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Revisiting Pattern Avoidance and Quasisymmetric Functions

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Abstract. Let $\mathfrak{S}_n$ be the nth symmetric group. Given a set of permutations Π, we denote by $\mathfrak{S}_n(\Pi)$ the set of permutations in $\mathfrak{S}_n$ which avoid Π in the sense of pattern avoidance. Consider the generating function $Q_n(\Pi) = \sum_\sigma F_{\operatorname{Des}\sigma}$ where the sum is over all $\sigma \in \mathfrak{S}_n(\Pi)$ and $F_{\operatorname{Des}\sigma}$ is the fundamental quasisymmetric function corresponding to the descent set of σ. Hamaker, Pawlowski, and Sagan introduced $Q_n(\Pi)$ and studied its properties, in particular, finding criteria for when this quasisymmetric function is symmetric or even Schur nonnegative for all $n \geq 0$. The purpose of this paper is to continue their investigation by answering some of their questions, proving one of their conjectures, as well as considering other natural questions about $Q_n(\Pi)$. In particular, we look at Π of small cardinality, superstandard hooks, partial shuffles, Knuth classes, and a stability property.

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1. Introduction

Let $\mathfrak{S}_n$ denote the symmetric group of all permutations $\pi = \pi_1\pi_2\ldots\pi_n$ of the set $[n] := \{1, 2, \ldots, n\}$. We sometimes insert commas between the elements of π or enclose them in parentheses to improve readability. We also use the notation $[m, n] = \{m, m+1, \ldots, n\}$. Given any sequence of distinct real numbers σ its *standardization*, $\operatorname{std}\sigma$, is the permutation obtained by replacing its smallest element by 1, its next smallest by 2, and so forth. We say that $\sigma \in \mathfrak{S}_n$ *contains* $\pi \in \mathfrak{S}_k$ *as a pattern* if there is some subsequence σ' of σ with $\operatorname{std}\sigma' = \pi$. If no such subsequence exists, then σ *avoids* π. For a set of permutations Π, we let

$$\mathfrak{S}_n(\Pi) = \{\sigma \in \mathfrak{S}_n \mid \sigma \text{ avoids every } \pi \in \Pi\}$$

and

$$\overline{\mathfrak{S}}_n(\Pi) = \mathfrak{S}_n - \mathfrak{S}_n(\Pi) = \{\sigma \in \mathfrak{S}_n \mid \sigma \text{ contains some } \pi \in \Pi\}.$$

We omit the set braces in Π if it contains only one permutation. For example, $\sigma = 25143$ contains $\pi = 132$, because $\operatorname{std} 254 = 132$, but $\sigma \in \mathfrak{S}_n(123)$, since σ contains no increasing subsequence with three elements. More information about pattern avoidance can be found in the book of Bóna [2].

Let $\mathbf{x} = \{x_1, x_2, \dots\}$ be a countably infinite set of variables. An element of the formal power series ring $\mathbb{R}[[\mathbf{x}]]$ is a *symmetric function* if it is of bounded degree and invariant under permutations of the variables. Bases for the vector space of symmetric functions homogeneous of degree n are indexed by integer partitions $\lambda = (\lambda_1, \lambda_2, \dots, \lambda_l)$ of n which we denote $\lambda \vdash n$. We use Greek letters near the middle of the alphabet to denote partitions and also use the multiplicity notation i^{k_i} if a part i of λ is repeated k_i times. In particular, we will be interested in the basis m_λ of monomial symmetric functions which is obtained by symmetrizing the monomial $x_1^{\lambda_1} x_2^{\lambda_2} \dots x_l^{\lambda_l}$, as well as the Schur functions s_λ about which we say more below. As an example

$$m_{(2,1)} = x_1^2 x_2 + x_2^2 x_1 + x_1^2 x_3 + x_3^2 x_1 + x_2^2 x_3 + x_3^2 x_2 + \cdots .$$

For information about symmetric functions as well as related material concerning Young tableaux and the Robinson–Schensted correspondence (which we use throughout), the reader can consult the texts of Sagan [9] or Stanley [11].

An element of $\mathbb{R}[[\mathbf{x}]]$ is *quasisymmetric* if it is invariant under bijections between subsets of the variables that preserve the order of the subscripts. The *algebra of quasisymmetric functions*, QSym, are those power series which are quasisymmetric and of bounded degree. They were first explicitly introduced by Gessel [5] and have since found many applications; see [10] or [11]. Bases for the vector space of quasisymmetric functions of degree n are indexed by compositions (ordered partitions) $\alpha = (\alpha_1, \alpha_2, \dots, \alpha_l)$ of n and we use the notation $\alpha \models n$ as well as multiplicity notation. To distinguish compositions from partitions, we use letters from the beginning of the Greek alphabet for compositions. The monomial quasisymmetric M_α function is formed by quasisymmetrizing the monomial $x_1^{\alpha_1} x_2^{\alpha_2} \dots x_l^{\alpha_l}$; for example

$$M_{(1,2)} = x_1 x_2^2 + x_1 x_3^2 + x_2 x_3^2 + \cdots .$$

Note that

$$m_\lambda = \sum_\alpha M_\alpha, \tag{1}$$

where the sum is over all compositions α obtained by rearranging the parts of λ.

There is another important basis for the quasisymmetric functions. To define it, note that there is a bijection between compositions $\alpha \models n$ and subsets $S \subseteq [n-1]$ given by

$$(\alpha_1, \alpha_2, \dots, \alpha_l) \mapsto \{\alpha_1,\ \alpha_1 + \alpha_2,\ \dots,\ \alpha_1 + \alpha_2 + \cdots + \alpha_{\ell-1}\}. \tag{2}$$

The *fundamental quasisymmetric function* associated with $S \subseteq [n-1]$ is

$$F_S = \sum x_{i_1} x_{i_2} \dots x_{i_n},$$

where the sum is over indices satisfying $i_1 \le i_2 \le \dots \le i_n$ and $i_j < i_{j+1}$ if $j \in S$. To illustrate, if $S = \{1\} \subseteq [2]$, then

$$F_S = x_1 x_2^2 + x_1 x_3^2 + x_2 x_3^2 + \dots + x_1 x_2 x_3 + x_1 x_2 x_4 + x_1 x_3 x_4 + x_2 x_3 x_4 + \cdots .$$

We also denote F_S by F_α if S corresponds to α under the bijection above. The expansion of a fundamental quasisymmetric function in terms of monomials is

$$F_\alpha = \sum_{\beta \le \alpha} M_\beta, \tag{3}$$

where $\beta \le \alpha$ means that β is a refinement of α. In the example above, we see that $F_{(1,2)} = M_{(1,2)} + M_{(1^3)}$.

We study certain quasisymmetric functions related to pattern avoidance which were introduced by Hamaker, Pawlowski, and Sagan [6]. Related work has been done by Adin and Roichman [1] and by Elizalde and Roichman [3,4]. A permutation $\sigma \in \mathfrak{S}_n$ has *descent set*

$$\operatorname{Des} \sigma = \{i \mid \sigma_i > \sigma_{i+1}\} \subseteq [n-1].$$

Given a set of permutations Π, define

$$Q_n(\Pi) = \sum_{\sigma \in \mathfrak{S}_n(\Pi)} F_{\operatorname{Des} \sigma}.$$

In [6], they found many interesting Π, such that for all n, the function $Q_n(\Pi)$ is symmetric. In that case, they were also often able to show that $Q_n(\pi)$ is *Schur nonnegative* in that the coefficients of its expansion in the Schur basis are nonnegative. Our main motivation for the present work is to answer some of the questions asked by Hamaker–Pawlowski-Sagan and to prove one of their conjectures.

Our work will be simplified by using certain symmetries of permutations. A permutation $\pi = \pi_1 \pi_2 \dots \pi_k$ has *reversal* $\pi^r = \pi_k \dots \pi_2 \pi_1$ and *complement* $\pi^c = k+1-\pi_1, k+1-\pi_2, \dots, k+1-\pi_k$. We apply these operations to sets of permutations by applying them to each individual element of the set. Also, given a partition $\lambda = (\lambda_1, \dots, \lambda_l)$, we use the notation $\lambda^t = (\lambda_1^t, \dots, \lambda_m^t)$ for its *transpose* given by reflecting the Young diagram for λ across the diagonal. We write our Young diagrams in English notation with the first row on the top. We also give coordinates to elements of a Young diagram as in a matrix.

Proposition 1.1. [6] *If $Q_n(\Pi)$ is symmetric, then so are $Q_n(\Pi^r)$ and $Q_n(\Pi^c)$. In particular, if $Q_n(\Pi) = \sum_\lambda c_\lambda s_\lambda$ for certain coefficients c_λ, then*

$$Q_n(\Pi^r) = Q_n(\Pi^c) = \sum_\lambda c_\lambda s_{\lambda^t}.$$

$\square$

We make heavy use of the following result of Gessel [5]: Suppose that P is a *standard Young tableau (SYT) of shape* $\lambda \vdash n$; that is, a filling of the Young diagram of λ with the numbers in $[n]$ so that rows and columns increase. We indicate that P has shape λ by writing $\operatorname{sh} P = \lambda$. Let

$$\mathrm{SYT}(\lambda) = \{P \mid \operatorname{sh} P = \lambda\}$$

and $f^\lambda = \#\,\mathrm{SYT}(\lambda)$ where the hash sign denotes cardinality. The *descent set* of P is

$$\operatorname{Des} P = \{i \mid i+1 \text{ appears in a lower row than } i \text{ in } P\}.$$

This notion permits one to expand the Schur functions in terms of the fundamental quasisymmetric functions.

Theorem 1.2. [5] *For any* λ, *we have*

$$s_\lambda = \sum_{Q \in \mathrm{SYT}(\lambda)} F_{\operatorname{Des} Q}.$$

□

Certain properties of the Robinson–Schensted correspondence will be crucial. We only review the ones which we need here and the interested reader can find more detail in [9,11]. The Robinson–Schensted map is a bijection

$$\mathrm{RS} : \mathfrak{S}_n \to \bigcup_{\lambda \vdash n} \mathrm{SYT}(\lambda) \times \mathrm{SYT}(\lambda).$$

If $\mathrm{RS}(\pi) = (P, Q)$, then we write $P = P(\pi)$ and $Q = Q(\pi)$ and call P and Q the P-*tableau* and Q-*tableau* of π, respectively. We need the following properties of the map RS.

Theorem 1.3. *Suppose that* $\mathrm{RS}(\pi) = (P, Q)$.

(a) $\operatorname{Des} \pi = \operatorname{Des} Q$.
(b) *If* $\operatorname{sh} P = \lambda$, *then* λ_1 *is the length of a longest increasing subsequence of* π.
(c) $P(\pi^r) = (P(\pi))^t$.
(d) $\mathrm{RS}(\pi^{-1}) = (Q, P)$. □

Call two permutations π, σ *Knuth equivalent* if $P(\pi) = P(\sigma)$. Given an SYT denote by $K(P)$ the Knuth equivalence class of all permutations with $P(\pi) = P$. Similarly, if λ is a partition, we let

$$K(\lambda) = \{\pi \mid \operatorname{sh} P(\pi) = \lambda\}.$$

The following result follows easily from Theorems 1.2 and 1.3(a).

Corollary 1.4. [6] *Suppose that* $P \in \mathrm{SYT}(\lambda)$.

(a) $\sum_{\pi \in K(P)} F_{\operatorname{Des} \pi} = s_\lambda$.
(b) $\sum_{\pi \in K(\lambda)} F_{\operatorname{Des} \pi} = f^\lambda s_\lambda$. □

The rest of this paper is structured as follows. In the next section, we determine which Π of cardinality $\#\Pi \leq 2$ have $Q_n(\Pi)$ symmetric for all n. In Section 3, we answer a question of Hamaker–Pawlowski–Sagan concerning the coefficients in the Schur expansion of $Q_n(K(P))$ where $K(P)$ is the Knuth class of a superstandard hook tableau P. Section 4 is devoted to proving a conjecture in [6] about $Q_n(\Pi)$ where Π is a certain variant of a shuffle set called a partial shuffle. In Section 5, we study Π, such that $\mathfrak{S}_n(\Pi)$ is a union of Knuth classes for all n (which implies that $Q_n(\Pi)$ is Schur nonnegative). In doing so, we provide a simpler proof of a theorem in [6] when $\Pi = K(P)$ for a single SYT P, and also answer a question asked by the authors about the case when Π is a union of two Knuth classes. We end with a section about stability results.

2. Pattern Sets of Small Size

In this section, we answer the question: for which $\Pi \subseteq \mathfrak{S}_k$ with $\#\Pi = 1$ or 2 is $Q_n(\Pi)$ symmetric for all n? It turns out that this occurs exactly when $\Pi \subseteq \{\iota_k, \delta_k\}$ where ι_k and δ_k are the increasing and decreasing permutations in $\mathfrak{S}_k$, respectively. We need the following result which follows easily from Theorem 1.3 (b) and (c) and Corollary 1.4 (b).

Lemma 2.1. [6] *For $n \geq 0$, we have*

$$Q_n(\emptyset) = \sum_{\lambda} f^\lambda s_\lambda,$$

$$Q_n(\iota_k) = \sum_{\lambda_1 < k} f^\lambda s_\lambda,$$

$$Q_n(\delta_k) = \sum_{\lambda_1^t < k} f^\lambda s_\lambda,$$

where all three sums are over $\lambda \vdash n$ together with any additional restriction noted in the summation. □

We now turn to the case $\#\Pi = 1$. We need the following fact about fundamental quasisymmetric functions. Although it is quite simple, we have been unable to find a reference in the literature.

Lemma 2.2. *If $\alpha \models k$, then F_α is symmetric if and only if $\alpha = (k)$ or $\alpha = 1^k$.*

Proof. The reverse implication is trivial. We prove the "only if" statement by contradiction, assuming that there is a third α with F_α symmetric. But then, there are two compositions β, γ which are both rearrangements of the partition $\lambda = (2, 1^{k-2})$, such that β refines α, but γ does not. Considering the expansion of F_α into monomial quasisymmetric functions in (3), we see from (1) that we have some but not all the terms which would be needed to give the monomial symmetric function m_λ. This contradiction completes the proof. □

It is now a short step to the first main result of this section.

Theorem 2.3. *Suppose $\#\Pi = 1$. Then, $Q_n(\Pi)$ is symmetric for all n if and only if $\Pi = \{\iota_k\}$ or $\Pi = \{\delta_k\}$ for some k.*

Proof. The reverse implication follows from Lemma 2.1. For the other direction, suppose, towards a contradiction, that $Q_n(\pi)$ is symmetric for all n where $\pi \in \mathfrak{S}_k$, but $\pi \neq \iota_k, \delta_k$. Since $\mathfrak{S}_k(\pi) = \mathfrak{S}_k - \{\pi\}$, we have

$$Q_k(\pi) = Q_k(\emptyset) - F_{\mathrm{Des}\,\pi}.$$

From our assumption and Lemma 2.1, we have $Q_k(\pi)$ and $Q_k(\emptyset)$ are symmetric, so the same must be true of $F_{\mathrm{Des}\,\pi}$ where $\mathrm{Des}\,\pi \subseteq [k-1]$. However, $\pi \neq \iota_k, \delta_k$, so we must have $\mathrm{Des}\,\pi \neq \emptyset, [k-1]$. Translating this statement from subsets of $[k-1]$ to compositions of k using (2) gives a contradiction to Lemma 2.2. $\square$

We now consider the case $\#\Pi = 2$. If $\Pi = \{\iota_k, \delta_l\}$, then, by Theorem 1.3 (b) and (c), $Q_n(\Pi)$ is the union of Knuth classes corresponding to all $\mathrm{SYT}(\lambda)$ where $\lambda_1 < k$ and $\lambda_1^t < l$. Applying Corollary 1.4 (b), we get the following result which we record for future reference.

Lemma 2.4. *If $\Pi = \{\iota_k, \delta_l\}$ for any $k, l \geq 1$, then $Q_n(\Pi)$ is symmetric and Schur nonnegative for all n.*

We now prove an analogue of Lemma 2.2 for sums of two fundamental quasisymmetrics. In it, we make use of the lattice C_k of compositions of k ordered by refinement. Therefore, $\alpha \leq \beta$ and other notations refer to this partial order. We also sometimes use the notation $M(\alpha)$ for M_α and eliminate the commas in α for readability.

Lemma 2.5. *Suppose that $k \geq 5$ and $\alpha, \beta \models k$ are distinct partitions. Then, $F_\alpha + F_\beta$ is symmetric if and only if $\{\alpha, \beta\} = \{(k), (1^k)\}$.*

Proof. The reverse implication is easy to see. Therefore, we concentrate on proving the forward direction.

Label the atoms of C_k as $\alpha_i = 1^{i-1}21^{k-i-1}$ for $1 \leq i < k$. Expanding $F_\alpha + F_\beta$ in terms of monomial quasisymmetrics, we see that $\sum M(\alpha_i) + \sum M(\alpha_j)$ is symmetric where the two sums are over the sets defined by

$$A = \{\alpha_i \mid \alpha_i \leq \alpha\} \quad \text{and} \quad B = \{\alpha_j \mid \alpha_j \leq \beta\}.$$

Symmetry and the fact that $\{\alpha, \beta\} \neq \{(k), (1^k)\}$ imply that A and B are disjoint, nonempty, and $A \uplus B = \{\alpha_1, \ldots, \alpha_{k-1}\}$.

Without loss of generality, we can assume that $\alpha_1 \in A$. Since $B \neq \emptyset$, there is an index i, such that $\alpha_i \in A$ and $\alpha_{i+1} \in B$. Let i be the minimum such index. If $i \geq 2$, then $\alpha_1, \alpha_2 \in A$ and so $M(\alpha_1 \vee \alpha_2) = M(31^{k-3})$ is in the expansion of F_α. However, $\alpha_i \in A$ and $\alpha_{i+1} \in B$ which implies that $M(\alpha_i \vee \alpha_{i+1}) = M(1^{i-1}31^{k-i-2})$ is not in the expansion of $F_\alpha + F_\beta$. It follows that this expansion is not symmetric which is a contradiction. Therefore, it must be that $i = 1$.

We have shown that $\alpha_1 \in A$ implies $\alpha_2 \in B$. Similarly, $\alpha_2 \in B$ implies $\alpha_3 \in A$, and so forth. It follows that

$$A = \{\alpha_1, \alpha_3, \ldots\} \quad \text{and} \quad B = \{\alpha_2, \alpha_4, \ldots\}.$$

Since $k \geq 5$, we have $\#A \geq 2$ and $\#B \geq 2$. Thus, $M(\alpha_1 \vee \alpha_3) = M(2^2 1^{k-4})$ is in the expansion of F_α. However, $M(\alpha_1 \vee \alpha_4) = M(2121^{k-5})$ is not in the expansion of $F_\alpha + F_\beta$. This final contradiction finishes the proof. □

We can now prove the second main result of this section.

Theorem 2.6. *Suppose $k \geq 4$ and $\Pi \subseteq \mathfrak{S}_k$ with $\#\Pi = 2$. Then, $Q_n(\Pi)$ is symmetric for all n if and only if $\Pi = \{\iota_k, \delta_k\}$.*

Proof. The backward direction follows from Lemma 2.4. For the forward direction, it is easy to check by computer that this is true for $k = 4$, so we assume that $k \geq 5$. There are now two cases depending on $|\Pi \cap \{\iota_k, \delta_k\}|$.

First, consider $\Pi = \{\pi, \delta_k\}$, where $\pi \in \mathfrak{S}_k - \{\iota_k\}$. The other possibility when $|\Pi \cap \{\iota_k, \delta_k\}| = 1$ is handled similarly. If $Q_k(\Pi)$ is symmetric, then so is $F_{\mathrm{Des}\,\pi} + F_{[k-1]} = F_{\mathrm{Des}\,\pi} + s_{1^k}$. It follows that $F_{\mathrm{Des}\,\pi}$ is symmetric. But then, $\pi \in \{\iota_k, \delta_k\}$ by Theorem 2.3, which contradicts our choice of Π.

Now, assume that $\Pi = \{\pi, \sigma\}$ with $\Pi \cap \{\iota_k, \delta_k\} = \emptyset$. As in the previous paragraph, the fact that $Q_k(\Pi)$ is symmetric implies that so is $F_\alpha + F_\beta$ where $\alpha = \mathrm{Des}\,\pi$ and $\beta = \mathrm{Des}\,\sigma$. This gives a contradiction to Lemma 2.5. □

We note that this result is not true when $k = 3$. For example, $\Pi = \{213, 231\}$ has $Q_n(\Pi)$ symmetric and Schur nonnegative for all n. We also conjecture, based on computer experiments, that things change when $\#\Pi \geq 3$.

Conjecture 2.7. *Given $p \geq 3$, there is a K which is a function of p, such that if $\#\Pi = p$ and $\Pi \subseteq \mathfrak{S}_k$ for $k \geq K$, then $Q_n(\Pi)$ cannot be symmetric for all n.*

3. Superstandard Hooks

An SYT P of shape λ is called *row superstandard* if it is obtained by filling the first row with the integers in $[1, \lambda_1]$, then the second row with the entries $[\lambda_1 + 1, \lambda_1 + \lambda_2]$, and so on. *Column superstandard* is defined analogously using the columns. *Superstandard* means either row or column superstandard. A *superstandard hook* is a superstandard tableau of hook shape. For us, these tableaux are interesting because of the following result.

Theorem 3.1. [6] *For any SYT P, we have $\mathfrak{S}_n(K(P))$ is a union of Knuth classes for all n if and only if P is a superstandard hook.* □

We now answer Question 5.13 from [6]. Specifically, we know from the previous result and Corollary 1.4 (a) that for a superstandard hook P, the generating function $Q_n(K(P))$ is Schur nonnegative. So what do the coefficients in its Schur expansion count? By Proposition 1.1, it suffices to consider the row superstandard case. Given positive integers r, s and some SYT P, then an (r, s)-*ascending sequence* in P is a sequence of its elements

$$p_{1,r} = p_{i_1,j_1} < p_{i_2,j_2} < \cdots < p_{i_s,j_s}, \tag{4}$$

such that $1 = i_1 < i_2 < \cdots < i_s$. We let

$\mathrm{Av}_n(r, s) = \{P \in \mathrm{SYT}(n) \mid P$ does not contain an (r, s)-ascending sequence$\}$.

Theorem 3.2. *Let R be the row superstandard tableau of shape $(r, 1^{s-1})$. Then*

$$Q_n(K(R)) = \sum_{\lambda} c_\lambda s_\lambda,$$

where

$$c_\lambda = \text{the number of } P \in \mathrm{Av}_n(r,s) \text{ of shape } \lambda.$$

Proof. By Corollary 1.4(a) and Theorem 3.1, it suffices to show that $P \in \mathrm{SYT}(n)$ contains an (r,s)-ascending sequence if and only if some permutation with insertion tableau P contains an element of $K(R)$ as a pattern. Note that

$$K(R) = [(1, 2, \ldots, r-1) \sqcup\!\sqcup (r+s-1, r+s-2, \ldots, r+1)] \cdot r,$$

where $\sqcup\!\sqcup$ denotes shuffle and the multiplication sign denotes concatenation.

First consider the forward direction and suppose P contains an (r,s)-ascending sequence as in (4). Consider the row word ρ of P, i.e., the word obtained from P by concatenating the rows of P from bottom to top, reading each row left to right. Because of the restriction on the first coordinates of the subscripts, we see that $p_{i_s,j_s} > \cdots > p_{i_1,j_1} = p_{1,r}$ is a subsequence of ρ. Furthermore, the elements $p_{1,1} < p_{1,2} < \cdots < p_{1,r-1}$ come in that order before $p_{1,r}$ in ρ. The union of these two subsequences standardizes to an element of $K(R)$ which is what we wished to prove.

For the converse, consider the column word κ of P. Therefore, $\kappa = C_1 C_2 \ldots C_t$ where C_j is the jth column of P read in decreasing order. Let π be a copy of some element of $K(R)$ in κ. Therefore, π contains a decreasing subsequence of length s, say $p_{i_1,j_1} > \cdots > p_{i_s,j_s}$. We claim that $i_k > i_{k+1}$ for all k. For assume to the contrary that $i_k \le i_{k+1}$. But also $j_k \le j_{k+1}$ by the column ordering in P. This forces $p_{i_k,j_k} < p_{i_{k+1},j_{k+1}}$ which is a contradiction. Furthermore, p_{i_s,j_s} must be the end of an increasing subsequence of π of length r. Since κ lists columns in decreasing order, this forces $j_s \ge r$. Since $p_{1,r}$ is the minimum element in the columns weakly right of column r, we have the sequence

$$p_{1,r} < p_{i_{s-1},j_{s-1}} < \cdots < p_{i_1,j_1},$$

which is the desired (r,s)-ascending sequence in P. $\square$

4. Partial Shuffles

The goal of this section is to prove a generalization of Conjecture 4.2 in [6]. We first establish some definitions. For any $a \le n$, define the corresponding *partial shuffle* as

$$(1, 2, \ldots, \widehat{a}, \ldots, n) \mathbin{\overline{\sqcup\!\sqcup}} (a) = [(1, 2, \ldots, \widehat{a}, \ldots, n) \sqcup\!\sqcup (a)] - \iota_n,$$

where $\sqcup\!\sqcup$ denotes the standard shuffle and $\widehat{\ }$ denotes deletion of the indicated element. For example, we have

$$(1, 2, 3, \widehat{4}, 5, 6) \mathbin{\overline{\sqcup\!\sqcup}} (4) = \{\mathbf{4}12356, 1\mathbf{4}2356, 12\mathbf{4}356, 1235\mathbf{4}6, 12356\mathbf{4}\}.$$

We also define the set of *fattened hooks* to be

$$H_{n,k} = \{\lambda \vdash n \mid \lambda_2 < k\},$$

As an example, if $k = 1$ then $H_{n,1}$ consists of the single partition (n). Also, $H_{n,2}$ is the set of ordinary hooks. We also set

$$\Pi(a, b) := (1, \ldots, \widehat{a}, \ldots, a + b) \,\overline{\sqcup}\, (a),$$

where $a + b \geq 2$ with $a \geq 1$ and $b \geq 0$. We now prove the following theorem where the case when $b = 1$ was first stated in [6] as Conjecture 4.2.

Theorem 4.1. *Fix nonnegative integers a, b. Then*

$$Q_n(\Pi(a + 2, b)) = \sum_{\lambda \in H_{n,a+b+1}} f^{\overline{\lambda}} s_\lambda,$$

where $\overline{\lambda}$ is λ with λ_1 replaced by $\min\{\lambda_1, a + b\}$.

The remainder of this section is devoted to proving this theorem. The main idea behind this proof is to establish descent-preserving bijections between "consecutive" sets $\Pi(a + 2, b)$ and $\Pi(a + 1, b + 1)$. Doing this then reduces the problem to proving that

$$Q_n(\Pi(a + b + 2, 0))$$

has the desired Schur function decomposition. To construct our bijections, we begin with some definitions and lemmas. Using these bijections, we end this section with a detailed proof of Theorem 4.1.

For any $\sigma \in \mathfrak{S}_n$ and $a > 0$, we say that j is an *a-start* or an *a-end* in σ provided that there exists an occurrence $\sigma_{i_1} \ldots \sigma_{i_a}$ of ι_a in σ where $i_1 = j$ or $i_a = j$, respectively. We further say j is a *maximal* a-start provided that j is an a-start but not an $(a + 1)$-start. Likewise, we say that j is a *minimal* a-end provided that j is an a-end but not an $(a + 1)$-end. To deal with the border cases that will arise, we define $n + 1$ to be a *maximal 0-start* and set $\sigma_{n+1} := n + 1$. Similarly, we define 0 to be a *minimal 0-end* and set $\sigma_0 := 0$. Representing permutations σ via their *permutation diagrams*, i.e., the points (i, σ_i) in the first quadrant of the plane, we see that j is a maximal a-start if and only if j is an a-start and there is no a-start i, such that σ_i is northeast of σ_j. We make use of this description going forward. Last, for $a, b \geq 0$ define m to be an *(a, b)-middle* in σ provided that there exists an occurrence $\sigma_{i_1} \ldots \sigma_{i_{a+b+1}}$ of ι_{a+b+1} in σ where $i_{a+1} = m$.

In our first lemma, we record a few basic observations regarding these definitions.

Lemma 4.2. *Let $\sigma \in \mathfrak{S}_n$ and fix $a, b \geq 0$. Let $s_1 < s_2 < \cdots < s_r$ and $e_1 < e_2 < \cdots < e_t$ be the sequences of maximal b-starts and minimal a-ends in σ, respectively. We have the following properties.*

(i) *The sequences $\sigma_{s_1}, \sigma_{s_2}, \ldots$ and $\sigma_{e_1}, \sigma_{e_2}, \ldots$ are decreasing subsequences in σ.*

(ii) *For every* (a,b)*-middle* m*, there exists some* i*, so that*

$$s_{i-1} < m < s_i \quad \textit{and} \quad \sigma_m < \sigma_{s_i}, \tag{5}$$

and some j*, so that*

$$e_j < m < e_{j+1} \quad \textit{and} \quad \sigma_{e_j} < \sigma_m, \tag{6}$$

where we take $s_0 = 0$ *and* $e_{t+1} = n+1$*.*

Proof. The proof of (i) follows immediately from the definitions. To prove (5), observe that any (a,b)-middle m is, in particular, a $(b+1)$-start. As such, there is some i, so that σ_{s_i} is northeast of σ_m. Moreover, by (i), it follows that if we choose s_i to be the smallest such index $> m$ we arrive at our claim. A similar proof justifying (6) is left to the reader. □

For our next lemma, we define an *increasing interval* in a permutation σ to be an occurrence $\sigma_{i_1}, \ldots, \sigma_{i_k}$ of ι_k, such that $\{\sigma_{i_1}, \ldots, \sigma_{i_k}\} = [\sigma_{i_1}, \sigma_{i_k}]$.

Lemma 4.3. *Fix* $a+b \geq 1$ *with* $a, b \geq 0$ *and some permutation* σ*. Let* $s_1 < s_2 < \cdots < s_r$ *and* $e_1 < e_2 < \cdots < e_t$ *be the sequences of maximal* b*-starts and minimal* a*-ends, respectively, in* σ*.*

(i) *We have* $\sigma \in \mathfrak{S}_n(\Pi(a+2,b))$ *if and only if for each* $1 \leq i \leq r$ *the* (a,b)*-middles between* s_{i-1} *and* s_i *correspond to an increasing interval with largest value* $\sigma_{s_i} - 1$*.*

(ii) *We have* $\sigma \in \mathfrak{S}_n(\Pi(a+1,b+1))$ *if and only if for each* $1 \leq i \leq t$ *the* (a,b)*-middles between* e_i *and* e_{i+1} *correspond to an increasing interval with smallest value* $\sigma_{e_i} + 1$*.*

Proof. To prove the forward direction of (i), assume $\sigma \in \mathfrak{S}_n(\Pi(a+2,b))$ and consider some (a,b)-middle, so that $s_{i-1} < m < s_i$. By (5), we know $\sigma_m < \sigma_{s_i}$. It now suffices to prove that for any index y with $\sigma_m < \sigma_y < \sigma_{s_i}$ we have $m < y < s_i$, since, in this case, y is clearly an (a,b)-middle. The desired implication follows, because, otherwise, we obtain an occurrence of some forbidden pattern.

To establish the reverse direction of our first claim, consider some occurrence

$$\sigma_{x_1} \ldots \sigma_{x_{a+1}} \sigma_{x_{a+2}} \ldots \sigma_{x_{a+b+1}}$$

of ι_{a+b+1} in σ. It suffices to show that for any index y, such that $\sigma_{x_{a+1}} < \sigma_y < \sigma_{x_{a+2}}$ we have $x_{a+1} < y < x_{a+2}$. To see this, first observe that x_{a+1} is an (a,b)-middle. By (5), this means that there exists some i, so that $s_{i-1} < x_{a+1} < s_i$ with $\sigma_{x_{a+1}} < \sigma_{s_i}$. Also noting that x_{a+2} is a b-start, it follows that there is some j, so that σ_{s_j} is (weakly) northeast of $\sigma_{x_{a+2}}$. Hence, by our choice of i, we must have $s_i \leq s_j$ and

$$\sigma_{x_{a+1}} < \sigma_{x_{a+2}} \leq \sigma_{s_j} \leq \sigma_{s_i},$$

where the last inequality follows from the last observation and (i) of Lemma 4.2. It now follows from our assumption about the (a,b)-middles forming an increasing interval that $x_{a+1} < y < x_{a+2}$ as desired.

The proof of our second claim is analogous and is left to the reader. □

Using this lemma, we prove the existence of our descent-preserving bijections. Let us first consider a motivating example. Let $a+b+2=7$ with $a=3$ and $b=2$, so that $\Pi(a+2,b) = 123467 \sqcup\!\sqcup\, 5$. Now, consider the following $\Pi(a+2,b)$ avoiding permutation

$$2\ 11\ 6\ 4\ \underline{10}\ 3\ 13\ \underline{5}\ 14\ 1\ 15\ \overline{16}\ 7\ 8\ \overline{9}\ 18\ 17\ 12,$$

where the underlined entries correspond to minimal 3-ends, the overlined entries correspond to maximal 2-starts, and the circled entries correspond to (a,b)-middles. As proved in Lemma 4.3, note that the circled entries to the left of 16 form an increasing interval, as do the circled entries between 16 and 9. To transform this permutation into one avoiding $\Pi(a+1,b+1)$, we simply "shift down" the circled entries according to which 3-ends (underlined entries) they are between. Using fractional values to (temporarily) avoid adjusting all values simultaneously, we get

$$2\ 11\ 6\ 4\ \underline{10}\ 3\ 10.1\ \underline{5}\ 5.1\ 1\ 5.2\ \overline{16}\ 5.3\ 5.4\ \overline{9}\ 18\ 17\ 12,$$

and finally, after standardizing, we arrive at

$$2\ 14\ 10\ 4\ \underline{12}\ 3\ 13\ \underline{5}\ 6\ 1\ 7\ \overline{16}\ 8\ 9\ \overline{11}\ 18\ 17\ 15,$$

which is $\Pi(a+1,b+1)$-avoiding.

Lemma 4.4. *Fix nonnegative integers a,b, so that $a+b \geq 1$. There exists a descent-preserving bijection*

$$\Phi : \mathfrak{S}_n(\Pi(a+2,b)) \to \mathfrak{S}_n(\Pi(a+1,b+1))$$

Proof. To define our function Φ, fix some $\sigma \in \mathfrak{S}_n(\Pi(a+2,b))$, and let $s_1 < s_2 < \cdots$ and $e_1 < e_2 < \cdots$ be the sequence of maximal b-starts and minimal a-ends, respectively, in σ. Now, let $m_1 < m_2 < \cdots$ be the sequence of (a,b)-middles.

It follows from Lemma 4.3 that values of σ indexed by (a,b)-middles between s_{i-1} and s_i form an increasing sequence whose largest value is $\sigma_{s_i}-1$. Now, let σ' be the result of replacing these "middle" entries in σ with place-holding zeros. As no value deleted from σ is northeast of some s_j, the maximal b-starts of σ' are the same as those in σ.

Construct σ'' from σ' by replacing the k place-holding zeros between e_i and e_{i+1} with the increasing sequence

$$\sigma_{e_i} + \frac{1}{k+1}, \sigma_{e_i} + \frac{2}{k+1}, \ldots, \sigma_{e_i} + \frac{k}{k+1}.$$

Finally, let $\Phi(\sigma)$ be the standardization of σ''. A straightforward check similar to what was done in the previous paragraph shows that the maximal b-starts, minimal a-ends, and (a,b)-middles are invariant under this construction. As such, it follows from Lemma 4.3 that $\Phi(\sigma)$ is $\Pi(a+1,b+1)$-avoiding and hence well defined. As this construction is clearly invertible, it follows that this function is also bijective.

It remains to show that Φ is descent preserving. To see this, consider some (a,b)-middle m with

$$e_j < m < e_{j+1} \quad \text{and} \quad \sigma_{e_j} < \sigma_m.$$

First, observe that σ_{m+1} cannot be between σ_{e_j} and σ_m as otherwise $m+1$ is an (a,b)-middle which combined with the placement of σ_m would create an occurrence of some pattern in $\Pi(a+2,b)$. Thus, $m \in \operatorname{Des}\sigma$ if and only if $m \in \operatorname{Des}\Phi(\sigma)$. Similarly, one can check that the presence of $m-1$ in the descent set is preserved in passing from σ to $\Phi(\sigma)$. Since only (a,b)-middles move when applying Φ, this completes the proof that the map is descent preserving. $\square$

Proof of Theorem 4.1. By Lemma 4.3, it suffices to prove that $Q_n(\Pi(a+b+2,0))$ has the desired Schur decomposition. Observe that $\Pi(a+b+2,0) = K(R)$ where R is the row superstandard tableau of shape $(a+b+1,1)$. By Theorem 3.2, we then see that

$$Q_n(\Pi(a+b+2,0)) = \sum_{\lambda} c_\lambda s_\lambda,$$

where c_λ is the number of $P \in \mathrm{Av}_n(a+b+1,2)$ of shape λ. Now, from the definitions, we have $P \in \mathrm{Av}_n(a+b+1,2)$ if and only if $\{p_{1,a+b+1}, p_{1,a+b+2}, \ldots, p_{1,n}\} = [p_{1,a+b+1}, n]$. It follows that $\lambda_2 < a+b+1$, since we would have to have $p_{2,a+b+1} > p_{1,a+b+1}$ and all the elements greater than $p_{1,a+b+1}$ are to its right. Thus, $\lambda \in H_{n,a+b+1}$. Furthermore, there is a bijection between such tableaux and those of shape $\overline{\lambda}$ obtained by removing the elements $[p_{1,a+b+1}, n]$. Thus, $c_\lambda = f^{\overline{\lambda}}$ finishing the proof. $\square$

5. Pattern-Knuth Closed Classes

We say that $\Pi \subseteq \mathfrak{S}$ is *pattern-Knuth closed* if $\mathfrak{S}_k(\Pi)$ is a union of Knuth classes for all k. Equivalently, $\overline{\mathfrak{S}}_k(\Pi)$ is a union of Knuth classes for all k. Note that if Π is pattern-Knuth closed, then $Q_k(\Pi)$ is Schur nonnegative for all k. This concept was introduced and studied by Hamaker, Pawloski, and Sagan [6]. In this section we continue the investigation of this topic and, in doing so, answer one of their questions.

If $\Pi \subseteq \mathfrak{S}_k$ is pattern-Knuth closed, then, in particular, Π must be a union of Knuth classes. In [6], the authors characterized which $\Pi = K(S)$ for a single SYT S are pattern-Knuth closed. It turns out that this happens precisely when S is a superstandard hook. In Theorem 5.11 below, we give an augmented version of their result and give a more conceptual proof. Our techniques are strong enough that in Theorem 5.13, we resolve the case where Π is a union of two Knuth classes which was left as Question 5.14 in [6]. We also discuss why these results do not seem to generalize to unions of more than two Knuth classes.

We first recall another useful characterization of a Knuth class. Consider positive integers $a < b < c$. A *Knuth move* in a permutation π consists of replacing a factor (adjacent subsequence) of the form acb with one of the form

cab, or vice-versa. One is also permitted to exchange factors of the form bac and bca.

Theorem 5.1. [7] *Two permutations are Knuth equivalent if and only if one can be transformed into the other by a series of Knuth moves.* □

We begin with an elementary property of pattern-Knuth closed sets.

Proposition 5.2. *If* Π *and* Π' *are pattern-Knuth closed, then so is* $\Pi \cup \Pi'$.

Proof. Observe that $\mathfrak{S}_n(\Pi \cup \Pi') = \mathfrak{S}_n(\Pi) \cap \mathfrak{S}_n(\Pi')$. As both $\mathfrak{S}_n(\Pi)$ and $\mathfrak{S}_n(\Pi')$ are unions of Knuth classes, it follows that there intersection must be as well. In other words, $\Pi \cup \Pi'$ is pattern-Knuth closed. □

Let ξ and ζ are (not necessarily disjoint) subsequences of a permutation σ. Define $\xi \cup \zeta$ to be the subsequence of σ whose elements consist of those of ξ together with those of ζ. We write $\xi \uplus \zeta$ if ξ and ζ are disjoint. The *shape* of a permutation, $\operatorname{sh}\sigma$, is the shape of its output tableaux under the Robinson–Schensted map. Finally, let $\operatorname{lis}\sigma$ (respectively, $\operatorname{lds}\sigma$) stand for the length of a longest increasing (respectively, decreasing) subsequence of σ.

Lemma 5.3. *Suppose that* $\sigma = \iota \uplus \delta$ *where* ι *is increasing of length* a *and* δ *is decreasing of length* b. *Then* $\operatorname{sh}\sigma$ *is one of the following:*

$$(a, 1^b),\ (a+1, 1^{b-1}),\ (a, 2, 1^{b-2}).$$

Proof. Because of the hypothesis, we have $\operatorname{lis}\sigma = a$ or $a+1$ and $\operatorname{lds}\sigma = b$ or $b+1$. We have three cases.

If $\operatorname{lds}\sigma = b+1$, then the first column of $\operatorname{sh}\sigma$ is of length $b+1$ by Theorem 1.3 (b) and (c). Furthermore, the length of the first row of $\operatorname{sh}\sigma$ is at least a. However, $\#\sigma = a+b$, so we must have $\operatorname{sh}\sigma = (a, 1^b)$. Similarly, if $\operatorname{lis}\sigma = a+1$, then this forces $\operatorname{sh}\sigma = (a+1, 1^{b-1})$. Finally, assume $\operatorname{lds}\sigma = b$ and $\operatorname{lis}\sigma = a$. Therefore, the first column of $\operatorname{sh}\sigma$ has b elements and the first row has a, giving a total of $a+b-1$ entries. Thus, the remaining entry must be in the $(2,2)$ box. □

We point out that the third case of this lemma can occur. In fact, the smallest example where this case is needed is when $\sigma = 65127843$. Here, σ contains a unique increasing sequence of length $a = 4$, namely 1278, and a unique decreasing sequence of length $b = 4$, namely 6543, and $\operatorname{sh}\sigma = (4, 2, 1, 1)$.

For any partition μ, recall that

$$K(\mu) = \bigcup_{T \in \mathrm{SYT}(\mu)} K(T)$$

and define

$$\mathfrak{S}_n(\mu) = \{\lambda \vdash n \mid \lambda \not\supseteq \mu\}.$$

Theorem 5.4. *Let* $\mu = (a, 1^b)$ *and* $\tau = (a, 2, 1^{b-1})$. *Then,* $\Pi = K(\mu) \cup K(\tau)$ *is pattern-Knuth closed and*

$$Q_n(\Pi) = \sum_{\lambda \in \mathfrak{S}_n(\mu)} f^\lambda s_\lambda.$$

Proof. To prove both statements, it suffices to show that for any permutation σ, we have $\sigma \in \overline{\mathfrak{S}}_n(\Pi)$ if and only if $\operatorname{sh}\sigma \supseteq (a, 1^b)$. For the forward direction, let κ be a copy of an element of either $K(\mu)$ or $K(\tau)$ in σ. Then, κ, and hence, σ contains an increasing subsequence of length a and a decreasing subsequence of length $b+1$. It follows that $\operatorname{sh}\sigma \supseteq (a, 1^b)$ as desired.

For the reverse, the assumption on $\operatorname{sh}\sigma$ means that σ has a subsequence $\kappa = \iota \cup \delta$ where ι is increasing of length a and δ is decreasing of length $b+1$. If the union is disjoint, then by the previous lemma, we must have $\operatorname{sh}\kappa$ is one of $(a, 1^{b+1})$, $(a+1, 1^b)$, or $(a, 2, 1^{b-1})$. In the third case, we are done, since σ contains an element of $K(\tau)$. If we are in one of the first two cases, then one can remove an element of δ or ι, respectively, to show that σ contains an element of $K(\mu)$. If the union is not disjoint, then ι and δ must overlap in precisely one element and an argument as in the proof of the lemma shows that $\operatorname{sh}\kappa = (1^a, b)$, finishing the proof. $\square$

We now begin our characterization of certain pattern-Knuth closed sets with some critical definitions. We call the descents of π^{-1} the *i-descents* of π and denote the set of all i-descents by $\operatorname{iDes}(\pi)$. (We note that some authors define this idea as *left descents.*) An equivalent definition of i-descents is the following: $a \in \operatorname{iDes}(\pi)$ if and only if $a+1$ is to the left of a in π. We say that a set of permutations $\Pi \subseteq \mathfrak{S}_n$ is *i-descent consistent* provided that $\operatorname{iDes}(\pi) = \operatorname{iDes}(\sigma)$ for all $\pi, \sigma \in \Pi$ and write $\operatorname{iDes}(\Pi)$ for this common set of i-descents. Equivalently, $\Pi \subseteq \mathfrak{S}_n$ is i-descent consistent if and only if $\Pi \subseteq D_J^{-1}$ for some $J \subseteq [n-1]$, where

$$D_J = \{\pi \in \mathfrak{S}_n \mid \operatorname{Des}(\pi) = J\}, \tag{7}$$

and we take the inverse of a set by taking the inverse of each of its elements. Recall that for any $S \in \operatorname{SYT}(n)$, we have $\operatorname{iDes}(\sigma) = \operatorname{Des}(S)$ for each $\sigma \in K(S)$ by Theorem 1.3 (a) and (d). Furthermore, for any $J \subseteq [n-1]$

$$\bigcup_S K(S) = D_J^{-1},$$

where the union is over all $S \in \operatorname{SYT}(n)$ where $\operatorname{Des}(S) = J$. Therefore, Knuth classes give a natural example of i-descent consistent sets. The next lemma will be important in our characterization of the pattern-Knuth closed sets which consist of a single Knuth class.

Lemma 5.5. *Fix $S \in \operatorname{SYT}(n)$. Then, the following are equivalent:*

(i) $K(S) = D_J^{-1}$ *for some* $J \subseteq [n-1]$,
(ii) $K(S) = D_J^{-1}$ *where* $J = [1, k]$ *or* $[k, n-1]$ *for some* $1 \le k \le n-1$, *and*
(iii) S *is a superstandard hook.*

Proof. Clearly, (ii) implies (i). The fact that (iii) implies (ii) follows directly from the definition of a superstandard hook. The proof that (i) implies (iii) is by contradiction. Assuming that S is either a non-superstandard hook or not of hook shape, it is easy to find another tableau T with $\operatorname{Des} T = \operatorname{Des} S = J$, so that $K(T) \subseteq D_J^{-1}$. Since Knuth classes are disjoint, this implies $K(S) \subset D_J^{-1}$ which is the desired contradiction. $\square$

The fact that Knuth classes are i-descent consistent gives a criterion for determining when two permutations π and σ have distinct insertion tableaux: If $\mathrm{iDes}(\pi) \neq \mathrm{iDes}(\sigma)$, then $P(\pi) \neq P(\sigma)$. We make repeated use of this criterion in what follows.

Another related notion needed is that of swap closure. This concept is due to Joel Lewis [8]. For any $\pi \in \mathfrak{S}_n$, the operation of interchanging adjacent elements π_i and π_{i+1} in π where $|\pi_i - \pi_{i+1}| > 1$ is called a *swap*. We say that two permutations are *swap equivalent* if one can be obtained from the other via a sequence of swaps. A set of permutations is called *swap closed* if it is closed under this equivalence relation. In what follows, we restrict our attention to swaps involving the largest element n. As such, we define $\overrightarrow{\pi}$ to be the result of swapping n with its right neighbor. In the case that this neighbor is $n-1$ or n is the rightmost element of π, we set $\overrightarrow{\pi} = \pi$. We also define $\overset{\twoheadrightarrow}{\pi}$ to be the permutation obtained from π by removing n from its position and placing it on the right end of π. We define $\overleftarrow{\pi}$ and $\overset{\twoheadleftarrow}{\pi}$ analogously.

The relationship between swap closure and the sets D_J^{-1} is given by the following lemma. The following result was also obtained by Lewis but not published [8].

Lemma 5.6. *Let* $\emptyset \neq \Pi \subseteq \mathfrak{S}_n$. *Then,* Π *is swap closed if and only if*

$$\Pi = \bigcup_{i=1}^{s} D_{J_i}^{-1}$$

for some $J_1, \ldots, J_s \subseteq [n-1]$.

Proof. We first claim that if $\emptyset \neq \Pi \subseteq D_J^{-1}$ for some $J \subseteq [n-1]$ and Π is swap closed, then $\Pi = D_J^{-1}$. Let $J = \{j_1, \ldots, j_k\}$ and define

$$\pi_J := j_k + 1, \ldots, n, j_{k-1} + 1, \ldots j_k, \ldots, j_1 + 1, \ldots, j_2, 1, \ldots, j_1 \in D_J^{-1}.$$

We claim that π_J is swap equivalent to each $\sigma \in D_J^{-1}$. As $\mathrm{iDes}(\sigma) = J$, we have the maximal increasing subsequence $1 \ldots j_1$ in σ. Since $j_1 + 1$ is to the left of j_1, we can move the elements $j_1, j_1 - 1, \ldots, 1$ in that order to the end of σ by a sequence of swaps leaving all the other elements of σ in the same relative positions. Repeating this process yields π_J. Thus, all the elements $\sigma \in D_J^{-1}$ are swap equivalent. Since $\Pi \neq \emptyset$ and is a swap closed subset of D_J^{-1}, we conclude that $\Pi = D_J^{-1}$.

Now, assume that Π is nonempty and swap closed. By an argument similar to the previous paragraph, for each $J \subseteq [n-1]$, such that $\Pi \cap D_J^{-1} \neq \emptyset$, we have $D_J^{-1} \subseteq \Pi$. The forward direction of our lemma now follows. Since swaps interchange elements that differ by at least 2, we see that iDes is invariant under swaps. As each $D_{J_i}^{-1}$ is the set of all permutations with i-descent set J_i, the reverse direction follows. □

We now wish to make a connection between pattern-Knuth closure and swap closure. Note that the second statement of this result follows from the first and the previous lemma.

Theorem 5.7. *If $\Pi \subseteq \mathfrak{S}_n$ is both pattern-Knuth closed and i-descent consistent, then Π is swap closed. Furthermore, when $\Pi \neq \emptyset$, we have $\Pi = D_J^{-1}$ for some $J \subseteq [n-1]$.*

To prove this theorem, we begin with some preliminaries. Given a permutation $\pi \in \mathfrak{S}_n$, and integers $1 \leq i \leq n$ and $1 \leq m \leq n$, we can construct a new permutation σ by standardizing

$$\pi_1, \ldots, \pi_{i-1}, m^+, \pi_i, \ldots, \pi_n,$$

where $m^+ := m + 1/2$. For simplicity, we refer to this operation by saying that σ is the result of *adding m^+ to π in position i*. Of course, we may also add $m^- = m - 1/2$ to σ where this is defined analogously. When adding an element to a permutation, we take the standardization to be implicit. For example, if we add 3^- to $\pi = 132$ in position 3, we write $133^-2 \in \mathfrak{S}_4$ instead of $1432 \in \mathfrak{S}_4$ and refer to 3^- or 3 instead of 3 or 4, respectively.

We write $\pi - m$ to denote the permutation obtained by deleting the value m from π. When subtracting an element, we always refer to the elements of $\pi - m$ in their standardized form, as opposed to the convention for adding an element. For example, if $\pi \in \mathfrak{S}_n$, then in $\pi - (n-1)$, the element $n-1$ is where n is in π. When m is the largest value, we instead write $\widehat{\pi}$ for $\pi - n$ and extend this notation to sets of permutations in the usual way. Finally, we also use this notation in the context of standard Young tableaux. For any $S \in \mathrm{SYT}(n)$, we define $\widehat{S}$ to be the standard Young tableau obtained by deleting n from S.

Lemma 5.8. *Assume that $\pi \in \mathfrak{S}$ contain the subsequence $m-3, m-1, m, m-2$ for some m. Then,* $\mathrm{iDes}(\pi - x) = \mathrm{iDes}(\pi - (m-2))$ *if and only if $x = m-2$.*

Proof. Because of the given configuration of elements, it follows that we have $m-3, m-2 \notin \mathrm{iDes}(\pi - (m-2))$. If $x < m-2$, then we see $m-3 \in \mathrm{iDes}(\pi - x)$, since $m-1$ is to the left of $m-2$ in π. Similarly, if $x > m-2$, then $m-2 \in \mathrm{iDes}(\pi - x)$, since both $m-1$ and m are to the left of $m-2$ in π. The lemma now follows. $\square$

Lemma 5.9. *Let $\Pi \subseteq \mathfrak{S}_n$ be pattern-Knuth closed. Assume that Π is such that $n-1 \notin \mathrm{iDes}(\sigma)$ for all $\sigma \in \Pi$. Then, $\overrightarrow{\pi} \in \Pi$ for all $\pi \in \Pi$.*

Proof. Fix $\pi \in \Pi$ with $\pi_i = n$, so that $\pi_{i+1} < n$. Now, add n^- to π in position $i+2$ and use it to interchange n and π_{i+1} via a Knuth move to obtain

$$\rho := \ldots, \pi_{i+1}, n, n^-, \ldots \in \overline{\mathfrak{S}}_{n+1}(\Pi).$$

Since all patterns in Π have length n, there exists some x, such that $\rho - x \in \Pi$. If $x \neq n, n^-$, then $n-1 \in \mathrm{iDes}(\rho - x)$ in which case $\rho - x \notin \Pi$. Therefore, x is one of n, n^- and $\overrightarrow{\pi} = \rho - x \in \Pi$. $\square$

Next, we prove a lemma to help with our inductive proofs of both Theorems 5.7 and 5.13.

Lemma 5.10. *Assume that $\Pi \subseteq \mathfrak{S}_n$ is pattern-Knuth closed with the property that for each $\pi \in \Pi$, we have either $\overset{\twoheadrightarrow}{\pi} \in \Pi$ or $\overset{\twoheadleftarrow}{\pi} \in \Pi$. Then, $\widehat{\Pi}$ is pattern-Knuth closed.*

Proof. To show that $\widehat{\Pi}$ is pattern-Knuth closed, take $\sigma \in \overline{\mathfrak{S}}_k(\widehat{\Pi})$ for some k and consider any $\rho \in \mathfrak{S}_k$ Knuth-equivalent to σ with the aim of showing $\rho \in \overline{\mathfrak{S}}_k(\widehat{\Pi})$. In particular, assume that σ contains the pattern $\widehat{\pi} \in \widehat{\Pi}$. Now, consider the case when $\overset{\twoheadrightarrow}{\pi} \in \Pi$ and observe that the concatenation $\sigma, k+1$ contains $\overset{\twoheadrightarrow}{\pi}$ and hence $\sigma, k+1 \in \overline{\mathfrak{S}}_{k+1}(\Pi)$. Since ρ and σ are Knuth-equivalent, we have $\rho, k+1$ and $\sigma, k+1$ are too. As Π is pattern-Knuth closed $\rho, k+1 \in \overline{\mathfrak{S}}_{k+1}(\Pi)$. Hence, $\rho \in \overline{\mathfrak{S}}_k(\widehat{\Pi})$ as needed.

The case when $\overset{\twoheadleftarrow}{\pi} \in \Pi$ follows by an analogous argument, and so, the details are omitted. □

We are now in a position to prove Theorem 5.7.

Proof of Theorem 5.7. The second assertion follows from the first and Lemma 5.6. The first statement of the theorem certainly holds when $n = 1$. We now proceed by induction on n with $\Pi \subseteq \mathfrak{S}_n$ where $n > 1$. By considering Π^r if necessary, we may assume $n - 1 \notin \mathrm{iDes}(\Pi)$.

It follows from Lemma 5.9 that $\overrightarrow{\pi} \in \Pi$ for all $\pi \in \Pi$. Therefore, we can always swap n to the right and remain in Π. Hence, $\overset{\twoheadrightarrow}{\pi} \in \Pi$ for all $\pi \in \Pi$, since $n-1$ is to the left of n in π. Therefore, we know by Lemma 5.10 that $\widehat{\Pi} \subseteq \mathfrak{S}_{n-1}$ is pattern-Knuth closed. Clearly, $\widehat{\Pi}$ is i-descent consistent, and so, we conclude by induction that $\widehat{\Pi}$ is swap closed. This means that if we take any $\overset{\twoheadrightarrow}{\pi} \in \Pi$ and swap elements neither of which are n, then the result is in Π. Consequently, it now suffices to show that $\overleftarrow{\pi} \in \Pi$ for any $\pi \in \Pi$. To this end, fix $\pi \in \Pi$ and set $\pi_i = n$ where $\pi_{i-1} \leq n-2$. We consider two cases.

Case 1: $n - 2 \notin \mathrm{iDes}(\Pi)$

In this case, $n-2, n-1, n$ is a subsequence of π. Add $(n-1)^-$ to π in position $i+1$ and use it to interchange π_{i-1} and n via a Knuth move to obtain

$$\rho := \ldots, n-2, \ldots, n-1, \ldots, n, \pi_{i-1}, (n-1)^-, \ldots \in \overline{\mathfrak{S}}_{n+1}(\Pi).$$

Let x be such that $\rho - x \in \Pi$, so that $\mathrm{iDes}(\rho - x) = \mathrm{iDes}(\Pi)$. Observe that $\rho - (n-1)^- = \overleftarrow{\pi}$, and hence, $\mathrm{iDes}(\rho - (n-1)^-) = \mathrm{iDes}(\Pi)$. By Lemma 5.8, it now follows that we must have $x = (n-1)^-$ proving, in this case, that $\overleftarrow{\pi} \in \Pi$.

Case 2: $n - 2 \in \mathrm{iDes}(\Pi)$

Add n^- to π in position $i-1$ and use it to interchange n and π_{i-1} via a Knuth move to obtain

$$\rho := \ldots, n-1, \ldots, n^-, n, \pi_{i-1}, \ldots \in \overline{\mathfrak{S}}_{n+1}(\Pi),$$

where $n-2$ is not shown, but is to the right of $n-1$. Let x be such that $\rho - x \in \Pi$ and $\mathrm{iDes}(\rho - x) = \mathrm{iDes}(\Pi)$. As $n - 2 \in \mathrm{iDes}(\Pi)$, it follows that $x = n-1, n^-$, or n. If $x = n, n^-$, we are done as $\overleftarrow{\pi} = \rho - x \in \Pi$.

If $x = n-1$, then $n-2$ must be to the right of n in ρ, so that $n - 2 \in \mathrm{iDes}(\rho - x)$. As n and $n-2$ are to the right of n^- in ρ, it follows that $\rho - n^-$ is obtainable from $\rho - (n-1)$ by swapping $n-1$ left. Define $\sigma = \rho - (n-1)$, so

$$\sigma = \ldots, \pi_{i-2}, n-1, n, \pi_{i-1}, \ldots \in \Pi$$

where $n-2$ is not shown, but is to the right of n. It now suffices to show that we can swap $n-1$ to the left an arbitrary number of times and stay in Π. Add $(n-1)^-$ to σ in position $i-2$ and apply a Knuth move to obtain

$$\ldots,(n-1)^-,n-1,\pi_{i-2},n,\pi_{i-1},\ldots \in \overline{\mathfrak{S}}_{n+1}(\Pi).$$

As $n-2 \in \mathrm{iDes}(\Pi)$, we must delete $y=(n-1)^-, n-1$, or n to obtain a pattern in Π. If y is one of the first two, we are done. If $y=n$, then, by the second paragraph in this proof, we can swap n to the right once, so that we obtain σ with $n-1$ and π_{i-2} interchanged which must still be in Π. Repeating this argument demonstrates that we can swap $n-1$ left as needed. $\square$

In [6], the authors prove in Theorem 5.8 that $K(T)$ is pattern-Knuth closed if and only if T is a superstandard hook. We are now in position to give a more conceptual explanation as to why this theorem holds as well as place its statement in a more general framework.

Theorem 5.11. *Suppose that $\Pi = K(S)$ for some $S \in \mathrm{SYT}(n)$. Then, the following are equivalent:*

(i) Π *is pattern-Knuth closed,*
(ii) Π *is swap closed,*
(iii) $\Pi = D_J^{-1}$ *where $J=[1,k]$ or $J=[k,n-1]$ for some k, and*
(iv) S *is a superstandard hook.*

Proof. Our definition of Π implies that Π is i-descent consistent. The implication (i) implies (ii) follows directly from Theorem 5.7. The fact that (ii) implies (iii) follows from the fact that Π is i-descent consistent as well as Lemmas 5.5 and 5.6. The implication (iii) implies (iv) also follows from Lemma 5.5. Finally, the fact that (iv) implies (i) is given a straightforward explanation in the paper of [6, Proposition 5.9]. $\square$

We now characterize pattern-Knuth closed classes that are unions of two Knuth classes, answering a question in [6]. We say that $S \neq T \in \mathrm{SYT}(n)$ are an *i-descent-complete* pair if $K(S) \cup K(T) = D_J^{-1}$ for some $J \subseteq [n-1]$. We start by characterizing such pairs.

Lemma 5.12. *Suppose that $S \neq T \in \mathrm{SYT}(n)$. Then, S and T are an i-descent-complete pair if and only if*

$$S = \begin{array}{|c|c|c|c|c|}\hline 1 & 2 & \cdots & k & n \\ \hline k+1 \\ \cline{1-1} \vdots \\ \cline{1-1} n-1 \\ \cline{1-1}\end{array} \qquad T = \begin{array}{|c|c|c|c|}\hline 1 & 2 & \cdots & k \\ \hline k+1 & n \\ \cline{1-2} \vdots \\ \cline{1-1} n-1 \\ \cline{1-1}\end{array} \tag{8}$$

where $2 \le k \le n-2$ and $\mathrm{Des}(S) = \mathrm{Des}(T) = [k, n-2]$, or

$$S = \begin{array}{|c|c|c|c|c|}\hline 1 & 2 & k & \cdots & n \\ \hline 3 \\ \cline{1-1} \vdots \\ \cline{1-1} k-1 \\ \cline{1-1}\end{array} \qquad T = \begin{array}{|c|c|c|c|c|}\hline 1 & 2 & k+1 & \cdots & n \\ \hline 3 & k \\ \cline{1-2} \vdots \\ \cline{1-1} k-1 \\ \cline{1-1}\end{array} \tag{9}$$

where $4 \leq k \leq n$ *and* $\mathrm{Des}(S) = \mathrm{Des}(T) = [2, k-2]$, *or* S *and* T *are transposes of these tableaux.*

Proof. The reverse direction follows from a straightforward check. To prove the forward direction, let $J \subseteq [n-1]$ be such that $K(S) \cup K(T) = D_J^{-1}$. As $S \neq T$, it follows from Lemma 5.5 that neither S nor T can be superstandard. Furthermore, by considering reverses if necessary, we may assume $1 \notin J$, so that 2 is in the first row of both S and T. It now suffices to show that no other pairs of tableaux other than those displayed in (8) or (9) are i-descent-complete pairs.

To this end, observe that the mapping between hook tableaux and subsets of $[n-1]$ given by Des is a bijection. As $K(S) \cup K(T)$ is the set of all permutations with i-descent set J, it follows that either S or T is hook shape. We take S to be of hook shape. In particular as S is not superstandard, its leftmost column has length at least 2. Now, consider adjacent elements a and b with $2 \leq a < b$ in the top row of S. If $b > a+1$, then $b-1$ must be in the first column of S, so that the tableau obtained by moving b into position $(2,2)$ is standard and has descent set J. Likewise, consider adjacent elements c and d with $2 \leq c < d$ in the first column of S. If $d > c+1$, then the tableau obtained by moving d into position $(2,2)$ is standard and also has descent set J. As there are exactly two tableaux with descent set J, we must have

$$S = \begin{array}{|c|c|c|c|c|c|c|}\hline 1 & 2 & \cdots & a & b & \cdots & n \\ \hline a{+}1 \\ \cline{1-1} a{+}2 \\ \cline{1-1} \vdots \\ \cline{1-1} b{-}1 \\ \cline{1-1} \end{array},$$

for some $a \geq 2$ and $a+1 < b \leq n$. If $a = 2$, then this results in the second pair in the statement of the theorem. If $a \geq 3$, then we claim that $b = n$. For if $b \leq n-1$, then we have at least two additional tableaux with descent set J. Namely, we have the tableau S' obtained by moving b into position $(2,2)$ and the tableau S'' obtained from S' by moving $b+1$ into positions $(2,3)$ where the fact that $a \geq 3$ guarantees that S'' is standard. When $b = n$, we get the first pair of tableaux in the statement of the theorem. This completes our proof. □

We are now ready to state our second main theorem of this section.

Theorem 5.13. *Suppose that* $\Pi = K(S) \cup K(T)$ *where* $S \neq T \in \mathrm{SYT}(n)$. *The following are equivalent:*

(i) Π *is pattern-Knuth closed,*
(ii) Π *is swap closed,*
(iii) $\Pi = D_J^{-1} \cup D_L^{-1}$ *where either* $J \neq L$ *are of the form given in Lemma 5.5, or* $J = L$ *is of the form given in Lemma 5.12, and*
(iv) S *and* T *are either distinct superstandard hooks, the tableaux pairs displayed in Lemma 5.12, or their transposes.*

In light of Theorems 5.11 and 5.13, one would hope that pattern-Knuth closure would, in general, be equivalent to swap closure. Unfortunately, this is not true. For example, take: $\Pi = K(T_1) \cup K(T_2) \cup K(T_3)$, where

$$T_1 = \begin{array}{|c|c|c|}\hline 1 & 2 & 4 \\ \hline 3 \\ \cline{1-1}\end{array} \quad T_2 = \begin{array}{|c|c|c|}\hline 1 & 3 & 4 \\ \hline 2 \\ \cline{1-1}\end{array} \quad T_3 = \begin{array}{|c|c|c|}\hline 1 & 2 & 3 \\ \hline 4 \\ \cline{1-1}\end{array}\,.$$

A computer check shows that $\mathfrak{S}_5(\Pi)$ is a union of Knuth classes, and so, by Lemma 6.1 below, we know that this Π is pattern-Knuth closed. On the other hand, this Π is not swap closed as $3124 \in K(T_1)$, but performing a swap gives $3142 \in K(T_4)$, where

$$T_4 = \begin{array}{|c|c|}\hline 1 & 2 \\ \hline 3 & 4 \\ \hline\end{array}\,.$$

We now turn to the proof Theorem 5.13. The crux of this proof is in demonstrating that i) implies ii). We begin by building up the required lemmas to show this implication. In what follows, we denote Knuth equivalence by $\sim$.

Lemma 5.14. *Fix $S \in \mathrm{SYT}(n)$. The following are equivalent:*

(i) *there exists some $\pi \in K(S)$ with $\pi_n = n$,*

(ii) *$K(\widehat{S}) = \widehat{K(S)}$ and n is in the top row of S, and*

(iii) *$\overset{\twoheadrightarrow}{\sigma} \in K(S)$ for all $\sigma \in K(S)$.*

Proof. Before proving any of the above implications observe that if we apply the Robinson–Schensted algorithm to any $\sigma \in \mathfrak{S}_n$ and keep track of only the values $< n$, it is clear that $P(\widehat{\sigma}) = \widehat{P(\sigma)}$. Thus, in general, $\widehat{K(S)} \subseteq K(\widehat{S})$.

We now prove (i) implies (ii). By the existence of $\pi \in K(S)$ with $\pi_n = n$, it is clear that $S = P(\pi)$ has n in the top row. From the observation in the first paragraph, to establish the equality, we need only show that $K(\widehat{S}) \subseteq \widehat{K(S)}$ and we know $P(\widehat{\pi}) = \widehat{P(\pi)} = \widehat{S}$. Now, pick $\sigma \in K(\widehat{S})$ so that $\sigma \sim \widehat{\pi}$, and hence, $\sigma, n \sim \pi$. Consequently, $P(\sigma, n) = P(\pi) = S$. Therefore, $\sigma \in \widehat{K(S)}$ as needed.

We now prove that (ii) implies (iii). By (ii), we see that for any $\sigma \in K(S)$, we have $\widehat{\sigma} \in \widehat{K(S)} = K(\widehat{S})$. Therefore, $P(\overset{\twoheadrightarrow}{\sigma})$ is obtained by adding n to the end of the top row of $\widehat{S}$. As n is in the top row of S, it follows that $P(\overset{\twoheadrightarrow}{\sigma}) = S$.

The fact that (iii) implies (i) is clear. $\square$

Lemma 5.15. *Fix $n \geq 1$. Let $S, T \in \mathrm{SYT}(n)$ be of hook shape with the property that $\widehat{S}$ and $\widehat{T}$ are superstandard hooks. If $\Pi = K(S) \cup K(T)$ is pattern-Knuth closed, then S and T are both superstandard.*

Proof. When $n \leq 3$, all tableaux are superstandard, and so, the result follows trivially. When $n = 4$, a computer check demonstrates the theorem. Therefore, we may assume $n \geq 5$. For a contradiction assume T is not superstandard. By considering Π^r if necessary, we may further assume n is in the top row of T and that the row word of T is

$$\rho(T) = n-1, \ldots, k+1, 1, 2, \ldots, k, n$$

where $2 \le k \le n-2$. We now consider three cases.
Case 1: $2 < k < n-2$

Add $(n-1)^-$ to $\rho(T)$ in position 1 to obtain

$$(n-1)^-, n-1, \ldots, k+1, 1, 2, \ldots, k, n \in \overline{\mathfrak{S}}_{n+1}(\Pi).$$

Via Knuth moves slide $n-1$ to position $n-1$ and then, via another Knuth move, interchange n and k to obtain

$$\tau := (n-1)^-, n-2, \ldots, k+1, 1, 2, \ldots, k-1, n-1, n, k \in \overline{\mathfrak{S}}_{n+1}(\Pi).$$

There exists some x, such that $\tau - x \in \Pi$. Because $k < n-2$, the decreasing prefix $(n-1)^-, n-2, \ldots, k+1$ has length at least 2. Now, a straightforward check shows that if $x \neq k$, then $P(\tau - x)$ is not of hook shape, since k bumps either $n-1$ or n into position $(2,2)$. When $x = k$, we see that $P = P(\tau - x)$ has hook shape with top row $1, 2, \ldots, k-1, n-1, n$. The fact that $2 < k < n-2$ means that $\widehat{P}$ is not superstandard, and so, $P \neq S, T$. This contradiction shows that the lemma holds in this case.
Case 2: $k = n-2$

Here, $\rho(T) = n-1, 1, 2, \ldots, n-2, n \sim 1, 2, \ldots, n-3, n-1, n, n-2$. Now, add $(n-2)^+$ in position $n-3$ and use it to interchange $n-3$ and $n-1$ to obtain

$$\begin{aligned} &1, 2, \ldots, n-4, (n-2)^+, n-1, n-3, n, n-2 \\ &\quad \sim 1, 2, \ldots, n-4, (n-2)^+, n-1, n, n-3, n-2 \in \overline{\mathfrak{S}}_{n+1}(\Pi). \end{aligned}$$

Denoting the last permutation by τ, there exists x be such that $\tau - x \in \Pi$. If $x \neq n-2, n-3$, then the shape of $P = P(\tau - x)$ is $(n-2, 2)$ which is not a hook. If $x = n-2$ or $n-3$, then

$$P = \begin{array}{|c|c|c|c|c|c|c|} \hline 1 & 2 & \ldots & n-4 & n-3 & n-1 & n \\ \hline n-2 \\ \cline{1-1} \end{array}\ .$$

Since $n \ge 5$, we see that the top row contains 1 and 2 and $n-1 \ge 4$ but not $n-2$. Therefore, $\widehat{P}$ is not superstandard, and hence, $P \neq S, T$. We conclude, in this case, that the lemma holds.

Case 3: $k = 2$

Here, $\rho(T) = n-1, \ldots, 3, 1, 2, n$. Now, add $(n-2)^+$ in position $n-1$ and use it to interchange 2 and n to obtain

$$\begin{aligned} &n-1, \ldots, 3, 1, (n-2)^+, n, 2 \\ &\quad \sim n-2, n-1, n, 1, (n-2)^+, n-3, \ldots, 3, 2 \in \overline{\mathfrak{S}}_{n+1}(\Pi). \end{aligned}$$

Denoting the last permutation by τ, there must exist some x, so that $\tau - x \in \Pi$. Note that the first 6 terms of τ are order isomorphic to 356142 whose insertion tableau is not of hook shape. Therefore, x must be one of the first 6 terms. If $x \neq 1$, then the remaining 5 elements in $\tau - x$ insert to a tableau which is not of hook shape. Therefore, $x = 1$ and $\tau - 1 = n-3, n-1, n, n-2, n-4, \ldots, 1$. It is now easy to check that if $P = P(\tau - 1)$, then $\widehat{P}$ is not superstandard and, consequently, $P \neq S, T$. This completes the final case and the proof of the lemma. $\square$

In what follows, we denote the symmetric difference of two sets by $\triangle$.

Lemma 5.16. *Suppose that $n \geq 4$. Let $\Pi = K(S) \cup K(T)$ be pattern-Knuth closed where $S, T \in \mathrm{SYT}(n)$. Assume that $n-1 \notin \mathrm{Des}(S)$ and $n-1 \in \mathrm{Des}(T)$ and $1 \in \mathrm{Des}(S) \triangle \mathrm{Des}(T)$. Then, there exists some $\pi \in K(S)$ and $\sigma \in K(T)$ with $\pi_n = n$ and $\sigma_1 = n$.*

Proof. We first prove the existence of such a $\pi \in K(S)$. Choose $\pi \in K(S)$ that maximizes i where $\pi_i = n$. Towards a contradiction, assume $i < n$, so that $\pi_{i+1} \leq n-2$, since $n-1 \notin \mathrm{Des}(S)$. Now, add $(n-1)^-$ in position i and use it to interchange n and π_{i+1} via a Knuth move to obtain

$$\rho := \ldots, n-1, \ldots, (n-1)^-, \pi_{i+1}, n, \ldots \in \overline{\mathfrak{S}}_{n+1}(\Pi).$$

Let x be such that $\rho - x \in \Pi$. We claim that $\rho - x \notin K(T)$. For if this was to occur, we must take $x = 1$ or 2, since $1 \in \mathrm{Des}(S) \triangle \mathrm{Des}(T)$. However, because $n \geq 4$, we would then have $n-1 \notin \mathrm{iDes}(\rho - x)$, whereas $n-1 \in \mathrm{Des}(T)$. We conclude that $\rho - x \in K(S)$.

Next observe that $x \neq n$ as otherwise $n-1 \in \mathrm{iDes}(\rho - x)$, whereas $n-1 \notin \mathrm{Des}(S)$. Consequently, n sits in position $i+1$ or $i+2$ in $\rho - x \in K(S)$ depending on whether x is to the left or the right of n in ρ, respectively. This contradicts our choice of π proving the first claim in this lemma.

The above argument, when applied to $\Pi^r = K(S^r) \cup K(T^r)$, establishes the second claim and completes our proof. $\square$

Proof of Theorem 5.13. We first show that (ii) implies (iii). As Π is swap closed, Lemma 5.6 together with the fact that Knuth classes are i-descent consistent implies that $\Pi = D_J^{-1} \cup D_L^{-1}$ for some $J, L \subseteq [n-1]$. If $J \neq L$, it further follows that $K(S) = D_J^{-1}$ and $K(T) = D_L^{-1}$. Hence, J, L are as given in Lemma 5.5. If $J = L$, then S and T are an i-descent-complete pair and J, L are given by Lemma 5.12. The implication that (iii) implies (iv) also follows from these two lemmas.

Now, assume (iv) with the goal of showing (i). It follows from Lemmas 5.5 and 5.12 that $\Pi = D_J^{-1} \cup D_L^{-1}$ for some $J, L \subseteq [n-1]$. As sets of the form D_J^{-1} are pattern-Knuth closed by Lemma 5.7 in [6], (i) follows from Proposition 5.2.

It remains to show that (i) implies (ii). Observe that when $\mathrm{Des}(S) = \mathrm{Des}(T)$,, the result follows from Theorem 5.7. Therefore, we may assume $\mathrm{Des}(S) \neq \mathrm{Des}(T)$. In light of Lemmas 5.5 and 5.6, it suffices to show that S and T are superstandard hooks. We proceed by induction on n. Since all tableaux are superstandard hooks when $n \leq 3$, we take $n \geq 4$.

First, assume that $n-1$ is in neither $\mathrm{Des}(S)$ nor $\mathrm{Des}(T)$. By (repeated application of) Lemma 5.9, we have

$$\overset{\twoheadrightarrow}{\pi} \in \Pi \tag{10}$$

for each $\pi \in \Pi$. By Lemma 5.10, $\widehat{\Pi}$ is pattern-Knuth closed. It also follows from (10) that n must be in the top row of either S or T. We consider two cases. If n is in the top row of both S and T, then it is the last element of both $\rho(S)$ and $\rho(T)$. By Lemma 5.14, see that

$$\widehat{\Pi} = \widehat{K(S)} \cup \widehat{K(T)} = K(\widehat{S}) \cup K(\widehat{T}). \tag{11}$$

By induction, we conclude that $\widehat{S}$ and $\widehat{T}$ are superstandard hooks. As n is in the top rows of S and T, we further see S and T are hook shape. Finally, Lemma 5.15 implies that S and T are both superstandard hooks.

Next, assume that n is in the top row of S but not T. Set $\rho = \widehat{\rho(T)}, n$, so that $\rho \in \Pi$ by (10). Let R be the insertion tableau of ρ. Clearly, $R = S$ or T, but as n is in the top row R, we conclude $R = S$. Additionally, notice that ρ is the reading word for R and that R is obtained by moving n in T to the top row of T. As $n-1 \notin \mathrm{Des}(T)$, it follows that $\mathrm{Des}(T) = \mathrm{Des}(R) = \mathrm{Des}(S)$. This contradicts the fact that $\mathrm{Des}(S) \neq \mathrm{Des}(T)$, and so, we conclude that this case cannot occur.

At this point, we may assume that $n-1 \in \mathrm{Des}(S) \triangle \mathrm{Des}(T)$ for if $n-1$ is in both sets, then repeating the above argument on $\Pi^r = K(S^r) \cup K(T^r)$ disposes of that case. In fact, we can assume even more. As Knuth moves commute with complementation, we see that Π^c is the union of Knuth classes and is pattern-Knuth closed. Therefore, the above argument, when applied to Π^c, deals with the cases when $1 \notin \mathrm{Des}(S) \triangle \mathrm{Des}(T)$. In what remains, we assume that $1, n-1 \in \mathrm{Des}(S) \triangle \mathrm{Des}(T)$.

Without loss of generality, assume $n-1 \notin \mathrm{Des}(S)$ and $n-1 \in \mathrm{Des}(T)$. It now follows from Lemma 5.16 that there exists $\pi \in K(S)$ and $\sigma \in K(T)$, so that $\pi_n = n$ and $\sigma_1 = n$. Hence, Lemma 5.14 applied to $K(S)$ and $K(T)^r$ tells us that for each $\zeta \in K(S)$ and $\xi \in K(T)$, we have

$$\overset{\twoheadrightarrow}{\zeta} \in K(S) \quad \text{and} \quad \overset{\twoheadleftarrow}{\xi} \in K(T)$$

and that Eq. (11) holds here, as well. Lemma 5.10 now gives us that $\widehat{\Pi}$ is pattern-Knuth closed. As $1 \in \mathrm{Des}(\widehat{S}) \triangle \mathrm{Des}(\widehat{T})$, we see that $\mathrm{Des}(\widehat{S}) \neq \mathrm{Des}(\widehat{T})$, and so, we may conclude by induction that $\widehat{S}$ and $\widehat{T}$ are superstandard hooks. As $\pi_n = n$ and $\sigma_1 = n$, it follows that S and T are hook shape. Finally, Lemma 5.15 implies that S and T are superstandard hooks as needed. □

6. Stability

In Question 7.1 in [6], the authors ask if $Q_n(\Pi)$ being symmetric or Schur nonnegative for n up to some bound would force it to continue to be so for all n. We now show the converse of this question is false by showing that $Q_n(\Pi)$ can be Schur nonnegative for all sufficiently large n without being so for some smaller value of n. In particular, we show that this is true for

$$\Pi_0 = K(3,1,1) - K(P_0), \tag{12}$$

where

$$P_0 = \begin{array}{|c|c|c|} \hline 1 & 2 & 4 \\ \hline 3 & \multicolumn{2}{c}{} \\ \cline{1-1} 5 & \multicolumn{2}{c}{} \\ \cline{1-1} \end{array} \; .$$

We need the following result.

Lemma 6.1. [6] *The set* Π *is pattern-Knuth closed if and only if* $\mathfrak{S}_n(\Pi)$ *is a union of Knuth classes for* $n \leq M+1$ *where* M *is the maximum length of a permutation in* Π. □

We also need the following criterion.

Lemma 6.2. *Given* Π *and* Π' *nonempty sets of permutations, we let*

$$M = \max\{\#\pi \mid \pi \in \Pi \cup \Pi'\}$$

where $\#\pi$ *is the length of* π. *If there is an* $N \geq M$, *such that*

$$\mathfrak{S}_N(\Pi) = \mathfrak{S}_N(\Pi'),$$

then:

$$\mathfrak{S}_n(\Pi) = \mathfrak{S}_n(\Pi')$$

for all $n \geq N$.

Proof. It suffices to prove that if $\sigma \in \mathfrak{S}_n$ contains a copy of some $\pi \in \Pi$, then σ also contains a $\pi' \in \Pi'$ as the converse statement follows by symmetry. Since $n \geq N \geq M$, there is a subsequence τ of σ of length N containing π. Since $\mathfrak{S}_N(\Pi) = \mathfrak{S}_N(\Pi')$, we have that τ must also contain a copy of some $\pi' \in \Pi'$. Therefore, σ contains π' and we are done. □

Now, consider $K(3,1,1)$. One can check by computer that $\mathfrak{S}_n(K(3,1,1))$ is a union of Knuth classes for $n \leq 6$. It follows from Lemma 6.1 that $K(3,1,1)$ is pattern-Knuth closed, so that $Q_n(K(3,1,1))$ is Schur nonnegative for all n.

By contrast, using the computer again, we see that $Q_6(\Pi_0)$ is not even symmetric where Π_0 is defined by (12). On the other hand, another computer check shows that $\mathfrak{S}_7(\Pi_0) = \mathfrak{S}_7(K(3,1,1))$. Therefore, by Lemma 6.2, we conclude that $\mathfrak{S}_n(\Pi_0) = \mathfrak{S}_n(K(3,1,1))$ for $n \geq 7$. It follows from the previous paragraph that $Q_n(\Pi_0)$ is Schur nonnegative for $n \geq 7$.

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