Pattern frequency sequences and internal zeros

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This paper is dedicated to the memory of Rodica Simion
who did some seminal work in the area of pattern avoidance

Abstract

Let q be a pattern and let $S_{n,q}(c)$ be the number of n-permutations having exactly c copies of q. We investigate when the sequence $(S_{n,q}(c))_{c\geq 0}$ has internal zeros. If q is a monotone pattern it turns out that, except for $q = 12$ or 21, the nontrivial sequences (those where n is at least the length of q) always have internal zeros. For the pattern $q = 1(l+1)l\ldots 2$ there are infinitely many sequences which contain internal zeros and when $l = 2$ there are also infinitely many which do not. In the latter case, the only possible places for internal zeros are the next-to-last or the second-to-last positions. Note that by symmetry this completely determines the existence of internal zeros for all patterns of length at most three.

1 Introduction

Let $q = q_1q_2\ldots q_l$ be a permutation in the symmetric group S_l. We call l the *length* of q. We say that the permutation $p = p_1p_2\ldots p_n \in S_n$ *contains a q-pattern* if and only if there is a subsequence $p_{i_1}p_{i_2}\ldots p_{i_l}$ of p whose elements are in the same relative order as those in q, i.e.,

$$p_{i_j} < p_{i_k} \text{ if and only if } q_j < q_k$$

whenever $1 \leq j,k \leq l$. For example, 41523 contains exactly two 132-patterns, namely 152 and 153. We let

$$c_q(p) = \text{the number of copies of } q \text{ in } p,$$

so that $c_{132}(41523) = 2$. Permutations containing a given number of q-patterns have been extensively studied recently [1–11].

In this paper, we consider permutations with a given number of q-patterns from a new angle. Let

$$S_{n,q}(c) = \text{the number of } n\text{-permutations with exactly } c \text{ patterns of type } q.$$

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For n and q fixed, the sequence $(S_{n,q}(c))_{c\geq 0}$ is called the *frequency sequence* of the pattern q for n. Clearly this sequence consists entirely of zeros if n is less than the length of q and so we call these sequences *trivial* and all others *nontrivial.* We also say that an n-permutation p is *q-optimal* if there is no n-permutation with more copies of q than p, and let

$$M_{n,q} = c_q(p) \text{ for an optimal } p.$$

The only q for which the frequency sequence is well understood is $q = 21$ (or equivalently $q = 12$). Occurences of this pattern are called inversions. It is well known [12] that for all n, the frequency sequence of inversions is log-concave, and so is unimodal and has no internal zeros. An integer c is called an *internal zero* of the sequence $(S_{n,q}(c))_{c\geq 0}$ if for some c we have $S_{n,q}(c) = 0$, but there exist c_1 and c_2 with $c_1 < c < c_2$ and $S_{n,q}(c_1)$, $S_{n,q}(c_2) \neq 0$.

When q has length greater than 2, numerical evidence suggests that the frequency sequence of q will no longer be unimodal, let alone log-concave. In fact, internal zeros seem to be present in most frequency sequences.

In the rest of this paper we study the frequency sequences of the monotone pattern $q = 12\ldots l$ and the pattern $q = 1(l+1)l\ldots 2$. We will show that in the first case, when $l \geq 3$ (the case $l = 2$ has already been mentioned) the nontrivial sequences always have internal zeros. For $1(l+1)l\ldots 2$-patterns there are infinitely many n where the sequence has internal zeros. For the 132-pattern there are also infinitely many n where the sequence has no internal zeros. Furthermore, internal zeros can only appear in positions $M_{n,132} - 1$ or $M_{n,132} - 2$.

2 The monotone case

We will now consider the sequence $(S_{n,q}(c))_{c\geq 0}$ where $q = 12\ldots l$. For later reference, we single out the known case when $l = 2$ discussed in the introduction.

Proposition 2.1 *The sequence $(S_{n,12}(c))_{c\geq 0}$ has no internal zeros (and is, in fact, log concave). The unique optimal permutation is $p = 12\ldots n$ with*

$$M_{n,12} = \binom{n}{2} \quad \diamond$$

It turns out that this is the only monotone pattern (aside from 21) whose sequence has no internal zeros. To prove this result, define an *inversion* (respectively, *noninversion*) in $p = p_1p_2\ldots p_n$ to be a pair (p_i, p_j) such that $i < j$ and $p_i > p_j$ (respectively, $p_i < p_j$).

Theorem 2.2 *Let $q = 12\ldots l$ where $l \geq 3$. Then in S_n, the unique optimal permutation is $p = 12\ldots n$ and*

$$M_{n,12\ldots l} = \binom{n}{l}.$$

The set of permutations having the next greatest number of copies of q are those obtained from p by an adjacent transposition and this number of copies is

$$\binom{n-1}{l} + \binom{n-2}{l-1}. \tag{1}$$

Proof: Consider any $r \in S_n$ different from p. Then r has an inversion (r_i, r_j). So the number of copies of q in r is the number not containing r_i plus the number which do contain r_i. The permutations in the latter case cannot contain r_j. So (1) gives an upper bound for the number of copies of q which is strict unless r has exactly one inversion. The theorem follows. $\diamond$

Corollary 2.3 *Let $q = 12\ldots l$ where $l \geq 3$. Then for $n \geq l$ the sequence $(S_{n,12\ldots l}(c))_{c\geq 0}$ has internal zeros.*

Proof: From the previous theorem, we see that the number of zeros directly before $S_{n,q}(M_{n,q}) = 1$ is

$$\binom{n}{l} - \binom{n-1}{l} - \binom{n-2}{l-1} = \binom{n-2}{l-2} \geq n-2 \geq 1$$

since $n \geq l \geq 3$. $\diamond$

For use in the 132 case, we record the following observation.

Lemma 2.4 *For any integer c with $0 \leq c \leq \binom{n}{2}$ there is a permutation $p \in S_n$ having c copies of the pattern 21 and no copies of 132.*

Proof: We induct on n. The result is clearly true if $n \leq 2$. Assuming it is true for $n-1$, first consider $c \leq \binom{n-1}{2}$ and let $p \in S_{n-1}$ satisfy the lemma. Then the concatenation $pn \in S_n$ works for such c. On the other hand, if $\binom{n-1}{2} < c \leq \binom{n}{2}$ then consider $c' = c - (n-1) \leq \binom{n-1}{2}$. Pick $p \in S_{n-1}$ with c' copies of 21 and none of 132. Then $np \in S_n$ is the desired permutation. $\diamond$

3 The case $q = 1(l+1)l\ldots 2$ and layered patterns

The rest of this paper is devoted to the study of the frequency sequences of the patterns $1(l+1)l\ldots 2$ for $l \geq 2$. To simplify notation, write $F_{n,1(l+1)l\ldots 2}$ for the sequence $(S_{n,1(l+1)l\ldots 2}(c))_{c\geq 0}$. One crucial property of these patterns is that they are layered. This section gives an overview of some important results on layered patterns.

A pattern is *layered* if it is the concatenation of subwords (the *layers*) where the entries decrease within each layer, and increase between the layers. For example, 3 2 1 5 4 8 7 6 9 is a layered pattern with layers 3 2 1, 5 4, 8 7 6, and 9. Layered patterns are examined in Stromquist's work [14] and in Price's thesis [9]. The most important result for our current purposes is the following theorem.

Theorem 3.1 ([14]) *Let q be a layered pattern. Then the set of q-optimal n-permutations contains at least one layered permutation.*

Layered $1(l+1)l\ldots 2$-optimal permutations have a simple recursive structure. This comes from the fact, which we will use many times, that to form a $1(l+1)l\ldots 2$ pattern in a layered permutation one must take a single element from some layer and l elements from a subsequent layer

Proposition 3.2 *Let p be a layered $1(l+1)l\ldots 2$-optimal n-permutation whose last layer is of length m. Then the leftmost $k = n-m$ elements of p form a $1(l+1)l\ldots 2$-optimal k-permutation.*

Proof: Let D_k be the number of $1(l+1)l\ldots 2$-copies of p that are disjoint from the last layer. The number of $1(l+1)l\ldots 2$-copies of p is clearly $k\binom{m}{l} + D_k$. So once k is chosen, p will have the maximum number of copies only if D_k is maximal. $\diamond$

We point out that the proof of this proposition uses the fact that $1(l+1)l\ldots 2$ has only two layers, the first of which is a singleton. Let $M_n = M_{n,1(l+1)l\ldots 2}$. Then the previous proposition implies that

$$M_n = \max_{1\leq k<n}\left(M_k + k\binom{m}{l}\right). \tag{2}$$

The integer k for which the right hand side attains its maximum will play a crucial role throughout this paper. Therefore, we introduce specific notation for it.

Definition 3.3 *For any positive integer n, let $k_n = k_{n,1(l+1)l\ldots 2}$ be the positive integer for which $M_n = \max_k(M_k + k\binom{m}{l})$ is maximal. If there are several integers with this property, then let k_n be the largest among them.*

In other words, k_n is the largest possible length of the remaining permutation after removing the last layer of a $1(l+1)l\ldots 2$-optimal n-permutation p. When there is no danger of confusion, we will only write k to simplify notation. We will also always use $m = n-k$ to denote the length of the last layer of p.

4 Construction of Permutations with a given number of copies of $q = 132$

We will first show that if $q = 132$ then there are infinitely many integers n such that F_n does not have internal zeros. We will call such an integer, or its corresponding sequence, *NIZ (no internal zero)*, and otherwise *IZ*. Our strategy is recursive: We will show that if k_n is NIZ, then so is n. As $k_n < n$, this will lead to an infinite sequence of NIZ integers. There is a problem, however. In order for this strategy to work, we must ensure that given k, then there is an n such that $k = k_n$. This is the purpose of the following theorem which is in fact true for the general pattern $q = 1(l+1)l\ldots 2$.

Theorem 4.1 *For $k_n = k_{n,1(l+1)l\ldots 2}$, the sequence $(k_n)_{n\geq 1}$ diverges to infinity and satisfies*

$$k_n \leq k_{n+1} \leq k_n + 1$$

for all $n \geq l+1$. So, since $k_{l+1} = 1$, for all positive integers k there is a positive integer n so that $k_n = k$.

The next section is devoted to a proof of this theorem. We suggest that the reader assume the result now and continue with this section to preserve continuity. We now consider the case $q = 132$ which behaves differently from $q = 1(l+1)l\ldots 2$ for $l \geq 3$. This is essentially due to the difference between the patterns $q = 12$ and $q = 12\ldots l$ for $l \geq 3$ as seen in Proposition 2.1 and Theorem 2.2. First we note the useful fact that

$$M_{k,132} \geq \binom{k-1}{2} \tag{3}$$

which follows by considering the permutation $1k(k-1)(k-2)\cdots 32$.

Theorem 4.2 *For $q = 132$ There are infinitely many NIZ integers.*

Proof: It is easy to verify that $n = 4$ is NIZ. So, by Theorem 4.1, it suffices to show that if $k_n \geq 4$ is NIZ then so is n. To simplify notation in the two proofs which follow, we will write k for $k_{n,132}$, M_n for $M_{n,132}$, and so forth.

Now given c with $0 \leq c \leq M_n = M_k + k\binom{m}{2}$ we will construct a permutation $p \in S_n$ having c copies of 132. Because of (3) and $k \geq 4$ we have $M_k \geq k-1$. So it is possible to write c (not necessarily uniquely) as $c = ks + t$ with $0 \leq s \leq \binom{m}{2}$ and $0 \leq t \leq M_k$. Since k is NIZ, there is a permutation $p' \in S_k$ with $c_{132}(p') = t$. Also, by Lemma 2.4, there is a permutation in S_m with no copies of 132 and s copies of 21. Let p'' be the result of adding k to every element of that permutation. Then, by construction, $p = p'p'' \in S_n$ and $c_{132}(p) = ks + t = c$ as desired. $\diamond$

One can modify the proof of the previous theorem to locate precisely where the internal zeros could be for an IZ sequence. We will need the fact (established by computer) that for $n \leq 12$ the only IZ integers are 6, 8, and 9, and that they all satisfied the following result.

Theorem 4.3 *For any positive integer n, the sequence $F_{n,132}$ does not have internal zeros, except possibly for $c = M_{n,132} - 1$ or $c = M_{n,132} - 2$, but not both.*

Proof: We prove this theorem by induction on n. As previously remarked, it is true if $n \leq 12$. Now suppose we know the statement for all integers smaller than n, and prove it for n. If n is NIZ, then we are done.

If n is IZ then, by the proof of Theorem 4.2, $k = k_n$ is IZ. So $k \geq 6$ and we have $M_k \geq k+2$ by (3). Now take c with $0 \leq c \leq M_n - 3$ so that we can write $c = ks + t$ with $0 \leq s \leq \binom{m}{2}$ and $0 \leq t \leq M_k - 3$. Since the portion of F_k up to $S_k(M_k - 3)$ has no internal zeros by induction, we can use the same technique as in the previous theorem to construct a permutation p with $c_{132}(p) = c$ for

c in the given range. Furthermore, this construction shows that if $S_k(M_k - i) \neq 0$ for $i = 1$ or 2 then $S_n(M_n - i) \neq 0$. This completes the proof. $\diamond$

5 The sequence $(k_n)_{n \geq l+1}$ for $q = 1(l+1)l \ldots 2$

For the rest of this paper, all invariants will refer to the pattern $q = 1(l+1)l \ldots 2$ unless explicitly stated otherwise.

In order to prove Theorem 4.1, we first need a lemma about the lengths of various parts of a $1(l+1)l \ldots 2$-optimal permutation p. In all that follows, we use the notation

$$\begin{aligned} b &= \text{the length of the penultimate layer of } p \\ a &= \text{the length of the permutation gotten by removing the last two layers of } p \\ &= n - m - b \\ &= k - b. \end{aligned}$$

Also observe that the sequence $(M_n)_{n \geq l+1}$ is strictly increasing. This is because when $n \geq l+1$, any layered $1(l+1)l \ldots 2$-optimal permutation $p \in S_n$ contains at least one copy of $1(l+1)l \ldots 2$. So inserting $n+1$ in front of any layer contributing to the $(l+1)l \ldots 2$ portion of some copy results in a permutation with more $1(l+1)l \ldots 2$-patterns than p. It follows from (2) that $m \geq l$ for $n \geq l+1$, a fact that will be useful in proving the following result.

Lemma 5.1 *Let $q = 1(l+1)l \ldots 2$, $k = k_{n,q}$, and $n \geq l+1$. Then we have the following inequalities*

(i) $b \leq m$,

(ii) $a \leq (m - l + 1)/l$,

(iii) $k < n/l$, *so in particular* $k < m$,

(iv) $m \leq l(n+1)/(l+1)$.

Proof: The basic idea behind all four of the inequalities is as follows. Let p' be the permutation obtained from our $1(l+1)l \ldots 2$- optimal permutation p by replacing its last two layers with a last layer of length m' and a next-to-last layer of length b'. Then in passing from p to p' we lose some $1(l+1)l \ldots 2$-patterns and gain some. Since p was optimal, the number lost must be at least as large as the number gained. And this inequality can be manipulated to give the one desired.

For the details, the following chart gives the relevant information to describe p' for each of the four inequalities. In the second case, the last two layers of p are combined into one, so the value of b'

is irrelevant.

m'	b'	number of gained $1(l+1)l\dots 2$-patterns $\leq$ number of lost $1(l+1)l\dots 2$-patterns
b	m	$m\binom{b}{l} \leq b\binom{m}{l}$
$b+m$	—	$a\left(\binom{m+b}{l} - \binom{m}{l} - \binom{b}{l}\right) \leq b\binom{m}{l}$
$m+1$	$b-1$	$(a+b-1)\binom{m}{l-1} \leq a\binom{b-1}{l-1} + \binom{m}{l}$
$m-1$	$b+1$	$a\binom{b}{l-1} + \binom{m-1}{l} \leq (a+b)\binom{m-1}{l-1}$

Now (i) follows easily by cancelling $bm/l!$ from the inequality in the first row of the table.

From the second line of the table, we have

$$ab\binom{m}{l-1} \leq a\sum_{i=1}^{l-1}\binom{b}{i}\binom{m}{l-i} = a\left(\binom{m+b}{l} - \binom{m}{l} - \binom{b}{l}\right) \leq b\binom{m}{l},$$

and cancelling $b\binom{m}{l-1}$, which is not zero because $m \geq l$, gives us (ii).

To prove (iii) we induct on n. If $n = l+1$, then we must have $p = 1(l+1)l\dots 2$, so $k = 1 < (l+2)/l = (n+1)/l$. Now we assume $n > l+1$.

If $k < l+1$, then the leftmost k elements of p contain no copies of $1(l+1)l\dots 2$, so we may replace them with any k-permutation and still have p optimal. Therefore we may pick $b = 1$ and $a = k-1$, and thus the second row of the table shows

$$\frac{l(k-1)}{m-l+1}\binom{m}{l} = (k-1)\binom{m}{l-1} = (k-1)\left(\binom{m+1}{l} - \binom{m}{l} - \binom{1}{l}\right) \leq \binom{m}{l},$$

so $k \leq (m+1)/l \leq (n+1)/l$, as desired.

If $k \geq l+1$, recall that from Proposition 3.2, the leftmost $k = a+b$ elements of p form a $1(l+1)l\dots 2$-optimal permutation, so we may, without loss, choose a maximal and thus assume that $a = k_k$.

From the third line of the chart, we have

$$\frac{l(k-1)}{m-l+1}\binom{m}{l} = (a+b-1)\binom{m}{l-1} \leq a\binom{b-1}{l-1} + \binom{m}{l}.$$

Using (i) we get that $\binom{b-1}{l-1} \leq \binom{m-1}{l-1} = \frac{l}{m}\binom{m}{l}$. Substituting this in the previous equation, cancelling $\binom{m}{l}$, and solving for k gives

$$k \leq \frac{m+1}{l} + \frac{a(m-l+1)}{m}$$

Since $k \geq l+1$, we have by induction that $a = k_k < k/l$. Substituting and solving for k again and then cancelling $m+1$, we get $k < \frac{m}{l-1}$. A final substitution of $m = n-k$ results in (iii).

For (iv), notice that the last row of the table gives

$$\binom{m-1}{l} \leq a\binom{b}{l-1} + \binom{m-1}{l} \leq (a+b)\binom{m-1}{l-1} = (n-m)\binom{m-1}{l-1}. \tag{4}$$

so cancelling $\binom{m-1}{l-1}$ gives $n-m \geq (m-l)/l$, which can be converted to the desired inequality. $\diamond$

We now turn to the proof of Theorem 4.1. First note that, by Lemma 5.1 (iv), we have

$$k = n - m \geq \frac{n-l}{l+1}. \tag{5}$$

So $(k_n)_{n\geq 1}$ clearly diverges to infinity. For our next step, we prove that $(k_n)_{n\geq 1}$ is monotonically weakly increasing. The following definition and notation will be useful in this task.

Definition 5.2 *Let $p_{n,i}$ denote an n-permutation whose last layer is of length $n-i$, and whose leftmost i elements form a $1(l+1)l\ldots 2$-optimal i-permutation, and let $c_{n,i} = c_{1(l+1)l\ldots 2}(p_{n,i})$.*

Note that

$$c_{n,i} = M_i + i\binom{n-i}{l}.$$

Proposition 5.3 *For $q = 1(l+1)l\ldots 2$ and all integers $n \geq l+1$, we have $k_n \leq k_{n+1}$.*

Proof: Let $k = k_n$. It suffices to show that $c_{n+1,k} > c_{n+1,i}$ for all $i < k$. This is equivalent to showing that

$$M_k + k\binom{n-k+1}{l} > M_i + i\binom{n-i+1}{l}. \tag{6}$$

However, by definition of k, we know that for all $i < k$,

$$M_k + k\binom{n-k}{l} \geq M_i + i\binom{n-i}{l}. \tag{7}$$

Subtracting (7) from (6), we are reduced to proving $k\binom{n-k}{l-1} > i\binom{n-i}{l-1}$. We will induct on $k-i$. If $k-i=1$, then we would like to show that

$$\frac{k(n-k-l+2)}{n-k+1}\binom{n-k+1}{l-1} = k\binom{n-k}{l-1} > (k-1)\binom{n-k+1}{l-1},$$

so it suffices to show that $k < (n+1)/l$, which follows from Lemma 5.1 (iii).

For $k-i>1$ we have, by induction, that $k\binom{n-k}{l-1} > (i+1)\binom{n-i-1}{l-1}$, so it suffices to show that

$$\frac{(i+1)(n-i-l+1)}{(n-i)}\binom{n-i}{l-1} = (i+1)\binom{n-i-1}{l-1} > i\binom{n-i}{l-1},$$

which simplifies to $(i+1) < (n+1)/l$, and this is is true because $i+1 \leq k$. $\diamond$

The proof of the upper bound on k_{n+1} is a bit more involved but follows the same general lines as the previous demonstration. Note that this will finish the proof of Theorem 4.1.

Lemma 5.4 *For $q = 1(l+1)l \dots 2$ and all integers $n \geq l+1$, we have $k_n \leq k_{n+1} \leq k_n + 1$.*

Proof: Induct on n. The lemma is true for $n = l+1$ since $k_{l+1} = k_{l+2} = 1$. Suppose the lemma is true for integers smaller than or equal to n, and prove it for $n+1$. For simplicity, let $k = k_n$, $m = n - k$, and $c_i = c_{n+1,i}$. Since we have already proved the lower bound, it suffices to show that

$$c_i \geq c_{i+1} \text{ for } k+1 \leq i < \left\lfloor \frac{n+1}{l} \right\rfloor \text{ with strict inequality for } i = k+1. \tag{8}$$

Note that we do not have to consider $i \geq \lfloor (n+1)/l \rfloor$ because of Lemma 5.1 (iii).

We prove (8) by induction on i. For the base case, $i = k+1$, we wish to show

$$M_{k+1} + (k+1)\binom{m}{l} > M_{k+2} + (k+2)\binom{m-1}{l}. \tag{9}$$

But since $p_{n,k}$ is optimal by assumption, we have

$$M_k + k\binom{m}{l} > M_{k+1} + (k+1)\binom{m-1}{l}. \tag{10}$$

Subtracting (10) from (9) and rearranging terms, it suffices to prove

$$\binom{m-1}{l-1} \geq (M_{k+2} - M_{k+1}) - (M_{k+1} - M_k). \tag{11}$$

First, if $k < l+1$, then (11) is easy to verify using Lemma 5.1 (iii) and the values $M_{l+2} = l+1$, $M_{l+1} = 1$, and $M_k = 0$ for $k \leq l$. Therefore we may assume that $k \geq l+1$. Let $p' \in S_k$, $p'' \in S_{k+1}$, and $p''' \in S_{k+2}$ be layered $1(l+1)l \dots 2$-optimal permutations having last layer lengths m', m'', and m''', respectively, as short as possible. Also let $k' = k - m'$, $k'' = k+1-m''$, and $k''' = k+2-m'''$. We would like to be able to assume the lemma holds for these permutations, and thus we would like to have $k+2 \leq n$. But by Lemma 5.1 (iii) we have $k+2 < n/2+2 \leq n$ if $n \geq 4$. Since $n \geq l+1$ this holds for $l \geq 3$ and the case $l = 2, n = 3$ is easy to check directly. Therefore we may assume that p', p'', and p''' all satisfy the lemma.

If $m'' = m'+1$ then let x be the largest element in the last layer of p'' (namely $x = k+1$). Otherwise, $m'' = m'$ and removing the last layer of both p' and p'' leaves permutations in $S_{k-m'}$ and $S_{k-m'+1}$, respectively. So we can iterate this process until we find the single layer where p' and p'' have different lengths (those lengths must differ by 1) and let x be the largest element in that layer of p''. Similarly we can find the element y which is largest in the unique layer were p'' and p''' have different lengths.

Now let

$$\begin{aligned}
r &= \text{the number of } 1(l+1)l\dots 2\text{-patterns in } p''' \text{ containing neither } x \text{ nor } y,\\
s &= \text{the number of } 1(l+1)l\dots 2\text{-patterns in } p''' \text{ containing } x \text{ but not } y,\\
t &= \text{the number of } 1(l+1)l\dots 2\text{-patterns in } p''' \text{ containing } y \text{ but not } x, \text{ and}\\
u &= \text{the number of } 1(l+1)l\dots 2\text{-patterns in } p''' \text{ containing both } x \text{ and } y.
\end{aligned}$$

Note that there is a bijection between the $1(l+1)l\ldots2$-patterns of p''' not containing y and the $1(l+1)l\ldots2$-patterns of p''. A similar statement holds for p'' and p'. So

$$M_k = r, \quad M_{k+1} = r+s, \quad M_{k+2} = r+s+t+u.$$

Note also that $s \geq t$ because increasing the length of the layer of x results in the most number of $1(l+1)l\ldots2$-patterns being added to p'. It follows that $(M_{k+2}-M_{k+1})-(M_{k+1}-M_k) = t+u-s \leq u$.

By Lemma 5.1 (iii), $k < m$, so to obtain (11) it suffices to show that $u \leq \binom{k}{l-1}$. But $\binom{k}{l-1}$ is the total number of subsequences of p''' having length $l+1$ and containing x and y. So the inequality follows.

The proof of the induction step is similar. Assume that (8) is true for $i-1$ so that

$$M_{i-1} + (i-1)\binom{r+1}{l} \geq M_i + i\binom{r}{l}. \tag{12}$$

where $r = n+1-i$. We wish to prove

$$M_i + i\binom{r}{l} \geq M_{i+1} + (i+1)\binom{r-1}{l}. \tag{13}$$

Subtracting as usual and simplifying, we need to show

$$2\binom{r-1}{l-1} - (i-1)\binom{r-1}{l-2} \geq (M_{i+1}-M_i) - (M_i - M_{i-1}).$$

Proceeding exactly as in the base case, we will be done if we can show that

$$\frac{2r-l-il+i+1}{r-l+1}\binom{r-1}{l-1} = 2\binom{r-1}{l-1} - (i-1)\binom{r-1}{l-2} \geq \binom{i-1}{l-1}.$$

Because $i < \lfloor \frac{n+1}{l} \rfloor$ we have $r \geq i$, so it suffices to show that

$$\frac{2r-l-il+i+1}{r-l+1} \geq 1.$$

This simplifies to showing that $i \leq (r+i)/l = (n+1)/l$, and this is guaranteed by our choice of i. $\diamond$

The following lemma contains two inequalities essentially shown in the proof of Lemma 5.4 which we will need to use again.

Lemma 5.5 *If $q = 1(l+1)l\ldots2$ then $0 \leq (M_{i+2}-M_{i+1}) - (M_{i+1}-M_i) \leq \binom{i}{l-1}$.*

Proof: For the upper bound, recall that $\binom{i}{l-1}$ is the total number of subsequences of p''' of length $l+1$ containing x and y while the double difference just counts those subsequences corresponding to the pattern $q = 1(l+1)l\ldots2$. For the lower bound, we showed that

$$(M_{i+2}-M_{i+1}) - (M_{i+1}-M_i) = t+u-s.$$

Recall that $t+u$ is the total contribution of y in p''', and s is the total contribution of x in p''. Therefore $t+u-s \geq 0$, as otherwise one could create a permutation with more $1(l+1)l\ldots2$-patterns than p''' by inserting a new element in the same layer as x. $\diamond$

6 The sequence $(c_{n,i})_{i=1}^{n-1}$ for $q = 1(l+1)l\dots 2$

Now that we have completed the proof of Theorem 4.1, we turn our attention to the tools which will enable us to show that there are infinitely many IZ integers. As before, all invariants are for $q = 1(l+1)l\dots 2$ unless otherwise stated.

For $l = 2$, we will need the following lemma.

Lemma 6.1 *For all n, we have $M_{n+1,132} - M_{n,132} \leq 5n^2/16$.*

Proof: Let $k = k_n$. We induct on n. It is easy to check the base cases $n = 1, 2$. Note that by Theorem 4.1, either $k_{n+1} = k$ or $k_{n+1} = k+1$. If $k_{n+1} = k$, then we have

$$M_{n+1} - M_n = nk - k^2$$

and maximizing this as a function of k gives

$$M_{n+1} - M_n \leq \frac{n^2}{4} \leq \frac{5n^2}{16}.$$

If $k_{n+1} = k+1$, then we have

$$M_{n+1} - M_n = M_{k+1} - M_k + \binom{n-k}{2}.$$

By induction, we have $M_{k+1} - M_k \leq \frac{5k^2}{16}$, and thus we have that

$$M_{n+1} - M_n \leq \frac{13k^2}{16} + \frac{n^2 + k - n - 2kn}{2}.$$

By Lemma 5.1 (iii) and (iv), this function is to be maximized on the interval $[(n-2)/3, n/2)$ and for $n \geq 3$ this maximum occurs at $k = (n-2)/3$. So

$$M_{n+1} - M_n \leq \frac{37n^2 - 4n + 4}{144} \leq \frac{5n^2}{16},$$

as desired. $\diamond$

Definition 6.2 *For $q = 1(l+1)l\dots 2$ and any positive integer n, let l_n be the least integer i greater than k_n such that $c_{n,i} \leq c_{n,i+1}$. If there is no integer with this property, let $l_n = n-1$.*

Do not confuse l_n, which will always be subscripted, with the length-related parameter l , which will never be. Our next result shows that the sequence $(c_{n,i})_{i=1}^{n-1}$ is "bimodal" with a maximum at $i = k_n$ and a minimum at $i = l_n$.

Theorem 6.3 *For $q = 1(l+1)l\ldots 2$ and all positive integers n, we have the following three results about the shape of $(c_{n,i})_{i=1}^{n-1}$*

(i) $c_{n,i} \le c_{n,i+1}$ *for all* $i < k_n$,

(ii) $c_{n,i} \ge c_{n,i+1}$ *for all* $k_n \le i < l_n$,

(iii) $c_{n,i} \le c_{n,i+1}$ *for all* $i \ge l_n$.

Proof: For (i) we induct on n. The claim is true trivially for $n < l+1$ since then $c_{n,i} = 0$ for all i, so we will assume $n \ge l+1$. If $i = k_n - 1$ then the claim is true by definition. If $i < k_n - 1$ then $i < k_{n-1}$ by Theorem 4.1 and we are able to apply induction. We would like to show that

$$M_i + i\binom{n-i}{l} \le M_{i+1} + (i+1)\binom{n-i-1}{l}$$

and we know by induction that

$$M_i + i\binom{n-i-1}{l} \le M_{i+1} + (i+1)\binom{n-i-2}{l}.$$

Subtracting as usual, we are reduced to showing that $i\binom{n-i-1}{l-1} \le (i+1)\binom{n-i-2}{l-1}$. This further reduces to $i \le (n-l)/l$ which is true by Lemma 5.1 (iii) and the fact that $i < k_n - 1$.

Statement (ii) is implied by the definition of l_n, so we are left with (iii). By the definition of l_n we have that $c_{n,l_n} \le c_{n,l_n+1}$, so it suffices to show that for all $i \ge l_n$, if $c_{n,i} \le c_{n,i+1}$ then $c_{n,i+1} \le c_{n,i+2}$. Subtracting in the usual way, we are reduced to showing that

$$(M_{i+2} - M_{i+1}) - (M_{i+1} - M_i) \ge \frac{2n - 2l - i(l+1)}{l-1}\binom{n-i-2}{l-2}. \tag{14}$$

Since we know that $(M_{i+2} - M_{i+1}) - (M_{i+1} - M_i) \ge 0$ by Lemma 5.5, our approach will be to show that $2n - 2l - i(l+1) \le 0$ for $i \ge l_n$ by showing that

$$l_n \ge (2n - 2l)/(l+1). \tag{15}$$

Before we prove (15), we will need the following two facts.

$$l_n \ge n/l \text{ and } l_n \ge l_{n-1}.$$

The first fact follows from our proof of Lemma 5.4, in which we showed that $c_{n,i} \ge c_{n,i+1}$ for $k_n \le i < \lfloor n/l \rfloor$. So to prove the second fact, it suffices to show that $c_{n-1,i} > c_{n-1,i+1}$ implies $c_{n,i} > c_{n,i+1}$ for $i \ge n/l$. This is proved in exactly the same way as (i) with all the inequalities reversed.

Now we are ready to prove (15). First we tackle the case where $l \ge 3$ by induction. If $n \le 3$ then $(2n - 2l)/(l+1) \le 0$ and we are done. So suppose $n \ge 4$. If $l_{n-2} > (n-1)/2$, then since $l_n \ge l_{n-2}$ and

$l \geq 3$ we have $l_n > (2n-2l)/(l+1)$ as desired. Hence we may assume that $(n-2)/l \leq l_{n-2} \leq (n-1)/2$. In this case we claim that $l_n \geq l_{n-2}+1$, which will imply (15) by induction.

Let $i = l_{n-2}$. We want to show that

$$M_i + i\binom{n-i}{l} > M_{i+1} + (i+1)\binom{n-i-1}{l},$$

and we have

$$M_{i-1} + (i-1)\binom{n-i-1}{l} > M_i + i\binom{n-i-2}{l}.$$

Subtracting, it suffices to show that

$$(M_{i+1} - M_i) - (M_i - M_{i-1}) \leq i\binom{n-i-2}{l-2}.$$

By Lemma 5.5, $(M_{i+1} - M_i) - (M_i - M_{i-1}) \leq \binom{i-1}{l-1}$, so it suffices to show that

$$\binom{i-1}{l-1} \leq \frac{il-i}{n-i-l}\binom{n-i-2}{l-1}. \tag{16}$$

Since $i \geq (n-2)/l$, we have that $(il-i)/(n-i-l) \geq 1$, and since $i \leq (n-1)/2$, we have that $n-i-2 \geq i-1$, so (16) is true, and thus (15) holds.

For the case where $l = 2$, we examine the quadratics

$$d_i(n) = \frac{1}{2}n^2 - \left(2i + \frac{3}{2}\right)n + \left(M_{i+1} - M_i + \frac{3}{2}i^2 + \frac{5}{2}i + 1\right),$$

which agree with $c_{n,i+1} - c_{n,i}$, wherever both $c_{n,i+1}$ and $c_{n,i}$ are defined. We will also need to refer to the roots of $d_i(n)$, which occur at

$$r_i = 2i + \frac{3}{2} - \sqrt{i^2 + i + \frac{1}{4} - 2\left(M_{i+1} - M_i\right)}, \text{ and}$$

$$s_i = 2i + \frac{3}{2} + \sqrt{i^2 + i + \frac{1}{4} - 2\left(M_{i+1} - M_i\right)}.$$

Lemma 6.1 gives us that

$$r_i < (2 - \sqrt{3/8})i + 3/2, \tag{17}$$

so r_i and s_i are real numbers and for $i > 13$, $r_i < 3i/2$. These roots are important in our situation for the following reasons:

$$d_i(n) < 0 \text{ if and only if } r_i < n < s_i, \tag{18}$$

$$\text{for } i > 13 \text{ we have } n \geq s_i \text{ if and only if } i \leq k_n, \text{ and} \tag{19}$$

$$\text{for } l_n > 13 \text{ we have } n \leq r_{l_n}. \tag{20}$$

Statement (18) is easily verified. Assume to the contrary that the forward direction of (19) is not true, and thus $n \geq s_i$ but $i > k_n$. Let n' be such that $k_{n'} = i$. By Proposition 5.3, we have that $n' > n \geq s_i$, and thus $d_i(n') \geq 0$ by (18). However because $i = k_{n'}$, we have that $d_i(n') < 0$, a contradiction. To prove the reverse direction of (19), notice that if $i \leq k_n$ then by (i) and the definition of k_n, we must have that $d_i(n) \geq 0$. Therefore by (18), either $n \geq s_i$ (as we would like) or $n \leq r_i$, and by (17), it cannot be the case that $n \leq r_i$, as that would imply that $n \leq r_i < 3i/2 < 3k_n/2$ if $i > 13$, contradicting Lemma 5.1 (iii).

To prove (20), note that by (18) we cannot have $r_{l_n} < n < s_{l_n}$ as then we would have $d_{l_n}(n) < 0$, contradicting the definition of l_n. Also, we cannot have $n \geq s_{l_n}$ as then we would have $l_n \leq k_n$ by (19), again contradicting the definition of l_n. Hence we must have (20).

With these tools, (15) is easy to prove; we have $n \leq r_{l_n} < 3l_n/2$ for $l_n > 13$, and thus $l_n > 2n/3$, as desired. It is easily checked that $l_n > 2n/3$ for $l_n \leq 13$. $\diamond$

We will depend on the following lemma to find integers n with an internal zero at $M_n - 1$.

Lemma 6.4 *For $q = 1(l+1)l\ldots2$, $l \geq 2$ and all $n \geq 2l+2$, if $k_{n-2} = k-1$ and $k_{n-1} = k$, then $M_n - c_{n,i} > 1$ for all $i \neq k$, so in particular, $k_n = k$.*

Proof: By Theorem 6.3 it suffices to show the following inequalities:

$$c_{n,k} - c_{n,k-1} > 1, \tag{21}$$

$$c_{n,k} - c_{n,k+1} > 1, \tag{22}$$

and

$$c_{n,k} - c_{n,n-1} > 1. \tag{23}$$

Statement (23) is clear for $n \geq 2l+2$ because $c_{n,n-1} = M_{n-1}$, $(M_i - M_{i-1}) \geq (M_{i-1} - M_{i-2})$ for all i by Lemma 5.5, and $M_{l+2} - M_{l+1} = l$.

We prove (22) by induction on n. First, if $k < l$, then $M_{k-1} = M_k = M_{k+1} = 0$, so it suffices to show that

$$k\binom{n-k}{l} > (k+1)\binom{n-k-1}{l} + 1,$$

and since $k_{n-2} = k-1$, we have

$$(k-1)\binom{n-k-1}{l} > k\binom{n-k-2}{l}.$$

Subtracting that latter from the former, it suffices to show that

$$1 \leq k\binom{n-k-2}{l-2}.$$

So we're done in this case since $n - k \geq l$ which follows from $n \geq 2l+2$ and $k < l$.

Now assume that $k \geq l$, so we may prove (22) by showing the stronger statement that

$$c_{n,k} - c_{n,k+1} > \binom{k-2}{l-2},$$

and thus we would like to show that

$$M_k + k\binom{n-k}{l} > M_{k+1} + (k+1)\binom{n-k-1}{l} + \binom{k-2}{l-2},$$

and as $k_{n-2} = k-1$, we have

$$M_{k-1} + (k-1)\binom{n-k-1}{l} > M_k + k\binom{n-k-2}{l}.$$

Subtracting as usual, we are reduced to showing

$$(M_{k+1} - M_k) - (M_k - M_{k-1}) \leq k\binom{n-k-2}{l-2} - \binom{k-2}{l-2}.$$

By Lemma 5.1 (iii)

$$k\binom{n-k-2}{l-2} - \binom{k-2}{l-2} > k\binom{k-2}{l-2} - \binom{k-2}{l-2} = (l-1)\binom{k-1}{l-1}.$$

The upper bound in Lemma 5.5 now completes the proof of (22).

To prove (21), we want to show

$$M_k + k\binom{n-k}{l} > M_{k-1} + (k-1)\binom{n-k+1}{l} + 1,$$

and we are given

$$M_k + k\binom{n-k-1}{l} \geq M_{k-1} + (k-1)\binom{n-k}{l}.$$

Subtracting as usual, we are reduced to showing that

$$k\binom{n-k-1}{l-1} > (k-1)\binom{n-k}{l-1} + 1.$$

Cancelling $\binom{n-k-1}{l-1}$ and simplifying, it suffices to show that

$$n > lk + \frac{n-k-l+1}{\binom{n-k-1}{l-1}}. \tag{24}$$

By Lemma 5.1 (iii), $n \geq lk+1$, so it suffices to show that

$$n-k-l+1 < \binom{n-k-1}{l-1},$$

which is true for $l \geq 3$. For $l = 2$, note that proving (24) reduces to showing $n > 2k+1$. But $k_{n-1} = k$, so $k < (n-1)/2$ by Lemma 5.1 (iii) which is equivalent to the desired inequality. ◇

7 The poset connection

There is an intimate connection between partially ordered sets, called posets for short, and permutations. Using this connection, we will provide characterizations of all n-permutations p which have $c_{1(l+1)l\ldots2}(p) \geq M_{n,1(l+1)l\ldots2} - 1$ for $l \geq 2$. This will provide us with the tools we need to show that there are an infinite number of IZ sequences for each of these patterns. Any necessary definitions from the theory of posets that are not given here will be found in Stanley's text [13].

If P is a poset such that any two distinct elements of P are incomparable we say that P is an *antichain*. Since there is a unique unlabeled antichain on n elements, we denote this poset by $\mathcal{A}_n$.

Given posets P and Q, the *ordinal sum* of P and Q, denoted $P \oplus Q$, is the unique poset on the elements $P \cup Q$ where $x \leq y$ in $P \oplus Q$ if either

(i) $x, y \in P$ with $x \leq y$,

(ii) $x, y \in Q$ with $x \leq y$, or

(iii) $x \in P$ and $y \in Q$.

A poset P is *layered* if it is an ordinal sum of antichains, i.e. if $P = \mathcal{A}_{p_1} \oplus \mathcal{A}_{p_2} \oplus \cdots \oplus \mathcal{A}_{p_k}$ for some $p_1, \ldots, p_k$. To introduce a related notion, let $\max P$ denote the set of maximal elements of P and $\overline{P} = P \setminus (\max P)$. Then P is *LOT (layered on top)* if $P = \overline{P} \oplus \max P$. Note that if P is layered then it is LOT, but not conversely.

If $p = p_1 p_2 \ldots p_n$ is a permutation, then the corresponding poset P_p has elements $p_1, p_2, \ldots, p_n$ with partial order $p_i < p_j$ if (p_i, p_j) is a noninversion in p. So, for example, $P_{12\ldots n}$ is a chain, $P_{n\ldots21} = \mathcal{A}_n$ and $P_{1(l+1)l\ldots2} = \mathcal{A}_1 \oplus \mathcal{A}_l$. Clearly not every poset is of the form P_p for some p. In fact, the P_p are exactly the posets of dimension at most 2, being the intersection of the total orders $1 < 2 < \cdots < n$ and $p_1 < p_2 < \cdots < p_n$.

Given posets P and Q let

$$c_Q(P) = \text{the number of induced subposets of } P \text{ isomorphic to } Q.$$

Also, if $S \subseteq P$ then let

$$\begin{aligned} c_Q(P; S) &= \text{the number of induced } Q' \subseteq P \text{ with } Q' \cong Q \text{ and } S \cap Q' \neq \emptyset, \\ c_Q(P; \text{not } S) &= \text{the number of induced } Q' \subseteq P \text{ with } Q' \cong Q \text{ and } S \cap Q' = \emptyset. \end{aligned}$$

We will freely combine these notations and eliminate the subscript when talking about a fixed poset Q. So, for example,

$$c_Q(P; S, T) = \text{the number of induced } Q' \subseteq P \text{ with } Q' \cong Q \text{ and } S \cap Q', T \cap Q' \neq \emptyset.$$

We will also abbreviate $c_Q(P; \{x\})$ to $c_Q(P; x)$ and $c_Q(P; \text{not}\{x\})$ to $c_Q(P; \text{not } x)$, $c_Q(P; \{x\}, \{y\})$ to $c_Q(P; x, y)$, etc.

As with permutations, for any non-negative integer n we will let $M_{n,Q} = \max\{c_Q(P) : |P| = n\}$. We will say a poset P is Q-optimal if $c_Q(P) = M_{|P|,Q}$.

Stromquist proved Theorem 3.1 by first demonstrating the following stronger result.

Theorem 7.1 ([14]) *If Q is a LOT pattern, then there is some Q-optimal LOT poset P. The same holds with "LOT" replaced by "layered."*

To show that the sequences of the patterns $1(l+1)l\ldots 2$, for $l \geq 2$, have infinitely many IZ integers, we will need to know more about $\mathcal{A}_1 \oplus \mathcal{A}_l$-optimal posets. The best possible case would be if all (sufficiently large) $\mathcal{A}_1 \oplus \mathcal{A}_l$-optimal posets were layered. This is true for the pattern $P_{132} = \mathcal{A}_1 \oplus \mathcal{A}_2$, but not in general. For example, it can be computed that $P_{231} \oplus \mathcal{A}_8$ is $\mathcal{A}_1 \oplus \mathcal{A}_3$-optimal, but $P_{231} \oplus \mathcal{A}_8$ is not layered. Fortunately, we are able to show that all $\mathcal{A}_1 \oplus \mathcal{A}_l$-optimal posets are of the following slightly more general form.

Definition 7.2 *We say $P = P_1 \oplus P_2$ is an l-decomposition of P if P_2 is layered and for all $A \subseteq P$ with $A \cong \mathcal{A}_1 \oplus \mathcal{A}_l$ we have $|A \cap P_1| \leq 1$.*

The first part of this section concerns the proof of the following theorem.

Theorem 7.3 *If P is an $\mathcal{A}_1 \oplus \mathcal{A}_l$-optimal poset then P has an l-decomposition.*

After this proof we will investigate 'almost' $\mathcal{A}_1 \oplus \mathcal{A}_l$-optimal posets, that is, posets P with $c_{\mathcal{A}_1 \oplus \mathcal{A}_l}(P) = M_{|P|,\mathcal{A}_1 \oplus \mathcal{A}_l} - 1$.

If q and p are permutations, it is generally not the case that $c_{P_q}(P_p) = c_q(p)$. For example, $P_{231} \cong P_{312}$ and thus $c_{P_{231}}(P_{312}) = 1$, but $c_{231}(312) = 0$. However, there is an important case in which we do get equality.

Lemma 7.4 *If q and p are permutations then $c_q(p) \leq c_{P_q}(P_p)$. Furthermore, if either q or p is layered then $c_q(p) = c_{P_q}(P_p)$.*

Proof: The inequality follows from the fact that each copy of q in p gives rise to a copy of P_q in P_p. For the equality, if q is layered then it is the unique permutation giving rise to the poset P_q. So every copy of P_q in P_p corresponds to a copy of q in p and we are done. The only other case we need to consider is if p is layered and q is not. But then both sides of the equality are zero. ◇

This lemma and the preceding theorems imply several important features about the connection between pattern matching in posets and permutations. Given any pattern q, the first statement in Lemma 7.4 implies that $M_{n,q} \leq M_{n,P_q}$ for all n. If q is layered, then by Theorem 7.1 there is a layered P_q-optimal poset $P = \mathcal{A}_{p_1} \oplus \mathcal{A}_{p_2} \oplus \cdots \oplus \mathcal{A}_{p_k}$ for some positive integers $p_1, \ldots, p_k$. It follows that there is a layered permutation p such that $P_p \cong P$, namely p is the permutation whose layer lengths from left to right are $p_1, \ldots, p_k$. By the preceding lemma, $c_q(p) = c_{P_q}(P)$, so $M_{|P|,q} = M_{|P|,P_q}$.

Lemma 7.5 *For all patterns Q, the sequence $(M_{n,Q})_{n\geq|Q|}$ is positive and strictly increasing.*

Proof: We will write M_n for $M_{n,Q}$ and $c(P)$ for $c_Q(P)$. Given $n \geq |Q|$, it is easy to construct a poset P with $c(P) > 0$. So let P be a Q-optimal poset. Now there must be some $x \in P$ with $c(P;x) > 0$. Now adjoin an element y to P to form a poset P' with $a < b$ in P' if either

(i) $a, b \in P$ with $a < b$,

(ii) $a = y$, $b \in P$ with $x < b$, or

(iii) $b = y$, $a \in P$ with $a < x$.

Then

$$c(P') = c(P'; \text{not } y) + c(P'; y) = c(P) + c(P'; y) \geq c(P) + c(P; x) > c(P)$$

so $M_{n+1} \geq c(P') > c(P) = M_n$. $\diamond$

We now begin the proof of Theorem 7.3 by making a few definitions. If P is a poset and $x \in P$ then the *open down-set* generated by x is

$$P_{<x} = \{y \in P \ : \ y < x\}.$$

If $x, y \in \max P$ then let $P^{x\to y}$ be the unique poset on the same set of elements which satisfies

$$P^{x\to y}_{<z} = P_{<z} \text{ for } z \neq x \text{ and } P^{x\to y}_{<x} = P_{<y}.$$

Note that $P - x = P^{x\to y} - x$. The following lemma is essentially in Stromquist [14], but is not explicitly proved there, so we will provide a demonstration.

Lemma 7.6 *Let Q be a LOT pattern and P be any poset with $x, y \in \max P$. Then*

$$c_Q(P^{x\to y}) \geq c_Q(P) + c_Q(P; y) - c_Q(P; x).$$

Proof: As before, we write $c(P)$ for $c_Q(P)$. Since

$$\begin{aligned} c(P) &= c(P; \text{not } x) + c(P; x, \text{not } y) + c(P; x, y), \text{ and} \\ c(P^{x\to y}) &= c(P^{x\to y}; \text{not } x) + c(P^{x\to y}; x, \text{not } y) + c(P^{x\to y}; x, y), \end{aligned}$$

it is enough to show that

$$c(P^{x\to y}; \text{not } x) = c(P; \text{not } x), \tag{25}$$

$$c(P^{x\to y}; x, \text{not } y) \geq c(P; x, \text{not } y) + c(P; y) - c(P; x), \tag{26}$$

$$c(P^{x\to y}; x, y) \geq c(P; x, y). \tag{27}$$

First, (25) is clear since P and $P^{x\to y}$ agree on all subsets not including x.

Next, notice that

$$c(P; x, \text{not}\, y) + c(P; y) - c(P; x) = c(P; y, \text{not}\, x),$$

and thus to prove (26), it suffices to show that $c(P^{x\to y}; x, \text{not}\, y) \geq c(P; y, \text{not}\, x)$, but this is easy. Let $A \subseteq P$ with $y \in A$, $x \notin A$, and $(A, \leq) \cong Q$. Then $A' = A \cup \{x\} - y$ is an occurrence of Q in $P^{x\to y}$, i.e., $(A', \leq_{P^{x\to y}}) \cong Q$, so (26) is proved.

Finally, to prove (27), let $A \subseteq P$ be an occurrence of Q in P which contains x and y, i.e., $(A, \leq_P) \cong Q$. Then we have that $(A, \leq_{P^{x\to y}}) \cong Q$ as well. This is because $A_{<x} = A_{<y}$ in P since x, y are maximal and Q is LOT. So A forms an occurance of Q in $P^{x\to y}$, and thus (27) is proven. $\diamond$

For the rest of this section, let $Q_l = \mathcal{A}_1 \oplus \mathcal{A}_l$, $c(P) = c_{Q_l}(P)$ and $M_n = M_{n,Q_l}$.

Lemma 7.7 *Let P be a poset such that $|P| > l \geq 2$. If for some $a \geq 0$ and $x \in \max P$ we have*

(a) $P - x$ is LOT,

(b) $c_{Q_l}(P) = M_{|P|,Q_l} - a$, and

(c) $c_{Q_l}(P; x) = c_{Q_l}(P; y) - a$ for all $y \in \max P \setminus x$,

then P is LOT (and thus a is actually 0).

Proof: Choose $y \in \max P$ with $y \neq x$, let $m = |\max P|$ and $k = |\overline{P}| = |P| - m$. (If no such y exists, then then result is trivial.) First consider what happens when $m < l$. Then (a) implies that $c(P; y) = 0$ for all $y \in \max P \setminus x$. This forces $c(P; x) = -a = 0$ by (c). Now (b) yields $c(\bar{P}) = c(P) = M_{|P|}$, contradicting Lemma 7.5. So we may assume $m \geq l$.

Note that $c(P) = c(P-x) + c(P; x)$, and since $c(P; x) = c(P; y) - a = c(P; x, y) + c(P; \text{not}\, x, y) - a = c(P; x, y) + c(P - x; y) - a$ we get that

$$c(P) = c(P - x) + c(P - x; y) + c(P; x, y) - a.$$

Furthermore, since $P - x$ is LOT we get that

$$c(P - x) = c(\overline{P}) + \binom{m-1}{l} k,$$

and

$$c(P - x; y) = \binom{m-2}{l-1} k.$$

Also, since $P_{<x} \subseteq \overline{P} = P_{<y}$, we have that

$$c(P; x, y) = \binom{m-2}{l-2} |P_{<x}|, \text{ so}$$

$$c(P) = c(\overline{P}) + \left(\binom{m-1}{l}k + \binom{m-2}{l-1}k + \binom{m-2}{l-2}|P_{<x}|\right) - a. \tag{28}$$

Furthermore, since $P - x$ is LOT, $P^{x \to y}$ is LOT, so we have

$$c(P^{x\to y}) = c(\overline{P}) + \binom{m}{l}k.$$

Therefore

$$\begin{aligned} c(P^{x\to y}) - c(P) &= a + \left(\left(\binom{m}{l} - \binom{m-1}{l} - \binom{m-2}{l-1}\right)k - \binom{m-2}{l-2}|P_{<x}|\right) \\ &= a + \binom{m-2}{l-2}\left(k - |P_{<x}|\right). \end{aligned} \tag{29}$$

Furthermore, by Lemma 7.6 and assumptions (b) and (c) we have that $c(P^{x\to y}) \geq M_{|P|}$. So we must have $c(P^{x\to y}) = M_{|P|}$ and, by (b) again, $c(P^{x\to y}) - c(P) = a$. It follows that $\binom{m-2}{l-2}(k - |P_{<x}|) = 0$. Therefore since $m \geq l$ we have $\binom{m-2}{l-2} > 0$ and so $k = |P_{<x}|$. Also, because $P_{<x} \subseteq \overline{P}$, we have $P_{<x} = \overline{P}$ and thus P is LOT, as desired. $\diamond$

Definition 7.8 *For any poset P, let $\mu(P)$ be defined by*

$$\mu(P) = \max\{k \ : \ \text{there exists } S \subseteq \max P \text{ with } |S| = k \text{ such that if } x, y \in S \text{ then } P_{<x} = P_{<y}\}$$

Clearly $\mu(P) \leq |\max P|$, with equality if and only if P is LOT. It turns out that $\mu(P)$ is a useful statistic for induction. We now have all the necessary tools to prove Theorem 7.3.

Proof of Theorem 7.3: Notice that the claim is trivial for $|P| < l + 1$ as all posets on less than $l+1$ elements cannot have any Q_l-patterns and thus they have the trivial l- decomposition $P \oplus \emptyset$.

Assume to the contrary that the claim is not true and let P be a Q_l-optimal poset of least cardinality that does not have a LOT l-decomposition with $\mu(P)$ maximal over all such choices of P and $|P| \geq l + 1$. Let S be the set from Definition 7.8, $m = |\max P|$ and $k = |\overline{P}| = |P| - m$.

First, we claim that P is LOT. If not, then there is some element, say $x \in (\max P) \setminus S$. Also let $y \in S$. If $c(P; x) \neq c(P; y)$, then by Lemma 7.6 either $c(P^{x\to y}) > M_{|P|}$ or $c(P^{y\to x}) > M_{|P|}$, both contradictions, so $c(P; x) = c(P; y)$ and $P^{x\to y}$ is Q_l-optimal. Since $\mu(P^{x\to y}) > \mu(P)$, by our choice of P we know that $P^{x\to y}$ has an l-decomposition $P_1 \oplus P_2$.

If $P_2 = \emptyset$, then $c(P^{x\to y}) = 0$, so by Lemma 7.5, $|P| < l + 1$ (because $M_{l+1} = 1$), a contradiction to our choice of P.

Hence we may assume that $P_2 \neq \emptyset$, so $P^{x\to y}$ is LOT. As the only element P and $P^{x\to y}$ disagree on is x, we have that $P - x$ is LOT. Hence by Lemma 7.7, P is also LOT.

Now that we know that P is LOT, we get that $c(P) = c(\overline{P}) + \binom{m}{l}k$, so $\overline{P}$ is Q_l- optimal. By induction, $\overline{P}$ has an l-decomposition $\overline{P} = P_1 \oplus P_2$ and thus $P = P_1 \oplus (P_2 \oplus \max P)$ is an l- decomposition for P. $\diamond$

Note that by using the ideas in the last paragraph of this proof one may show that if $P = P_1 \oplus P_2$ is an l-decomposition for an Q_l-optimal poset P then $|P_1| < l+1$. Hence because all posets on less than three elements are layered, all P_{132}-optimal posets (and thus 132-optimal permutations) are layered. This observation will be useful in the following proof.

Theorem 7.9 *If P is such that $c_{Q_l}(P) = M_{|P|,Q_l} - 1$ then there is a poset Q with $|Q| = |P|$ and one of the following:*

(i) $c_{Q_l}(Q) = M_{|P|,Q_l} - 1$ and Q is LOT, or

(ii) Q is Q_l-optimal and $|\max Q| = l$, or

(iii) $l = 2$ and $|P| = 5$.

Proof: Assume that (i) does not hold and choose P with $c(P) = M_{|P|} - 1$ and $\mu(P)$ maximal over all such choices. Let $n = |P|$, $m = |\max P|$ and $k = |\overline{P}| = n - m$.

We must have $|c(P;x) - c(P;y)| \leq 1$ for all $x, y \in \max P$ as otherwise by Lemma 7.6 we would have either $c(P^{y \to x}) > M_n$ or $c(P^{x \to y}) > M_n$, a contradiction. Hence we have

$$\max\{c(P;x) - c(P;y) : x, y \in \max P\} \in \{0, 1\}.$$

First we tackle the easier case, where $\max\{c(P;x) - c(P;y) : x, y \in \max P\} = 1$. Pick two maximal elements of P, say $x, y \in \max P$, so that $c(P;y) - c(P;x) = 1$. By Lemma 7.6 we have that $c(P^{x \to y}) = M_n$, and thus by Theorem 7.3 we know $P^{x \to y}$ has an l-decomposition $P_1 \oplus P_2$. Since $c(P) = M_n - 1$, we must have $M_n > 0$, so we also have that $n \geq l + 1$ and $P_2 \neq \emptyset$. Therefore $P^{x \to y}$ and consequently $P - x$ are LOT. Hence by Lemma 7.7, P is LOT, a contradiction.

Now assume $\max\{c(P;x) - c(P;y) : x, y \in \max P\} = 0$. Let S be as in Definition 7.8, pick $x \in (\max P) \setminus S$ (x must exist as P is not LOT) and $y \in S$. Now $c(P;x) = c(P;y)$ and thus $c(P^{x \to y}) \geq M_n - 1$ by Lemma 7.6. However if $c(P^{x \to y}) = M_n - 1$ then we have contradicted our choice of P as $\mu(P^{x \to y}) > \mu(P)$. Therefore $c(P^{x \to y}) = M_n$ so by Theorem 7.3, $P^{x \to y}$ has an l-decomposition $P_1 \oplus P_2$. By the same reasoning as the previous case, $P_2 \neq \emptyset$, so again $P^{x \to y}$ and $P - x$ are both LOT.

Although we cannot apply Lemma 7.7 in this case, (29) still holds for P with $a = 0$, so

$$c(P^{x \to y}) - c(P) = 1 = \binom{m-2}{l-2} \left(k - |P_{<x}|\right).$$

Thus $P^{x \to y}$ is Q_l-optimal. Furthermore, we must have $\binom{m-2}{l-2} = 1$. If $l > 2$, this implies that $m = l$, so (ii) is true with $Q = P^{x \to y}$.

If $l = 2$ then we must have $k - |P_{<x}| = 1$, so there is precisely one element, say $z \in \overline{P} \setminus P_{<x}$. Since $P - x$ is LOT, z must lie in $\max \overline{P}$. Let $b = |\max \overline{P}|$. Then we have

$$c(P) = c(\overline{P}) + c(P - z; \max P) + c(P - x; z, \max P) + c(P; x, z) \tag{30}$$

Because $P - z$ is LOT, we have that $c(P - z; \max P) = \binom{m}{2}(k-1)$, and because $P - x$ is LOT we have that $c(P - x; z, \max P) = \binom{m-1}{2}$. Notice that because $P^{x \to y}$ is $\mathcal{A}_1 \oplus \mathcal{A}_2$-optimal, by the comment after the proof of Theorem 7.3, $P^{x \to y}$ is layered, and thus $\overline{P}$ is layered. Since the $\mathcal{A}_1 \oplus \mathcal{A}_2$-patterns in P containing both x and z are formed with exactly one element which lies in $P_{<z}$, $c(P; x, z) = k - b$. Finally, $c(P^{x \to y}) = c(\bar{P}) + \binom{m}{2}k$. Now combining all these c-values with equation (30) gives

$$c(P^{x \to y}) - c(P) = 1 = \binom{m}{2}k - \binom{m}{2}(k-1) - \binom{m-1}{2} - k + b, \tag{31}$$

so $k + 2 = b + m$. We have by Lemma 5.1 (iii) that $m > k$ and $b > k/2$ (this follows from the fact that $\overline{P}$ is layered and $\mathcal{A}_1 \oplus \mathcal{A}_2$-optimal), which forces $k \leq 3$. This in turn implies $|P| = k + m \leq 7$. Now it can be checked by direct computation that for $|P|$ in this range either the theorem is true vacuously or one of (i) to (iii) holds. $\diamond$

Theorem 7.10 *If there is an n-poset P with $c_{Q_l}(P) = M_{n,Q_l} - 1$ then there is an n-poset Q with $c_{Q_l}(Q) = M_{n,Q_l} - 1$ and*

(i) if $l > 2$ then Q is layered, or

(ii) if $l = 2$ then $Q = Q_1 \oplus Q_2$ where $|Q_1| \leq 5$ and Q_2 is layered.

Furthermore, in either case $Q = P_r \oplus \mathcal{A}_m$ for some permutation $r \in S_{n-m}$ and integer m which is positive unless $l = 2$ and $n = 5$.

Proof: Induct on n. If $n < l + 1$, then $M_n = 0$, so the theorem is true vacuously. If $n = l + 1$, then $M_n = 1$ and $c(\mathcal{A}_{l+1}) = 0 = M_n - 1$. Hence we may assume that $n > l + 1$.

If case (ii) of Theorem 7.9 is true, let Q be the poset guaranteed there, $k = |\overline{Q}|$ and $m = |\max Q| = l$. Then by Lemma 5.1 (iii), $k < (k + m)/l < 2m/m = 2$, so $n = k + m \leq l + 1$, a case we have already dealt with.

It is routine to check that the poset P_{15423} satisfies case (ii) of this theorem if case (iii) of Theorem 7.9 is true.

Therefore we may assume that case (i) of Theorem 7.9 is true, and thus there is a LOT n-poset Q so that $c(Q) = M_n - 1$. Since Q is LOT, $Q = \overline{Q} \oplus \max Q = \overline{Q} \oplus \mathcal{A}_m$. As $c(Q) = c(\overline{Q}) + k\binom{m}{l}$, we must have $c(\overline{Q}) \geq M_k - 1$. If $c(\overline{Q}) = M_k$, then by Theorem 3.1, there is some layered k-poset R so that $c(R) = M_k$, and thus $R \oplus \mathcal{A}_m$ is layered, $c(R \oplus \mathcal{A}_m) = M_n - 1$ and $R = P_r$ for some $r \in S_k$. If $c(\overline{Q}) = M_k - 1$, then by induction, there is some poset R, $|R| = k$, which satisfies this theorem. So $R \oplus \mathcal{A}_m$ is the desired poset. $\diamond$

Theorem 7.11 *For the pattern $q = 1(l+1)l \dots 2$, there are infinitely many IZ integers.*

Proof: Assume that the theorem is false. Since $S_6(M_6-1)=0$ for $l=2$ and $S_{l+2}(M_{l+2}-1)=0$ for $l\geq 3$, there must be some maximal $k\geq l+2$ so that $S_k(M_k-1)=0$. By Theorem 4.1, there is some n so that $k_{n-2}=k-1$ and $k_{n-1}=k$. Also note that since $k_n\geq k_{n-1}=k\geq l+2$, by Lemma 5.1 (iii) we have $n>lk_n\geq l(l+2)>2l+2$, so we may apply Lemma 6.4 to see that $k_n=k$.

By our choice of k, $S_n(M_n-1)\neq 0$, so there is some $p\in S_n$ so that $c(p)=M_n-1$. By Lemma 7.4, $c(P_p)=M_n-1$, and thus Theorem 7.10 produces a poset $Q=P_r\oplus\mathcal{A}_m$ for some $r\in S_{n-m}$ and integer m which is positive since $n>6$.

Let $\bar{k}=n-m$. By Theorem 7.1, there is a layered Q_l-optimal $\bar{k}$-poset R, and so we must have $c(R\oplus\mathcal{A}_m)\geq c(Q)$. Therefore, by Lemma 7.4, we have $c(R\oplus\mathcal{A}_m)=c_{n,\bar{k}}\geq c(Q)=M_n-1$, and thus the inequality in Lemma 6.4 implies that $\bar{k}=k$. However, if $\bar{k}=k$ then we have $c(r)=M_k-1$, contradicting our choice of k. $\diamond$

Numerical evidence and the contrast between Proposition 2.1 and Theorem 2.2 makes us suspect that Theorem 4.2 is not true for $q=1(l+1)l\ldots 2$, $l\geq 3$. In fact, we believe the following is true.

Conjecture 7.12 *The frequency sequence for $q=1(l+1)l\ldots 2$, $l\geq 3$ has internal zeros for all $n\geq l+1$.*

It would be interesting to find a proof of this conjecture. Perhaps a first step would be to find a simpler proof of Theorem 7.11.

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