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SIGNED POSETS

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Abstract : When the möbius function μ of a poset P has values $+1, 0, -1$ only, a signed graph $S(P)$ is defined by taking P as its vertex set and the sign of edge $e=xy$ as $\mu(x,y)$. A signed graph S is called balanced if $V(S)$ can be partitioned into subsets so that every positive edge and no negative edge joins two vertices in the same subset, and the number of such subsets is either one or two. When the number of subsets is not specified, S is called clusterable. We characterize balanced signed posets and discuss clusterable ones.

1. Introduction : There are a number of graphs that have been associated with a partially ordered set. The most famous of these is the Hasse diagram but recently other interesting graphs related to partial orders have been studied, for example, the upper bound, lower bound, and double bound graphs of McMorris and Zaslavsky (1984) We will define and investigate a new graph which is related to the möbius function of a partially ordered set. In order to do so, we need some definitions.

A graph G consists of a finite set of points $V(G)$ called *vertices* together with a prescribed set of (unordered) pairs of points $E(G)$ called *edges*. By convention $|V(G)|=p$ and $|E(G)|=q$. A *signed graph* S is a graph together with a function $f: E(G)\to\{+1,-1\}$, i. e., each edge is considered positive or negative. We can also consider a signed graph as a complete graph K_p (where every pair of vertices is an edge) together with a function

$$g: E(K_p)\to\{+1, 0, -1\} \text{ defined by } g(e)=\begin{cases} f(e) \text{ if } e\in E(G) \\ 0 \text{ otherwise}\end{cases} \quad \dots \quad (1)$$

This viewpoint will be useful later.

Signed triangles were first studied by Heider (1946) in psychology, where $V(G)$ represented a set of people with positive and negative edges corresponding to friendly and unfriendly relations between the pair involved. Signed graphs (with more than three vertices) were independently discovered in (Harary 1953) and the structure theorems for "balance" were derived. Cartwright and Harary (1956) subsequently developed "balance theory" as a mathematical model for cognitive structures in psychology. Many contributions to the theory of signed graphs have since appeared, both in the mathematical and social sciences [(Harary 1982), (Hage and Harary 1983), (Harary and Lindström 1981), (Zaslavasky 1981)].

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There is another area of mathematics where functions taking the values ± 1 and 0 often appear. A finite *partially ordered set* (*poset*) P is a finite set together with an asymmetric, reflexive, transitive partial order $\leqslant$ on its elements. The *möbius function* of P is a map $\mu : P \times P \rightarrow Z$ (the integers) defined recursively for all $(x, y) \in P \times P$ by

$$\mu(x, y)= \begin{cases} 0 & \text{if } x \nleq y \\ +1 & \text{if } x=y \\ -\sum\limits_{x \leq z < y} \mu(x, z) & \text{otherwise} \end{cases} \qquad \dots \quad (2)$$

This is a natural generalization of the möbius function from number theory and has been used extensively in combinatorics [(Harary and Palmer, 1973), (Rota and Sagan 1980), (Stanley 1972)]. An introduction to the subject is given by Rota (1964). It turns out that many posets have möbius functions that take the values ± 1 and 0; these will be called *signed posets*. It is our purpose to show that there is a natural correspondence between signed posets and signed graphs which has many interesting properties.

2. The Correspondence. Given a signed poset P, construct a signed graph $S(P)$ by letting the vertices of $S(P)$ be the elements of P and by defining the function g of (1) as

$$g(x, y)= \begin{cases} 0 & \text{if } x=y \\ \mu(x, y) & \text{otherwise} \end{cases} \qquad \dots \quad (3)$$

for all $x, y \in P$; $g(x, x)$ is set equal to 0 to eliminate loops from $S(P)$. Hence there will be an edge between distinct vertices x, y in $S(P)$ whenever $\mu(x, y) \neq 0$ in P, and in that case the sign of the edge will be that of $\mu(x, y)$. With this definition we see that if $M(P)$ is the matrix of the möbius function in the incidence algebra of P and $A(S(P))$ is the adjacency matrix of $S(P)$ then

$$A(S(P))=M(P)+M(P)^{t}-2I \qquad \dots \quad (4)$$

where t denotes transposition and I is the $|P| \times |P|$ identity matrix.

Given x, y in an arbitrary poset P, x *covers* y if $x>y$ and there is no $z \in P$ with $x>z>y$. The *Hasse diagram* of P, $H(P)$, has as vertices the elements of P with an edge from x down to y whenever x covers y. The Hasse diagrams of several signed posets and the corresponding signed graphs are shown in Figure 1. The reason for so displaying the graphs $S(P)$ will be given in the next section. Notice that if xy is an edge in $H(p)$ then xy is a negative edge in $S(P)$ but not conversely. A criterion for the converse will be given in Theorem 1.

We now substantiate our claim that there are many posets that are signed and hence can be turned into signed graphs. The simplest nontrivial example occurs when P is *totally ordered*, *i. e.*, for all $x, y \in P$ either $x \leqslant y$ or $y \leqslant x$. In this case P is called a *chain* C_r $(r=|P|-1)$ of *length* r. One can instantly compute its möbius function ;

$$\mu(x, y) = \begin{cases} +1 & \text{if} \quad x=y \\ -1 & \text{if} \quad x \text{ covers } \quad y \\ 0 & \text{otherwise.} \end{cases}$$

We immediately observe that C_r is a signed poset and that $H(C_r)$ is isomorphic to $-S(C_r)$.

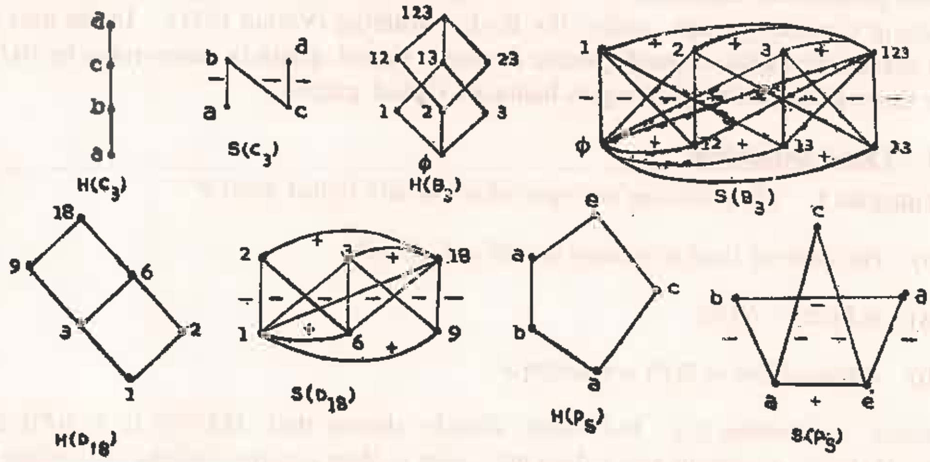

Figure 1. Some Hasse diagrams and their signed graphs

Given posets P, Q, the *cartesian product* $P \times Q = \{(x, y) \mid x \in P, y \in Q\}$ is a poset under the partial order $(x, y) \leqslant_{P\times Q} (u, v)$ whenever $x \leqslant_P u$ and $y \leqslant_Q v$. It is easy to show that

$$\mu_{P\times Q}((x, y), (u, v)) = \mu_P(x, u)\ \mu_Q\ (y, v) \qquad \dots \quad (6)$$

so any product of signed posets is a signed poset although $S(P \times Q)$ is not in general isomorphic to the graphical product $S(P) \times S(Q)$ of (Harary 1969, p. 22). In particular, any product of chains is a signed poset. Furthermore $\mu(x, y)$ depends only on the *interval* $[x, y] := \{z \in P \mid x \leqslant z \leqslant v\}$; hence any poset whose intervals are products of chains is a signed poset.

This last category includes a number of important posets :

(1) The *boolean algebras*, B_n, consisting of all subsets of an n-element set ordered by inclusion. Clearly $B_n \cong (C_1)^n$, and $S(B_n)$ can be viewed as a triangulation of the euclidean n-dimensional cube using the unique diagonal in each face all of whose direction cosines are non-negative ! If edge e is the diagonal of a face of dimension d then the sign of e is $(-1)^d$.

(2) The *divisor lattices*, D_n, of all divisors of a positive integer n with the partial order $d_1 \leqslant d_2$ whenever $d_1 \mid d_2$. If $n = \prod_{i=1}^{k} p_i^{\alpha_i}$ is the prime factorization of n, then $D_n \cong \prod_{i=1}^{k} C_{\alpha_i}$ and $S(D_n)$ is a triangulation of the $\prod_{i=1}^{k} \alpha_i$ k-cubes contained in a k-rectangle.

(3) *Trees* which are those posets whose Hasse diagram is a tree in the graph theoretical sense.

Other signed posets include the families of hooklength posets (Sagan 1979), posets associated with polynomial sequences of binomial type (Joni, Rota and Sagan 1981), (Sagan 1984) and posets of Coxeter groups under the Bruhat ordering (Verma 1971). In the next section we shall characterize those signed posets P whose signed graph is isomorphic to $H(P)$ itself and also those posets corresponding to balanced signed graphs.

3. Characterizations.

THEOREM 1. *The following are equivalent for any signed poset P*

(i) *The interval* $[x,y]$ *is a chain for all* $x \leqslant y$ *in* P,

(ii) $S(P) \cong -H(P)$,

(iii) *All the edges of* $S(P)$ *are negative.*

Proof. (i) implies (ii). We have already shown that $E(H(P)) \subseteq E(S(P))$ for any poset P. If $[x,y]$ is a chain and x does not cover y, then $\mu(x,y)=0$ so the only edges in $S(P)$ are those where x covers y.

(ii) implies (iii) at once.

(iii) implies (i). If the interval $[x,y]$ is not a chain, pick a minimal non-empty subinterval $[u,v] \subset [x,y]$ such that $[u,v]$ isn't a chain. Then there must be $n \geqslant 2$ chains from u to v. Also all the chains are disjoint except at u and v since $[u,v]$ is minimal. Thus we have $\mu(u,v)=n-1 \geqslant 1$, a contradiction. □

Theorem 1 raises the question : Which signed graphs having all negative edges come from a poset P ? As such a graph would be isomorphic to $-H(P)$, we have the next statement immediately.

COROLLARY 2. *Let* S *be a signed graph with only negative edges. If* $S=S(P)$ *for some poset* P, *then* S *has no triangles.* □

Although the condition that S be triangle free is necessary for a negative signed graph to arise from a poset, it is not sufficient. The falsity of the converse to Corollary 2 is shown by the following counterexample, pointed out to us by L. Babai.

Example. If S is a negative signed graph with $S=S(P)$ for some poset P then we can orient its edges so that (x,y) is an arc whenever y covers x. In view of Theorem 1 (i) and the asymmetry law for posets, there can be at most one directed path between any two vertices. However, we will produce a triangle free graph S such that any orientation of the graph contains the subgraph

Let n be a positive integer and set $[n] := \{1,2, \ldots, n\}$. The vertex set of S will be $V = [n] \times [5]$ and the edges will be of the form

$$(x,i)\,(y,i+1) \text{ where } x,y \in [n],\ i \in [5]$$

and addition is taken mod 5. This graph is called the lexicographic product of C_5 by $\bar{K}_n$; and clearly has no triangles.

Orient S arbitrarily. For $i=1,2, \ldots, 5$ let $V_i = V \times \{i\}$ and denote by E_i the edges oriented from V_i to V_{i+1}. We may assume, possibly after relabeling, that both $|E_1|$ and $|E_2| \geqslant n^2/2$. For $i=1,2, \ldots, n$, let $A_i = \{v \in V_2 \mid ((i,1), v) \text{ is an arc}\}$ (see Figure 1a) ; hence

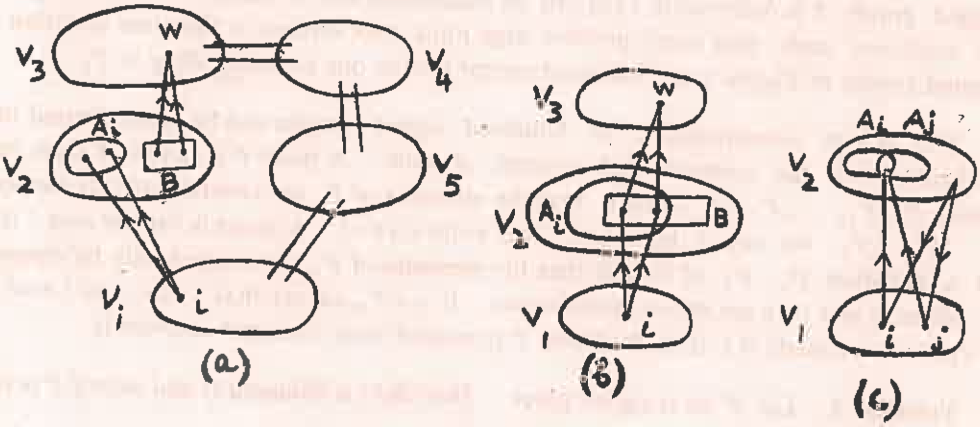

Figure 1

$$\sum_{i=1}^{n} |A_i| = |E_1| \geqslant n^2/2. \qquad \ldots \quad (7)$$

Also $|E_2| \geqslant n^2/2$ assures us that there is a $w \in V_3$ with at least $n/2$ edges directed towards it. So let $B = \{v \in V_2 \mid (v,w) \text{ is an arc}\}$ (see Figure 1a) ; hence

$$|B| \geqslant n/2. \qquad \ldots \quad (8)$$

Now assume there is no subgraph of the form

It follows that for all i

$$|A_i \cap B| \leqslant 1 \qquad \cdots \quad (9)$$

since otherwise we can have the situation of Figure 1b. Also for all $i \neq j$ we have

$$A_i \cap A_j = \phi \text{ or } A_i \cup A_j = V_2$$

since otherwise the situation of Figure 1c appears. However if $A_i \cup A_j = V_2$ for any pair i, j then (9) implies that $|B| \leqslant 2$ which contradicts (8) for $n \geqslant 5$. But if the A_i are pairwise disjoint then

$$\sum_{i=1}^{n} |A_i| \leqslant n, \text{ which contradicts (7) for } n \geqslant 2$$ □

Among all signed graphs, the family of balanced ones is particularly interesting. A signed graph S is *balanced* if $V(S)$ can be partitioned into at most two subsets C_1 and C_2 called *coalitions* such that every positive edge joins two vertices in the same coalition. All the signed graphs of Figure 1 are balanced except for the one corresponding to P_5.

The posets corresponding to balanced signed graphs can be characterized using a generalization of the conventional concept of rank. A poset P is *ranked* if there exists a partition $P_0, P_1, \cdots, P_r$ of P such that the elements of P_i are covered only by elements of P_{i+1}. If $x \in P_i$, we say x has *rank* i and write $r(x)=i$. A poset is *ranked mod 2* if there exists a partition P_0, P_1 of P such that the elements of P_i are covered only by elements of P_{i+1} where i and $i+1$ are taken modulo two. If $x \in P_i$ we say that x has *rank i mod 2* and write $r_2(x)=i$. Clearly if P is ranked then P is ranked mod 2 but not conversely.

THEOREM 3. *Let P be a signed poset. Then S(P) is balanced if and only if P is ranked mod 2, and for all x, y ∈ P*

$$\mu(x,y) = (-1)^{r_2(x)-r_2(y)} \text{ or } 0 .$$

Proof. This equivalence follows easily from the definitions above. □

More generally a signed graph S is *clusterable* if there is a partition $C_1, C_2, \ldots, C_n$ of $V(S)$ into coalitions such that, again, every positive edge joins two vertices in the same coalition. If n is the smallest number of possible sets in the partition then S is said to have *cluster number n*. The corresponding notion for posets, ranked mod n, is defined as expected. Note that a poset is ranked if and only if it is ranked mod n for all n. We have the next observation immediately.

THEOREM 4. *If P is a signed poset which is ranked mod n and*

$$\mu(x,y) = \begin{cases} +1 \text{ or } 0 \text{ if } x, y \text{ have the same rank} \\ -1 \text{ or } 0 \text{ if } x, y \text{ have different ranks} \end{cases}$$

then S(P) is clusterable with cluster number at most n. □

Unfortunately the converse is not true. For example, $S(P_8)$ is clusterable with cluster number 3 but P_8 is not ranked mod n for any n.

4. **Open problems.** We have constructed a function S from signed posets to signed graphs. There are many unanswered questions about this map.

(1) What is the image of $S(.)$, i.e., for which signed graphs S do we have $S(P)=S$ for some poset P? Note that if S is a negative graph which is both triangle-free and balanced then such a poset clearly exists

(2) For those graphs S in the image of $S(.)$, what is the number of posets P with $S(P)=S$? This is really four questions depending on whether the posets or the graphs are labeled or unlabeled. It is easy to see that

(a) if S is a negative signed labeled tree, then the number of labeled posets mapping to S is $2^{|E(S)|}$,

(b) if P^* is the dual of P then $S(P)=S(P^*)$; hence if $P \neq P^*$ then $|S(S^{-1}(P))| \geqslant 2$.

(3) The domain of $S(.)$ can be extended to all posets P if we permit the use of signed *multigraphs* (where there can be more than one edge between two vertices). Hence if $\mu(x,y)=m$ then there will be $|m|$ edges of the same sign as m between x and y in $S(P)$. All the questions asked above can now be reformulated in this context.

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