# A relative of the Thue-Morse Sequence

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**Abstract**

We study a sequence, $\mathbf{c}$, which encodes the lengths of blocks in the Thue-Morse sequence. In particular, we show that the generating function for $\mathbf{c}$ is a simple product.

Consider the sequence

$$\mathbf{c} : c_0, c_1, c_2, c_3, \ldots = 1, 3, 4, 5, 7, 9, 11, 12, 13, \ldots$$

defined to be the lexicographically least sequence of positive integers satisfying $n \in \mathbf{c}$ implies $2n \notin \mathbf{c}$. In fact, the lexicographic minimality of $\mathbf{c}$ makes it possible to replace the previous "implies" with "if and only if." Equivalently, $\mathbf{c}$ is defined inductively by $c_0 = 1$ and

$$c_{k+1} = \begin{cases} c_k + 1 & \text{if } (c_k+1)/2 \notin \mathbf{c} \\ c_k + 2 & \text{otherwise} \end{cases} \tag{1}$$

for $k \geq 0$. This sequence was the focus of a problem of C. Kimberling in the *American Mathematical Monthly* [6]. (In fact, he looked at the sequence $4c_0, 4c_1, 4c_2, \ldots$) The solution was given by D. Bloom [4]. Our Corollary 7 answers essentially the same question. Related results have recently been announced by J. Tamura [9].

At the 4è Colloque Séries Formelles et Combinatoire Algébrique (Montréal, June 1992) S. Plouffe and P. Zimmermann [8] posed the following problem. Show that the generating function for $\mathbf{c}$ is

$$\sum_{k\geq 0} c_k x^k = \frac{1}{1-x} \prod_{j\geq 1} \frac{1-x^{2e_j}}{1-x^{e_j}} = \frac{1}{1-x} \prod_{j\geq 1} (1 + x^{e_j}) \tag{2}$$

the sequence of exponents being

$$\mathbf{e} : e_1, e_2, e_3, e_4, \ldots = 1, 1, 3, 5, 11, 21, 43, \ldots$$

where $e_1 = 1$ and

$$e_{j+1} = \begin{cases} 2e_j + 1 & \text{if } j \text{ is even} \\ 2e_j - 1 & \text{if } j \text{ is odd} \end{cases} \tag{3}$$

for $j \geq 1$. They found this conjecture by using a method that goes back to Euler. First they assumed that the generating function was of the form

$$\prod_{j\geq 0} \frac{1-x^{a_j}}{1-x^{b_j}}$$

for a certain pair of sequences $a_j, b_j$. Then they took the logarithm to convert the product into a sum. Finally they used Möbius inversion to determine the candidate sequences. Details of this procedure can be found in the text of G. Andrews [2, Theorem 10.3].

The purpose of this note is to prove (2). Before doing this, however, we will show that $\mathbf{c}$ has a number of other interesting properties. Chief among these is the fact that $\mathbf{c}$ is closely related to the famous Thue-Morse sequence, $\mathbf{t}$. See the survey article of J. Berstel [3] for more information about $\mathbf{t}$.

First we need to have a characterization of the integers in the sequence $\mathbf{c}$.

**Proposition 1** *If $n$ is any positive integer then $n \in \mathbf{c}$ if and only if $n = 2^{2i}(2j+1)$ for some nonnegative integers $i$ and $j$.*

**Proof.** Every positive integer $n$ can be uniquely written in the form $n = 2^k(2j+1)$ where $k, j \geq 0$. We will proceed by induction on $k$.

If $k = 0$, then $n$ is odd. But then $n/2$ is not an integer, and so $n$ is in the sequence by definition (1).

Now assume that $k \geq 1$ and that the proposition holds for all powers less than $k$ of 2. If $k = 2i$ is even, then by induction we have $2^{2i-1}(2j+1) \notin \mathbf{c}$. So $n = 2^{2i}(2j+1) \in \mathbf{c}$ by (1). On the other hand, if $k = 2i+1$ is odd, then induction implies $2^{2i}(2j+1) \in \mathbf{c}$. Thus $n = 2^{2i+1}(2j+1) \notin \mathbf{c}$ as desired. ∎

Let $\chi$ be the characteristic function of $\mathbf{c}$, i.e.,

$$\chi(n) = \begin{cases} 1 & \text{if } n \in \mathbf{c} \\ 0 & \text{otherwise.} \end{cases}$$

Restating the previous proposition in terms of $\chi$ yields the next result.

**Lemma 2** *The function $\chi$ is uniquely determined by the equations*

$$\begin{aligned} \chi(2n+1) &= 1 \\ \chi(4n+2) &= 0 \\ \chi(4n) &= \chi(n). \quad \blacksquare \end{aligned}$$

Another way of obtaining the sequence $\chi(n)$ for $n \geq 1$ is as follows. Starting from the sequence

$$101 \bullet 101 \bullet 101 \bullet 101 \bullet \ldots$$

defined on the alphabet $\{0, 1, \bullet\}$, fill in the sucessive holes with the sucessive terms of the sequence itself, obtaining:

$$101110101011101 \bullet \ldots$$

Iterating this process infinitely many times (by inserting the initial sequence into the holes at each step), one gets a "Toeplitz transform" which is nothing but our sequence $\chi$. The proof of this fact is easily obtained using Lemma 2. See the article of J.-P. Allouche and R. Bacher [1] for more information about Toeplitz transformations.

The connection with the Thue-Morse sequence can now be obtained. This sequence is

$$\mathbf{t} : t_0, t_1, t_2, t_3, \ldots = 0, 1, 1, 0, 1, 0, 0, 1, \ldots$$

defined by the conditions

$$\begin{aligned} t_0 &= 0 \\ t_{2n+1} &\equiv t_n + 1 \pmod 2 \\ t_{2n} &= t_n. \end{aligned}$$

We will need a lemma relating $\mathbf{t}$ and $\chi$. All congruences in this and any future results will be modulo 2.

**Lemma 3** *For every positive integer, $n$, we have*

$$\chi(n) \equiv t_n + t_{n-1}.$$

**Proof.** This is a three case induction based on Lemma 2 and the definitions of $\chi$ and $\mathbf{t}$. We will only do one of the cases as the others are similar.

$$\begin{aligned} t_{4n} + t_{4n-1} &\equiv t_{2n} + t_{2n-1} + 1 \\ &\equiv t_n + t_{n-1} + 2 \\ &\equiv \chi(n) \\ &= \chi(4n). \quad \blacksquare \end{aligned}$$

Define $d_k$ to be the first difference sequence of $c_k$, i.e., $d_k = c_k - c_{k-1}$, for $k \geq 0$ $(c_{-1} = 0)$. So $\mathbf{d}$ is the sequence

$$d_0, d_1, d_2, d_3, d_4, \ldots = 1, 2, 1, 1, 2, 2, 2, 1, 1, 2, 1, \ldots$$

Note that from the definition of $\mathbf{c}$ in (1), the value of $d_k$ is either 1 or 2. Write the Thue-Morse sequence in term of its blocks

$$\mathbf{t} = 011010011 \ldots = 0^{d'_0} 1^{d'_1} 0^{d'_2} 1^{d'_3} \ldots$$

defining a sequence $d'_k$. It is this sequence that is related to our original one via the difference operator.

**Theorem 4** *For all $k \geq 0$ we have $d_k = d'_k$.*

**Proof.** Since both sequences consist of 1's and 2's, we need only verify that the 1's appear in the same places in both. It will be convenient to let $c'_k = \sum_{i \leq k} d'_i$. We now proceed by induction on $k$, assuming that $d_i = d'_i$ for $i \leq k$. Then, from the definitions,

$$d_{k+1} = 1 \Leftrightarrow \chi(c_k + 1). \tag{4}$$

But by the induction hypothesis, $c_k = \sum_{i\leq k} d_i = \sum_{i\leq k} d'_i = c'_k$. So, from equation (4),

$$\begin{aligned} d_{k+1} = 1 &\Leftrightarrow \chi(c'_k + 1) = 1 \\ &\Leftrightarrow t_{c'_k+1} + t_{c'_k} \equiv 1 \quad \text{(Lemma 3)} \\ &\Leftrightarrow t_{c'_k+1} \neq t_{c'_k} \\ &\Leftrightarrow d'_{k+1} = 1 \qquad\quad \text{(definitions)}. \quad \blacksquare \end{aligned}$$

S. Brlek [5] used the sequence **d** in calculating the number of factors of **t** of given length. The paper of A. de Luca and S. Varricchio [7] attacks the same problem in a different way.

Now if $n \in \mathbf{c}$ then we will consider its *rank*, $r(n)$, which is the function satisfying $c_{r(n)} = n$. Note that $r(n)$ is not defined for all positive integers $n$. In order to obtain a formula for $r(n)$, we will need a definition. Let the base 2 expansion of $n$ be

$$n = \sum_{i\geq 0} \epsilon_i 2^i$$

with the $\epsilon_i \in \{0, 1\}$ for all $i$. Define a function $s$ by

$$s(n) = \sum_{i\geq 0} (-1)^i \epsilon_i.$$

In other words, $s(n)$ is the alternating sum of the binary digits of $n$.

**Theorem 5** *If* $n \in \mathbf{c}$ *then*

$$r(n) = (2n + s(n))/3 - 1. \tag{5}$$

**Proof.** The proof will be by induction. From Proposition 1, $n \in \mathbf{c}$ if and only if $n$ is odd or $n = 2^{2i}(2j+1)$ where $i > 0$ and $j \geq 0$. To facilitate the induction, it will be convenient to split the odd numbers into two groups depending upon whether the highest power of 2 dividing $n+1$ is even or odd. So there will be three cases

1. $n = 2^{2i}(2j + 1)$
2. $n = 2^{2i}(2j + 1) - 1$
3. $n = 2^{2i-1}(2j + 1) - 1$

where $i > 0$ and $j \geq 0$. The arguments are similar, so we will only do the first case.

So suppose $n$ is even (remember that $i > 0$). Thus $n+1$ is odd and, by Proposition 1, we have $n+1 \in \mathbf{c}$. Since both $n$ and $n+1$ are in $\mathbf{c}$, the left side of equation (5) satisfies $r(n+1) = r(n)+1$. So, by induction, it suffices to show that $r'(n+1) = r'(n)+1$ where $r'(n)$ is the right side of this equation. Moreover, $n$ is a multiple of 4, hence $s(n+1) = s(n)+1$ (write down their binary expansions). Thus

$$\begin{aligned} r'(n+1) &= (2n+2+s(n+1))/3-1 \\ &= (2n+2+s(n)+1)/3-1 \\ &= (2n+s(n))/3 \\ &= r'(n)+1. \quad \blacksquare \end{aligned}$$

As straightforward corollaries we have the next two results.

**Corollary 6** *If $n \in \mathbf{c}$ then*

$$r(n) = 2n/3 + O(\log\ n)$$

*and $r(n)$ takes the value $2n/3$ infinitely often.* $\blacksquare$

**Corollary 7** *For any nonnegative integer $k$*

$$c_k = 3k/2 + O(\log\ k)$$

*and $c_k = 3k/2$ infinitely often.* $\blacksquare$

We shall now prove the identity (2). First we note a property of the exponents $e_j$ which is a simple consequence of their definition (3).

**Lemma 8** *For $k \geq 2$, let $f_k = \sum_{2 \leq j \leq k} e_j$. Then*

$$f_k = \begin{cases} e_{k+1} - 2 & \text{if } k \text{ is even} \\ e_{k+1} - 1 & \text{if } k \text{ is odd.} \quad \blacksquare \end{cases}$$

Finally, we come to the proof. We restate the generating function here for easy reference.

**Theorem 9** *The generating function for $\mathbf{c}$ is*

$$\sum_{k \geq 0} c_k x^k = \frac{1}{1-x} \prod_{j \geq 1} (1 + x^{e_j}).$$

**Proof.** It suffices to show that if $k \geq 2$ then

$$g_k(x) = \frac{1}{1-x}(1+x^1)(1+x^1)(1+x^3)\cdots(1+x^{e_k})$$

is the generating function for the sequence

$$1, 3, 4, 5, 7, \ldots, c_{f_k}, 2^k, 2^k, 2^k, \ldots$$

with $c_{f_k} = 2^k - 1$. The proof is an induction, breaking up into two parts depending on the parity of $k$. We will do the case where $k$ is odd. (Even $k$ is similar.) Now, by Lemma 8, $g_k(x)(1+x^{e_{k+1}})$ is the generating function for the sequence

$$1, 3, \ldots, c_{f_k}, 2^k+1, 2^k+3, \ldots, 2^k+c_{f_k}, 2^{k+1}, 2^{k+1}, \ldots$$

Using Proposition 1 and the fact that $k$ is odd, we see that $2^k+1 = c_{f_k+1}$ and $2^k + c_{f_k} = 2^{k+1} - 1 = c_{f_{k+1}}$. So we want to show that

$$c_{f_k+1}, c_{f_k+2}, \ldots, c_{f_{k+1}} = 2^k + c_0, 2^k + c_1, \ldots, 2^k + c_{f_k}.$$

But if $n < 2^k$, then the highest power of 2 dividing $n$ is equal to the highest power dividing $2^k + n$. Thus, by Proposition 1 again, $n \in \mathbf{c}$ if and only if $2^k + n \in \mathbf{c}$. This gives us the desired equality of the two sequences. ■

One possible generalization of $\mathbf{c}$ is the sequence $\mathbf{c}^{(\alpha)}$ defined by $n \in \mathbf{c}^{(\alpha)}$ if and only if $\alpha n \notin \mathbf{c}^{(\alpha)}$. Thus $\mathbf{c}$ is the special case $\alpha = 2$.

The following observation is a direct consequence of our definitions.

**Proposition 10** *If $\chi^{(\alpha)}(n)$ is the characteristic function of $\mathbf{c}^{(\alpha)}$, then the sequence $(\chi^{(\alpha)}(n))$ is the unique fixed point of the morphism*

$$\begin{aligned} 1 &\rightarrow 1^{\alpha-1}0 \\ 0 &\rightarrow 1^{\alpha-1}1 \end{aligned}$$

*which begins with 1.* ■

One can also see that $\mathbf{c}^{(\alpha)}$ satisfies analogs of many of our previous theorems. For example, if one defines $e_1^{(\alpha)} = 1$ and

$$e_{j+1}^{(\alpha)} = \begin{cases} \alpha e_j^{(\alpha)} + 1 & \text{if } j \text{ is even} \\ \alpha e_j^{(\alpha)} - 1 & \text{if } j \text{ is odd} \end{cases}$$

for $j \geq 1$, then the following result is a generalization of Theorem 9 and has an analogous proof.

**Theorem 11** *The generating function for $\mathbf{c}^{(\alpha)}$ is*

$$\frac{1}{1-x} \prod_{j \geq 1} \frac{1 - x^{\alpha e_j^{(\alpha)}}}{1 - x^{e_j^{(\alpha)}}}. \quad ■$$

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