2}{2} \right\rfloor} F_{n-j}(xq^j, yq^j) F_{n-j}(xq^{j+1}, yq^{j+1}).$$

$$F_n(x,y) = \sum_{k \geq 0} x^{n-2k} y^k (pq)^{\binom{n}{2} - k(n-k)} \begin{bmatrix} n-k \\ k \end{bmatrix}_{p,q}.$$

$$F_{2n}(x,y) = \sum_{k=0}^{n} x^{n-k} y^k (pq)^{\binom{n+k}{2} - nk} \begin{bmatrix} n \\ k \end{bmatrix}_{p,q} F_{n-k}(xq^{n+k}, yq^{n+k}).$$

$$\prod_{i=1}^{n} (1 + p^{n-i+1} q^i) = F_{n+1}(x,y)_{1,1} + \sum_{k=0}^{n-2} q^k F_k(xp^{n-k+1}, yp^{n-k+1})_{1,1} \prod_{i=k+3}^{n} (1 + p^{n-i+1} q^i).$$

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