Ordered set partition posets

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Abstract

The lattice of partitions of a set and its d-divisible generalization have been much studied for their combinatorial, topological, and respresentation-theoretic properties. An ordered set partition is a set partition where the subsets are listed in a specific order. Ordered set partitions appear in combinatorics, number theory, permutation polytopes, and the study of coinvariant algebras. The ordered set partitions of $\{1, \ldots, n\}$ can be partially ordered by refinement and then a unique minimal element attached, resulting in a lattice Ω_n. But this lattice has received no attention to our knowledge. The purpose of this paper is to provide the first comprehensive look at Ω_n. In particular, we determine its Möbius function, show that it admits a recursive atom ordering, and study the action of the symmetric group $\mathfrak{S}_n$ on associated homology groups, looking in particular at the multiplicity of the trivial representation. We also consider the related posets where every block has size either divisible by some fixed $d \geq 2$ or congruent to 1 modulo d.

1 Introduction

For nonnegative integers m, n with $m \leq n$ we will use the notation

$$[n] = \{1, 2, \ldots, n\} \quad \text{and} \quad [m, n] = \{m, m+1, \ldots, n\}. \tag{1}$$

If S is any finite set then we will use the notations $\#S$ or $|S|$ for the cardinality of S.

A *set partition* π of S is a a family of nonempty subsets $B_1, \ldots, B_k$ called *blocks* such that $S = \uplus_i B_i$ (disjoint union) and we write $\pi = B_1/\ldots/B_k$. We may leave out set braces and commas in examples. Note that the order of the blocks does not matter so that, for example,

$134/26/5 = 26/5/134$. The partition is *d-divisible* if d divides $\#B_i$ for all i. Partitions can be partially ordered by *refinement* where $B_1/\ldots/B_k \le C_1/\ldots/C_l$ if each C_j is a union of certain B_i. The poset of all partitions of $[n]$ ordered by refinement is a lattice denoted Π_n. Also, ordering the d-divisible partitions and adding a unique minimal element $\hat{0}$ results in a lattice $\Pi_n^{(d)}$. Notice that the symmetric group $\mathfrak{S}_n$ acts on both these lattices. The combinatorial, topological, and representation-theoretic properties of Π_n and $\Pi_n^{(d)}$ have been extensively studied. See, for example, [CHR86, Com25, EH18, HS15, HH03, Sun94b, Sun16, Wac96].

An *ordered set partition* of a finite set S is a sequence of non-empty subsets $\omega = (B_1, B_2, \ldots, B_k)$ with $\uplus_i B_i = S$, where the B_i are again called *blocks*. Note the use of parentheses and commas, as opposed to forward slashes, to indicate that now the order of the blocks matters. As with ordinary set partitions, we will often leave out the set braces and commas in each B_i. The ordered set partitions of [3] are displayed in Figure 1. We will use letters near the end of the Greek alphabet for ordered partitions. The number of blocks of ω is its *length*, denoted $\ell(\omega)$. The *Stirling numbers of the second kind* are

$$S(n,k) = \text{the number of unordered partitions of } n \text{ into } k \text{ blocks.}$$

It follows that

$$k!S(n,k) = \#\text{ of ordered partitions of } n \text{ into } k \text{ blocks.} \tag{2}$$

Ordered set partitions and Stirling numbers have connections to combinatorics, number theory, polyhedral theory, and coinvariant algebras. They appear in a closeted form as far back as a paper of Carlitz [Car33, equation (11)]. See the papers of Ishikawa, Kasraoui, and Zeng [IKZ08] or Sagan and Swanson [SS24] for history and references.

As in the unordered case, if $d \ge 1$ is an integer then we say that ω is *d-divisible* if d divides $\#B_i$ for all i. In this case we write

$$\omega \models_d S.$$

We will drop the d if $d = 1$ so that there is no restriction on the block sizes. To illustrate, the 2-divisible partitions of $\{a, b, c, d\}$ are

$$(ab, cd),\ (ac, bd),\ (ad, bc),\ (bc, ad),\ (bd, ac),\ (cd, ab),\ \text{and } (abcd).$$

We partially order the d-divisible partitions of S by insisting that $(B_1, B_2, \ldots, B_k)$ is covered by all elements of the form

$$(B_1, \ldots, B_{i-1}, B_i \uplus B_{i+1}, B_{i+2}, \ldots B_k) \text{ for } 1 \le i < k$$

and extending by transitivity. In other words, one is permitted to merge adjacent blocks, keeping the new block in the same relative position with the other blocks. From this one sees that $\omega \le \psi$ in this partial order if each block of ψ is a union of adjacent blocks of ω, and one block B of ψ is to the left of another C if the blocks of ω contained in B are to the left of those in C. Adding a unique minimal element $\hat{0}$ results in a poset which we will call $\Omega_S^{(d)}$. In the case $S = [n]$ we will write this as

$$\Omega_n^{(d)} = \{\hat{0}\} \ \uplus\ \{\omega \mid \omega \models_d [n]\}$$

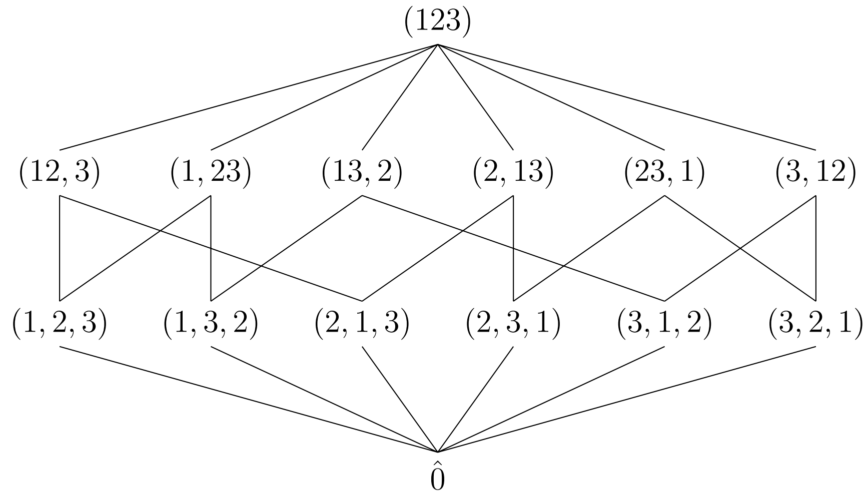

Figure 1: The poset Ω_3

where when $n = 0$ the second set is considered to be empty. We will shorten $\Omega_n^{(1)}$ to just Ω_n. And if we write $\omega \in \Omega_n^{(d)}$ then we are tacitly assuming that ω is an ordered set partition in $\Omega_n^{(d)}$, i.e., $\omega \neq \hat{0}$. If we write $x \in \Omega_n^{(d)}$ then x could be any element of the poset, including $\hat{0}$ and similarly for other letters near the end of the Latin alphabet. See Figure 1 for the Hasse diagram of Ω_3.

As far as we are aware, the posets $\Omega_n^{(d)}$ have not been discussed in the literature. The purpose of the current work is to study their combinatorial, topological, and representation-theoretic properties. The rest of this paper is organized as follows. In the next section we concentrate on the combinatorics of Ω_n. For example, we determine its Möbius function in Theorem 2.2 and show that it admits a recursive atom ordering in Theorem 2.5. Section 3 is devoted to generalizing some of the results in the previous section to $\Omega_n^{(d)}$. Interestingly, the Möbius value $\mu(\Omega_n^{(d)})$ is a d-divisible analogue of the Euler numbers introduced by Leeming and MacLeod [LM81], see Theorem 3.4 (b). The symmetric group $\mathfrak{S}_n$ on $[n]$ acts naturally on $\Omega_n^{(d)}$ by permuting the entries of the blocks. Section 4 looks at the induced action on various associated homology groups. As an illustration, Theorem 4.4 shows that the Frobenius characteristic of the action on the Whitney homology and its dual can be expressed in terms of complete homogeneous symmetric functions h_λ where λ an integer partition with all parts divisible by d. In Section 5, rank-selected and corank-selected subposets of $\Omega_n^{(d)}$ are considered. Theorem 5.4 gives a recurrence for the Frobenius characteristic of the corank-selected homology. Section 6 is devoted to studying the multiplicity $b_m(T)$ of the trivial representation of $\mathfrak{S}_{dm}$ acting on the homology of the subposet of $\Omega_{dm}^{(d)}$ selected by coranks T. In Theorem 6.1 we show that $b_m(T)$ counts permutations in $\mathfrak{S}_{m-1}$ with descent set T. A similar result, Theorem 6.3, is obtained for the action of $\mathfrak{S}_{dm-1}$ considered as the subgroup of $\mathfrak{S}_{dm}$ which fixes dm. Finally, Section 7 considers the combinatorics of the subposet of Ω_n where all the block sizes are congruent to 1 modulo d. In this case, the Möbius function is given up to sign by a generalization of the Catalan numbers as shown in Theorem 7.4.

2 Combinatorial properties of Ω_n

In our first result we will collect some elementary properties of the poset Ω_n. For more information about posets, including definitions of any undefined terms, see the texts of Sagan [Sag20, Chapter 5] or Stanley [Sta12, Chapter 3] Assume that $(P, \leq)$ is a poset. All posets in this work will be finite without further mention. If $x, y \in P$ with $x \leq y$ then we have the closed interval

$$[x, y] = \{z \mid x \leq z \leq y\}.$$

If P has a unique minimal element or unique maximal element then they are denoted $\hat{0}$ or $\hat{1}$, respectively. If P contains both of these elements it is said to be *bounded*. We write $x \lhd y$ if x is *covered* by y in P, that is, $x < y$ and there is no z with $x < z < y$. If P has a $\hat{0}$ then the *atoms* a of P are the elements covering $\hat{0}$. The poset is *ranked* if, for every $x \in P$, all maximal chains from $\hat{0}$ to x have the same length ℓ. In that case, we say that x has *rank* ℓ and write

$$\operatorname{rk} x = \ell.$$

We also define the rank of an interval $[x, y]$ to be

$$\operatorname{rk}(x, y) = \operatorname{rk} y - \operatorname{rk} x.$$

If P is ranked and has a unique maximal element $\hat{1}$ then we say that P is *graded*. In this case, the *corank* of $x \in P$ is

$$\operatorname{crk} x = \operatorname{rk} \hat{1} - \operatorname{rk} x.$$

Finally, we will need the *Boolean algebra*

$$\mathcal{B}_n = \{S \mid S \subseteq [n]\}$$

ordered by inclusion. The Boolean algebra is a *lattice* which is a poset where every pair of elements x, y has a greatest lower bound or *meet*, $x \wedge y$, and a least upper bound *or join*, $x \vee y$.

Note that, interestingly, the isomorphism in part (d) of the next result depends only on k (the number of blocks of the partition) and not on n (the sum of the parts).

Theorem 2.1. *The poset Ω_n satisfies the following.*

(a) It has $\hat{1} = ([n])$.

(b) Its atoms are the ω with $\#B = 1$ for all blocks B of ω.

(c) It is ranked. The rank of $\omega = (B_1, \ldots, B_k)$ is

$$\operatorname{rk} \omega = n - k + 1.$$

The number of ω at corank k is $(k+1)!S(n, k+1)$.

(d) For any $\omega = (B_1, \ldots, B_k)$ we have

$$[\omega, \hat{1}] \cong \mathcal{B}_{k-1}.$$

(e) For any orderdered partitions $\psi, \omega \in \Omega_n$ we have

$$[\psi, \omega] \cong \mathcal{B}_{\mathrm{rk}(\psi,\omega)},$$

(f) The poset Ω_n is an atomic lattice but it is not semimodular in general.

Proof. (a) We need to show that every ordered partition $\omega \in \Omega_n$ satisfies $\omega \leq ([n])$. But this clearly follows from the description of the partial order.

(b) Using the description of the partial order again, we have that ω will cover $\hat{0}$ if and only if no block of ω can be written as a disjoint union of two proper subsets. But this is equivalent to having all blocks of size one.

(c) Consider any maximal chain

$$C : \hat{0} = \omega_0 \lhd \omega_1 \lhd \ldots \lhd \omega_\ell = \omega.$$

By part (b), ω_1 has n singleton blocks. And we lose a block in each cover $\omega_i \lhd \omega_{i+1}$. So to end with an ordered partition with k blocks we must have

$$\ell = n - k + 1.$$

The result now follows from this discussion and equation (2).

(d) We will construct an anti-isomorphism $A : \mathcal{B}_{k-1} \to [\omega, \hat{1}]$. This will suffice since $\mathcal{B}_{k-1}$ is self dual. Take $\omega = (B_1, B_2, \ldots, B_k)$ and number the $k-1$ commas as $1, 2, \ldots, k-1$ from left to right to obtain

$$\omega = (B_1 \overset{1}{,} B_2 \overset{2}{,} \ldots, B_{k-1} \overset{k-1}{,} B_k).$$

Now given any $S \subseteq [k-1]$ we form the ordered partition $A(S)$ by removing any comma not labeled by an element of S and taking the disjoint union of any blocks no longer separated by commas. For ease of notation, we will often suppress the disjoint union signs and write $B_i B_{i+1}$ in place of $B_i \uplus B_{i+1}$. For example, if $k = 6$ then

$$\omega = (B_1 \overset{1}{,} B_2 \overset{2}{,} B_3 \overset{3}{,} B_4 \overset{4}{,} B_5 \overset{5}{,} B_6)$$

And the set $S = \{2, 5\}$ gives rise to the ordered set partition

$$A(S) = (B_1 \uplus B_2 \overset{2}{,} B_3 \uplus B_4 \uplus B_5 \overset{5}{,} B_6) = (B_1B_2 \overset{2}{,} B_3B_4B_5 \overset{5}{,} B_6).$$

It is easy to check that A is invertible and is a poset anti-isomorphism.

(e) This follows immediately from (d) and the fact that all intervals in a Boolean algebra are Boolean algebras.

(f) Since Ω_n has a $\hat{0}$, to show it is a lattice it suffices to show the existence of a join. Suppose $\psi, \omega \in \Omega_n$ where

$$\psi = (A_1, A_2, \ldots, A_k),$$
$$\omega = (B_1, B_2, \ldots, B_\ell).$$

Find the smallest i such that

$$A_1 \uplus A_2 \uplus \ldots \uplus A_i = B_1 \uplus B_2 \uplus \ldots \uplus B_j \tag{3}$$

for some j. Such an i exists because if $i = n$ then the union of A's is $[n]$. Call the union in (3) C_1 and removed the corresponding blocks from ψ and ω. Iterate this process to find C_2 and so forth. It is easy to see that $(C_1, C_2, \ldots C_m)$ is the join of ψ and ω.

To prove that Ω_n is atomic, consider any of its ordered set partitions $\omega = (B_1, B_2, \ldots, B_n)$. Construct a set of atoms $\mathcal{A}$ as follows. Put $a = (a_1, a_2, \ldots, a_n)$ into $\mathcal{A}$ precisely when $\{a_1, a_2, \ldots, a_i\} = B_1$ where $i = \#B_1$, and $\{a_{i+1}, a_{i+2}, \ldots, a_{i+j}\} = B_2$ where $j = \#B_2$, etc. It is easy to verify that $\bigvee \mathcal{A} = \omega$.

Finally to show that Ω_n is neither upper nor lower semimodular, we use the equivalent conditions in terms of covers. In the upper case, consider the atoms $\psi = (1, 2, 3, a_4, \ldots, a_n)$ and $\omega = (2, 3, 1, a_4, \ldots, a_n)$ where $a_4, \ldots, a_n$ forms an arbitrary permutation of $[4, n]$. Clearly, $\psi \wedge \omega = \hat{0}$ which they cover. But $\psi \vee \omega = (123, a_4, \ldots, a_n)$ which doees not cover ψ or ω. For lower semimodular, one uses the ordered set partitions $\psi' = (12, 3, b_4, \ldots, b_n)$ and $\omega' = (2, 13, b_4, \ldots, b_n)$. $\square$

The *Möbius* function of a finite poset P is a function from the closed intervals $[x, z]$ of P to the integers defined by $\mu(x, x) = 1$ and either of the two equivalent equations

$$\mu(x, z) = -\sum_{x \leq y < z} \mu(x, y) \tag{4}$$

$$= -\sum_{x < y \leq z} \mu(y, z) \tag{5}$$

for $x < z$. If P has a $\hat{0}$ and a $\hat{1}$ we write

$$\mu(P) = \mu(\hat{0}, \hat{1}).$$

It is a far-reaching generalization of the Möbius function in number theory. The second statement in the following theorem shows that (by definition) Ω_n is Eulerian. We give two proofs, one combinatorial and one geometric. For the latter, we need the fact that Ω_n is the face lattice of the permutahedron, the convex hull of the $n!$ points whose coordinates are the permutations in $\mathfrak{S}_n$; see the paper of Billera and Sarangarajan [BS96, Proposition 1.4]. From this it also follows that the order complex of Ω_n is homotopy equivalent to a single sphere in dimension $n - 2$.

Theorem 2.2. *We have*

$$\mu(\Omega_n) = (-1)^n \tag{6}$$

In fact, if $x \leq y$ are any elements of Ω_n, including $x = \hat{0}$, then

$$\mu(x, y) = (-1)^{\operatorname{rk} y - \operatorname{rk} x}. \tag{7}$$

Proof. We will give a combinatorial proof of (6). It is well known that if $S \in \mathcal{B}_k$ then

$$\mu(S, \hat{1}) = (-1)^{\operatorname{crk} S}.$$

We also have the classical identity

$$x^n = \sum_{k=0}^{n} S(n,k)x(x-1)\cdots(x-k+1)$$

and plugging in $x = -1$ gives

$$(-1)^n = \sum_{k=0}^{n} (-1)^k k! S(n,k).$$

Combining these facts with (c) and (d) from the previous theorem gives that in Ω_n

$$\begin{aligned}
\mu(\Omega_n) &= -\sum_{\hat{0}<\omega\leq\hat{1}} \mu(\omega,\hat{1}) \\
&= -\sum_{k=0}^{n-1} \sum_{\operatorname{crk}\omega=k} (-1)^k \\
&= -\sum_{k=1}^{n} (-1)^{k-1} k! S(n,k) \\
&= -(-1)^{n-1} \\
&= (-1)^n
\end{aligned}$$

as desired.

To demonstrate the more general (7), we appeal to geometry. As mentioned before the proof, Ω_n is the face lattice of the permutohedron. Since all such face lattices are Eulerian, we are done. □

We will need another form of equation (6) for the sequel. A *composition* of n is a sequence $\alpha = (\alpha_1, \alpha_2, \ldots, \alpha_k)$ of positive integers called *parts* with $\sum_i \alpha_i = n$. In this case we write $\alpha \models n$ and call $k = \ell(\alpha)$ the *length* of α. Any ordered set partition $\omega = (B_1, B_2, \ldots, B_k)$ has an associated composition called the *type* of ω and defined as

$$\operatorname{type}\omega = (\#B_1, \#B_2, \ldots, \#B_k).$$

For example $\operatorname{type}(245, 16, 3789) = (3, 2, 4)$. Given a composition $\alpha \models n$, the number of ordered set partitions of that type is clearly a multinomial coefficient

$$\#\{\omega \in \Omega_n \mid \operatorname{type}\omega = \alpha\} = \binom{n}{\alpha} = \frac{n!}{\prod_i \alpha_i}. \tag{8}$$

Corollary 2.3. *We have*

$$\sum_{\alpha\models n} (-1)^{\ell(\alpha)} \binom{n}{\alpha} = \sum_{k=1}^{n} \sum_{\substack{\alpha\models n \\ \ell(\alpha)=k}} (-1)^k \frac{n!}{\prod_i \alpha_i} = (-1)^n$$

Proof. By equation (8) the two summations are equal. So, we only need so show that the first equals $(-1)^n$. Using equations (6) and (7) as well as organizing by type gives

$$\begin{aligned}(-1)^n &= \mu(\Omega_n)\\ &= -\sum_{\hat{0}<\omega\leq\hat{1}} \mu(\omega,\hat{1})\\ &= -\sum_{\alpha\models n}\ \sum_{\text{type}\,\omega=\alpha} (-1)^{\ell(\alpha)-1}\\ &= \sum_{\alpha\models n}(-1)^{\ell(\alpha)}\binom{n}{\alpha}\end{aligned}$$

so we are done. □

We will now consider shellability questions for Ω_n. The face lattice of a convex polytope is always CL-shellable, see the paper of Björner and Wachs [BW83, Theorem 4.5]. Also CL-shellability is equivalent to having a recursive atom ordering [BW83, Theorem 3.1], defined as follows. Let $\mathcal{A}(P)$ be the atoms of a finite poset with a $\hat{0}$ and a $\hat{1}$. A linear ordering $a_1, a_2, \ldots, a_t$ of $\mathcal{A}(P)$ is a *recursive atom ordering* or *RAO* if

(R1) For all j, the interval $[a_j, \hat{1}]$ admits an RAO where the atoms coming first are those covering some a_i for $i < j$

(R2) For all $i < k$, if $a_i, a_k < y$ for some y then there exists a_j with $j < k$ and an $x \in P$ with

$$a_j, a_k \lhd x \leq y.$$

We wish to give an explicit RAO for Ω_n. Sagan [Sag86, Lemma 3] proved the following lemma which will be useful.

Lemma 2.4 ([Sag86])**.** *Suppose P is a poset such that $[a, \hat{1}]$ is a semimodular lattice for all $a \in \mathcal{A}(P)$. Then P admits an RAO if and only if some ordering of the atoms of P satisfies condition (R2) above.*

Theorem 2.5. *The poset Ω_n has an RAO and is thus both CL-shellable and Cohen-Macaulay.*

Proof. From Theorem 2.1 (d) we have that $[a, \hat{1}]$ is semimodular for all atoms $a \in \Omega_n$. So, by the previous lemma, it suffices to show (R2) for $\mathcal{A}(\Omega_n)$. By Theorem 2.1 (b), each atom has the form $a = (p_1, p_2, \ldots, p_n)$ and so can be identified with the permutation $p_1 p_2 \ldots p_n$.

We claim that the lexicographic order $\leq_l$ on permutations satisfies (R2). For take atoms $a_i, a_k \leq \omega$ where $a_i <_l a_k$ so that $i < k$. Write

$$\begin{aligned}a_i &= (p_1, p_2, \ldots, p_n),\\ a_k &= (r_1, r_2, \ldots, r_n).\end{aligned}$$

Take any $\omega \geq a_i, a_j$. Let s be the first index such that $p_s \neq r_s$. Since a_i is lexicographically smaller than a_k we must have $p_s < r_s$ Also, these two elements are in the same positions in their respective atoms and so they must be in the same block B of ω.

Now let $p_l, p_{l+1}, \ldots, p_m$ and $r_l, r_{l+1}, \ldots, r_m$ be the elements of B listed as they appear in a_i and a_k, respectively. By minimality of s, the element p_s must appear after r_s in a_k. Since $p_s < r_s$, there must must be a descent in the permutation $r_s, r_{s+1}, \ldots r_t$ where $r_t = p_s$. In other words, there is an index $u \in [s, t-1]$ such that $r_u > r_{u+1}$. Let a_j be a_k with elements r_u and r_{u+1} switched. So, by the previous inequality, $a_j \leq_l a_k$. Furthermore

$$\psi := a_j \vee a_k = (r_1, \ldots, r_{u-1}, r_u r_{u+1}, r_{u+2}, \ldots, r_n)$$

so that $a_j, a_k \lhd \psi$. Finally, since ψ is formed by merging two elements in the same block B of ω we have $\psi \leq \omega$, verifying (R2). $\square$

3 Combinatorial properties of $\Omega_n^{(d)}$

In this section, we will see that many of the properties of Ω_n carry over to $\Omega_n^{(d)}$. Throughout, we will assume that $d \geq 1$ is a divisor of $n \geq 0$. We start with an analogue of Theorem 2.1. To state it, we will need the *d-divisible Stirling numbers of the second kind* which are

$S^{(d)}(n, k) =$ the number of unordered partitions of n into k blocks all of size divisible by d.

The proofs are analogous to the $d = 1$ case and so are omitted. Alternatively, one can use the fact that $\Omega_n^{(d)}$ is a join sublattice of Ω_n.

Theorem 3.1. *If d divides n then the poset $\Omega_n^{(d)}$ satisfies the following.*

(a) It has $\hat{1} = ([n])$.

(b) Its atoms are the ω with $\#B = d$ for all blocks B of ω.

(c) It is ranked. The rank of $\omega = (B_1, \ldots, B_k)$ is

$$\operatorname{rk} \omega = n/d - k + 1.$$

The number of ω at corank k is $(k+1)! S^{(d)}(n, k+1)$.

(d) For any $\omega = (B_1, \ldots, B_k)$ we have

$$[\omega, \hat{1}] \cong \mathcal{B}_{k-1}.$$

(e) For any $\psi, \omega \in \Omega_n^{(d)}$ we have

$$[\psi, \omega] \cong \mathcal{B}_{\operatorname{rk}(\psi, \omega)},$$

(f) The poset $\Omega_n^{(d)}$ is an atomic lattice but it is not semimodular in general. $\square$

To calculate $\mu(\Omega_n^{(d)})$ we will need a generalization of the Euler numbers. The *(ordinary) Euler numbers*, E_n, can be defined as by the generating function

$$\sum_{n\geq 0} E_n \frac{x^n}{n!} = \tan x + \sec x.$$

These constants have a long and venerable history in combinatorics and number theory. Now given $d \geq 1$ we let ζ_d be a primitive dth root of unity. Define the *d-divisible Euler numbers*, $\mathcal{E}_n^{(d)}$, by

$$\sum_{n\geq 0} \mathcal{E}_n^{(d)} \frac{x^n}{n!} = \frac{d}{e^x + e^{\zeta_d x} + e^{\zeta_d^2 x} + \cdots + e^{\zeta_d^{d-1} x}} = \frac{1}{1 + x^d/d! + x^{2d}/2d! + \cdots}. \tag{9}$$

The $\mathcal{E}_n^{(d)}$ was first considered by Leeming and MacLeod [LM81] who called them generalized Euler numbers. They have since been studied by several people [Ges83, KL25, LM83, Sag25]. It is not hard to show that

$$\mathcal{E}_n^{(2)} = \begin{cases} (-1)^{n/2} E_n & \text{if } n \text{ is even.} \\ 0 & \text{if } n \text{ is odd.} \end{cases}$$

More generally, $\mathcal{E}_n^{(d)} = 0$ if d does not divide n. Note also that

$$\mathcal{E}_n^{(1)} = (-1)^n = \mu(\Omega_n^{(1)}).$$

As we will see shortly, this is not an accident. If $\mathfrak{S}_n$ be the symmetric group on $[n]$ then the *descent set* of $\pi = \pi_1\pi_2\ldots\pi_n \in \mathfrak{S}_n$ is

$$\operatorname{Des}\pi = \{i \mid \pi_i > \pi_{i+1}\}.$$

Let

$$A_{dn}^{(d)} = \{\pi \in \mathfrak{S}_{dn} \mid \operatorname{Des}\pi = \{d, 2d, \ldots, (n-1)d\}. \tag{10}$$

Sagan [Sag25, Theorem 3.1] proved the following.

Theorem 3.2 ([Sag25])**.** *Suppose $n \geq 0$ and $d \geq 1$.*

(a) We have

$$\mathcal{E}_n^{(d)} = \sum_{\omega \models_d [n]} (-1)^{\ell(\omega)}.$$

(b) We have

$$\mathcal{E}_{dn}^{(d)} = (-1)^n \# A_{dn}^{(d)}. \qquad \square$$

The other ingredient we will need is a variant of the notion of poset product introduced by Sundaram [Sun94a, pp. 287-288]. Let $(P, \leq_P)$ and $(Q, \leq_Q)$ be two posets, each having a minimum element. For any such poset P we let $P^- = P \setminus \{\hat{0}\}$. Then the *reduced product* of P and Q is

$$P \dot{\times} Q = \{\ (\hat{0}_P, \hat{0}_Q)\ \} \uplus (P^- \times Q^-),$$

where $\times$ is the usual poset product. Equivalently, $P\dot{\times}Q$ is the subposet of $P \times Q$ obtained by removing all elements of the form $(\hat{0}, q)$ or $(p, \hat{0})$. The notion of reduced product extends to products of three or more posets in the expected manner. It is easy to see that if $\omega \in \Omega_n^{(d)}$ has $\operatorname{type}\omega = (\alpha_1, \alpha_2, \ldots, \alpha_k)$ then

$$[\hat{0}, \omega] \cong \Omega_{\alpha_1}^{(d)} \dot{\times} \Omega_{\alpha_2}^{(d)} \dot{\times} \cdots \dot{\times} \Omega_{\alpha_k}^{(d)}. \tag{11}$$

We will need the following result of Sundaram [Sun94a, Remark 2.6.1].

Theorem 3.3 ([Sun94a])**.** *Let P, Q be posets both having a $\hat{0}$ and a $\hat{1}$. Then*

$$\mu(P\dot{\times}Q) = -\mu(P)\mu(Q).$$

□

We now have everything in place to compute $\mu(\Omega_n^{(d)})$.

Theorem 3.4. *In $\Omega_n^{(d)}$ where d divides n we have the following Möbius values.*

(a) For ψ, ω ordered partitions, $\psi \leq \omega$,

$$\mu(\psi, \omega) = (-1)^{\operatorname{rk}(\psi,\omega)}.$$

(b) For the full poset

$$\mu(\Omega_n^{(d)}) = \mathcal{E}_n^{(d)}$$

(c) For an ordered partition ω with $\operatorname{type}\omega = (\alpha_1, \alpha_2, \ldots, a_k)$ *we have*

$$\mu(\hat{0}, \omega) = (-1)^{k-1} \prod_{i=1}^{k} \mathcal{E}_{\alpha_i}^{(d)}.$$

Proof. (a) This follows immediately from Theorem 3.1, parts (d) and (e).

(b) Using part (a) and Theorem 3.2 (a) give

$$\begin{aligned} \mu(\Omega_n^{(d)}) &= -\sum_{\hat{0}<\omega\leq\hat{1}} \mu(\omega, \hat{1}) \\ &= -\sum_{\omega\models_d [n]} (-1)^{\ell(\alpha)-1} \\ &= \mathcal{E}_n^{(d)}. \end{aligned}$$

(c) This is an easy consequence of part (b), equation (11), and Theorem 3.3. □

We will now show that $\Omega_n^{(d)}$ admits an RAO.

Theorem 3.5. *The poset $\Omega_n^{(d)}$ where d divides n has an RAO and is thus both CL-shellable and Cohen-Macaulay.*

Proof. As in the proof of Theorem 2.5, by combining Theorem 3.1 (d) and Lemma 2.4, it suffices to show that some ordering of the atoms of $\Omega_n^{(d)}$ satisfies condition (2) in the definition of an RAO. And by Theorem 3.1 (b), these atoms are of the form $a = (B_1, B_2, \ldots, B_{n/d})$ where $\#B_i = d$ for all i. Write the elements of each B_i in decreasing order and let $B_i \leq_l B_j$ if B_i is less than or equal to B_j lexicographically. Finally, let $a \leq_l a'$ if the first blocks in which they differ are B and B', respectively, where $B \leq_l B'$.

Now suppose that $a_i, a_k \leq \omega$ where $i < k$ and

$$a_i = (P_1, P_2, \ldots, P_{n/d}),$$
$$a_k = (R_1, R_2, \ldots, R_{n/d}),$$

Letting s be the first index in which a_i differs from a_k, we must have $P_s <_l R_s$. Let $p \in P_s$ and $r \in R_s$ be the largest elements in which they differ. This forces $p < r$.

Since $a_i, a_k < \omega$, there must be a block $B \in \omega$ with $P_s, R_s \subseteq B$. Let $P_\ell, P_{\ell+1}, \ldots, P_m$ and $R_\ell, R_{\ell+1}, \ldots, R_m$ be the blocks of a_i and a_k, respectively, which are subsets of B where $\ell \leq s \leq m$. Since $p \notin R_s$ and we are comparing atoms lexicographically, there must be a $t > s$ with $p \in R_t$. If there is a lexicographic descent in the sequence $R_s, R_{s+1}, \ldots, R_t$ then we proceed as in the proof of the $d = 1$ case, Theorem 2.5.

Otherwise, we have $R_s < R_{s+1} <_l \ldots <_l R_t$. Letting $m_i = \max R_i$ for all i, this implies that $m_s < m_{s+1} < \ldots < m_t$. Combining this with the fact that $p < r \in R_s$ gives us $p < m_s \leq m_{t-1}$. Now consider the sets

$$R'_t = (R_t - \{p\}) \cup \{m_{t-1}\},$$
$$R'_{t-1} = (R_{t-1} - \{m_{t-1}\}) \cup \{p\}.$$

Note that since R'_{t-1} was constructed by removing the maximum of R_{t-1} and replacing it with a smaller element, we must have $R'_{t-1} <_l R_{t-1}$. Finally, let a_j be the atom obtained from a_k by replacing R_{t-1} and R_t by R'_{t-1} and R'_t, respectively. From the inequality on the $(t-1)$st blocks, it follows that we must have $a_j <_l a_k$. Also, the fact that $R_t \cup R_{t-1} = R'_t \cup R'_{t-1}$ shows that $a_j < \omega$. Now, a_j and a_k only differ in two adjacent blocks so that $a_j, a_k \lhd \psi$ for some ψ. Finally, since $a_j, a_k < \omega$ we are forced to have $\psi = a_j \vee a_k \leq \omega$, finishing the proof. □

4 The action of the symmetric group

In this section, we will study the action of the symmetric group $\mathfrak{S}_n$ on various homology groups associated with $\Omega_n^{(d)}$. For more about the theory of symmetric group representations see the books of James [Jam78], James and Kerber [JK81], Sagan [Sag01], or Serre [Ser77].

Let P be a bounded poset and consider the *proper part* of P which is

$$\overline{P} = P - \{\hat{0}, \hat{1}\}.$$

The set of all chains in $\overline{P}$ forms a simplicial complex called the *order complex* of P and denoted $\Delta(P)$. We write the ith (reduced) homology group of $\Delta(P)$ over the rationals as $\tilde{H}_i(P)$.

Suppose that P is Cohen-Macaulay. Then P is graded, so suppose $\operatorname{rk} \hat{1} = r$. In this case, all its homology groups vanish except in the top dimension, $r-2$, and $\tilde{H}_{r-2}(P)$ is a vector space of dimension $|\mu(P)|$. Furthermore, any group of automorphisms of P induces an action on $\tilde{H}_{r-2}(P)$.

Now consider the poset $\Omega_n^{(d)}$. By Theorem 3.5, this poset is Cohen-Macaulay. Clearly $\mathfrak{S}_n$ is the group of automorphisms of P and so acts on its homology. We first consider the case when $d = 1$ which is particularly simple.

Proposition 4.1. *The symmetric group $\mathfrak{S}_n$ acts on $\tilde{H}_{n-2}(\Omega_n)$ like the sign.*

Proof. By equation (6), $\tilde{H}_{n-2}(\Omega_n)$ is one-dimensional. So, the character of the $\mathfrak{S}_n$-action is completely determined by the trace of any transposition, say $\sigma = (1,2)$. By the Hopf trace formula (e.g. [Sun94a]), this trace is $(-1)^{n-2}\mu(\Omega_n^\sigma)$ where Ω_n^σ is the subposet of Ω_n fixed by $\sigma = (1,2)$. Now σ fixes an ordered set partition ω if and only if 1 and 2 are in the same block of ω. It follows that Ω_n^σ is isomorphic to Ω_{n-1}. Hence, using equation (6) again,

$$\operatorname{tr}(\sigma, \tilde{H}_{n-2}(\Omega_n)) = (-1)^{n-2}\mu(\Omega_{n-1}) = -1,$$

which proves the proposition. $\square$

For all d, we know that intervals in $\Omega_n^{(d)}$ are reduced products by (11). Because of this, we will need a key technical fact about how reduced products behave under group actions, which was established in [Sun94a, Proposition 2.5, Proposition 2.6].

Proposition 4.2. *[Sun94a] Let P_1 and P_2 be Cohen-Macaulay posets of ranks r_1 and r_2, respectively, Let G_i be a finite group of automorphisms of P_i for $i = 1, 2$. Then $P_1 \dot{\times} P_2$ is Cohen-Macaulay, and there is a $(G_1 \times G_2)$-isomorphism*

$$\tilde{H}_{r_1-2}(P_1) \otimes \tilde{H}_{r_2-2}(P_2) \cong \tilde{H}_{r_1+r_2-3}(P_1 \dot{\times} P_2). \qquad \square$$

The Whitney homology $WH_i(P)$ of a poset P was originally defined by Baclawski [Bac75]. The following equivalent definition of Whitney homology, due to Anders Björner, was shown to be useful for determining group actions on poset homology by Sundaram [Sun94b]. Let P be a ranked poset with least element $\hat{0}$, and let r be the length of a longest chain in P. The ith Whitney homology of P is related to the usual order homology of P by isomorphisms establishing that, for $0 \le i \le r$,

$$WH_i(P) \cong \bigoplus_{\operatorname{rk}(x)=i} \tilde{H}_{i-2}(\hat{0}, x).$$

Note that this implies that $WH_0(P)$ is the trivial module and

$$WH_r(P) \cong \tilde{H}_{r-2}(P).$$

These isomorphisms can be shown to commute with any group of automorphisms of P. See [Sun94a, Sun94b] . Because of this, we will replace $\cong$ with $=$ when dealing with Whitney homology and the corresponding simplicial homology groups. We will use the following important acyclicity property of Whitney homology, proved in [Sun94b, Lemma 1.1], which will permit us to obtain an expression for $WH_r(P)$ as an alternating sum of G-modules.

Proposition 4.3 ([Sun94b]). *Let P be a bounded poset, and let G be a group of automorphisms of P. Assume that the Whitney homology is free in all degrees. Then each Whitney homology module is a G-module, and as a virtual sum of G-modules one has*

$$WH_r(P) - WH_{r-1}(P) + \cdots + (-1)^r WH_0(P) = 0,$$

where r is the length of a longest chain in P.

It is often useful to examine the Whitney homology of the dual poset. For example, as noted in Theorem 3.1 (d), in the case of $\Omega_n^{(d)}$, the intervals $[\omega, \hat{1}]$ have a simple description as Boolean lattices. As in [Sun94b], we define the dual Whitney homology $WH^*(P)$ of a graded poset P of rank r to be the Whitney homology of the dual poset P^*, so that, for $0 \leq i \leq r$,

$$WH_i^*(P) = \bigoplus_{\mathrm{crk}(x)=i} \tilde{H}_{i-2}(x, \hat{1}).$$

We emphasize that ranks and coranks are all taken in P itself. As in the ordinary case, we have that $WH_0^*(P)$ is the trivial module and

$$WH_r^*(P) \cong \tilde{H}_{r-2}(P). \tag{12}$$

Now assume P is a Cohen-Macaulay poset of rank r. Then for every open interval (x, y) in P, nonvanishing homology can occur only in the top dimension. We will often simply write $\tilde{H}(x, y)$ for that homology group. Proposition 4.3 gives the following two formulas for the homology $\tilde{H}_{r-2}(P)$, which we will use repeatedly in what follows.

$$\tilde{H}_{r-2}(P) = WH_{r-1}(P) - WH_{r-2}(P) + \cdots + (-1)^{r-1} WH_0(P). \tag{13}$$

$$\tilde{H}_{r-2}(P) = WH_{r-1}^*(P) - WH_{r-2}^*(P) + \cdots + (-1)^{r-1} WH_0^*(P). \tag{14}$$

The action of the symmetric group Ω_n is most conveniently described using its Frobenius characteristic and symmetric functions. See the texts of Macdonald [Mac95], Sagan [Sag01], or Stanley [Sta99] for more information about these topics. Writing ch for the Frobenius characteristic, we see from Proposition 4.1 that

$$\mathrm{ch}\, \tilde{H}_{n-2}(\Omega_n) = e_n$$

where e_n is the elementary symmetric function of degree n in a countably infinite set of variables (which we will suppress).

Write

$$\beta_{dm}^{(d)} := \mathrm{ch}\, \tilde{H}_{m-2}(\Omega_{dm}^{(d)}). \tag{15}$$

Our goal is to derive a formula for $\beta_{dm}^{(d)}$ by first deriving the Frobenius characteristic for the Whitney homology modules of Ω_{dm}^{d}. To do so, we will need the complete homogeneous functions h_n. As usual, if $\lambda = (\lambda_1, \ldots, \lambda_k)$ is a partition then we let

$$h_\lambda = h_{\lambda_1} \cdots h_{\lambda_k}.$$

We extend this definition to compositions $\alpha = (\alpha_1, \ldots, \alpha_k)$ by letting

$$h_\alpha = h_{\alpha_1} \cdots h_{\alpha_k}.$$

We use the same conventions to define elementary symmetric functions e_λ and e_α as well as the Frobenius characteristics $\beta_\lambda^{(d)}$ and $\beta_\alpha^{(d)}$, where in the last two cases λ and α must have all parts divisible by d. Note that, although we use parentheses for both partitions and compositions, context should make it clear which is meant. Finally, if d is a positive integer and $\alpha = (\alpha_1, \ldots, \alpha_k)$ is a composition then we let

$$d\alpha = (d\alpha_1, \ldots, d\alpha_k)$$

and similarly for partitions $d\lambda$.

It will be convenient to have a notation for the set of compositions of a given integer of a given length. So we let

$$\mathcal{C}(n,k) = \{\alpha \models n \mid \ell(\alpha) = k\}.$$

Theorem 4.4. *Let $n = dm$ and consider $\Omega_{dm}^{(d)}$.*

(a) The dual Whitney homology of $\Omega_{dm}^{(d)}$ is given, for $0 \le k \le m-1$, by

$$\operatorname{ch} WH_k^*(\Omega_{dm}^{(d)}) = \sum_{\alpha \in \mathcal{C}(m,k+1)} h_{d\alpha},$$

and in the top dimension by

$$\operatorname{ch} WH_m^*(\Omega_{dm}^{(d)}) = \beta_{dm}^{(d)}.$$

In particular, it is a permutation module except in the top dimension.

(b) The Whitney homology of $\Omega_{dm}^{(d)}$ is given, for $0 \le k \le m-1$, by

$$\operatorname{ch} WH_{m-k}(\Omega_{dm}^{(d)}) = \sum_{\alpha \in \mathcal{C}(m,k+1)} \beta_{d\alpha}^{(d)},$$

and in the bottom dimension by

$$\operatorname{ch} WH_0(\Omega_{dm}^{(d)}) = h_{dm}.$$

(c) The top homology of $\Omega_{dm}^{(d)}$ is given by

$$\beta_{dm}^{(d)} = \sum_{k=1}^{m} (-1)^{m-k} \sum_{\alpha \in \mathcal{C}(m,k)} h_{d\alpha}. \tag{16}$$

It has dimension

$$\dim \tilde{H}_{m-2}(\Omega_{dm}^{(d)}) = \sum_{\omega \in \Omega_{dm}^{(d)}} (-1)^{m-\ell(\omega)}. \tag{17}$$

(d) The top homology of $\Omega^{(d)}_{dm}$ is also given by the recurrence

$$\beta^{(d)}_{dm} = (-1)^{m+1} h_{dm} + \sum_{k=2}^{m} (-1)^k \sum_{\alpha \in \mathcal{C}(m,k)} \beta^{(d)}_{d\alpha}. \tag{18}$$

Proof. (a) The formula in the top dimension follows immediately from (15) together with equation (12) applied to $\Omega^{(d)}_{dm}$.

For the summation formula, note first that, since $WH^*_0(P)$ is always the trivial module,

$$\operatorname{ch} WH^*_0(\Omega^{(d)}_{dm}) = h_{dm}$$

which agrees with the sum at $k = 0$. So assume $k \geq 1$ and consider an upper interval $[\omega, \hat{1}]$ in $\Omega^{(d)}_{dm}$, where $\omega = (B_1, \ldots, B_{k+1})$. Then ω has type $d\alpha$ where $\#B_i = d\alpha_i$ for all i, and thus $\alpha \vDash m$. By Theorem 3.1 (d) we have $[\omega, \hat{1}] \cong \mathcal{B}_k$, so its homology is one-dimensional. The stabiliser subgroup of ω is the Young subgroup

$$G_\omega := \times_{i=1}^{k+1} \mathfrak{S}_{B_i}, \tag{19}$$

where $\mathfrak{S}_{B_i}$ is the group of permutations on the elements of the block B_i. Arguing as in Proposition 4.1, one sees that G_ω acts trivially on the homology, because the group fixes all ordered partitions in $[\omega, \hat{1}]$. The orbit of such a ω under the $\mathfrak{S}_{dm}$-action consists of all ordered partitions ψ of type $d\alpha$. This is a transitive action, and therefore we have

$$\bigoplus_{\text{type } \psi = d\alpha} \tilde{H}_{k-2}(\psi, \hat{1}) = 1 \uparrow^{\mathfrak{S}_{dm}}_{G_\omega},$$

which has Frobenius characteristic $\prod_{i=1}^{k+1} h_{d\alpha_i}$. It follows that, for $1 \leq k \leq m-1$,

$$\operatorname{ch} WH^*_k(\Omega^{(d)}_{dm}) = \sum_{\operatorname{rk} \psi = m-k} \operatorname{ch} \tilde{H}_{k-2}(\psi, \hat{1}) = \sum_{\alpha \in \mathcal{C}(m,k+1)} h_{d\alpha},$$

since elements at corank k have $k+1$ blocks.

(b) The formula for dimension 0 follows immediately from the fact that $WH_0(\Omega^{(d)}_{dm})$ is the trivial module.

For the other dimensions, we need to examine the lower intervals $[\hat{0}, \omega]$. Again, suppose $\omega = (B_1, \ldots, B_k)$ with composition type $d\alpha = (d\alpha_1, \ldots, d\alpha_k)$ where $\#B_i = d\alpha_i$ and thus $\alpha \vDash m$. By the isomorphism (11) we have that

$$[\hat{0}, \omega] \cong \Omega^{(d)}_{d\alpha_1} \dot{\times} \cdots \dot{\times} \Omega^{(d)}_{d\alpha_k}.$$

Also, from equation (19), we have that $G_\omega = \times_{i=1}^{k} \mathfrak{S}_{B_i}$ is the stabilizer of ω, and hence of $[\hat{0}, \omega]$. By definition (15), each $\mathfrak{S}_{B_i}$ acts on the homology of the corresponding component $\Omega^{(d)}_{d\alpha_i}$ of the reduced product like the representation whose Frobenius characteristic is $\beta^{(d)}_{d\alpha_i}$.

Now we invoke Proposition 4.2. It follows, by collecting orbits as in (a), that the action of $\mathfrak{S}_{dm}$ on the orbit of ω is the induced module $\otimes_{i=1}^{k} \tilde{H}(\Omega^{(d)}_{d\alpha_i}) \uparrow^{\mathfrak{S}_{dm}}_{G_\omega}$, and hence its Frobenius characteristic is $\beta^{(d)}_{d\alpha}$. Since a partition with k blocks has rank $m-k+1$, we have shown that

$$\operatorname{ch} WH_{m-k+1}(\Omega^{(d)}_{dm}) = \sum_{\alpha\in\mathcal{C}(m,k)} \beta^{(d)}_{d\alpha},$$

and reindexing gives the desired result.

(c) Combining definition (15), equation (14), and part (a), we obtain

$$\beta^{(d)}_{dm} = \sum_{k=0}^{m-1}(-1)^{m-k-1}\operatorname{ch} WH^*_k(\Omega^{(d)}_{dm}) = \sum_{k=0}^{m-1}(-1)^{m-k-1}\sum_{\alpha\in\mathcal{C}(m,k+1)} h_{d\alpha},$$

which is equivalent to (16) after re-indexing. Taking dimensions in (16) gives (17), since

$$\#\{\omega \models_d [dm] \mid \ell(\omega)=k\} = \sum_{\alpha\in\mathcal{C}(m,k)} \binom{dm}{d\alpha},$$

and the multinomial coefficient is the dimension of the induced module $1\uparrow^{\mathfrak{S}_{dm}}_{G_\omega}$.

(d) We use equation (13) and the fact that $WH_0(\Omega^{(d)}_{dm})$ is the trivial module to obtain

$$\begin{aligned}
\beta^{(d)}_{dm} &= \sum_{k=0}^{m-1}(-1)^k \operatorname{ch} WH_{m-1-k}(\Omega^{(d)}_{dm})\\
&= (-1)^{m-1}h_{dm} + \sum_{k=0}^{m-2}(-1)^k \sum_{\alpha\in\mathcal{C}(m,k+2)} \beta^{(d)}_{d\alpha}\\
&= (-1)^{m+1}h_{dm} + \sum_{k=2}^{m}(-1)^k \sum_{\alpha\in\mathcal{C}(m,k)} \beta^{(d)}_{d\alpha}.
\end{aligned}$$

This completes the proof. □

Before moving on, we would like to make some remarks about this result. First, it is easy to use equation (16) to calculate the $\beta^{(d)}_{dm}$ in terms of complete homogeneous functions. As an example, here are the expressions for the first three values of m:

$$\beta^{(d)}_{d} = h_d,\ \beta^{(d)}_{2d} = h_d^2 - h_{2d},\ \beta^{(d)}_{3d} = h_d^3 - 2h_dh_{2d} + h_{3d}.$$

Next, combining (17) with Theorem 3.2 we see that

$$\dim \tilde{H}_{m-2}(\Omega^{(d)}_{dm}) = (-1)^m \mathcal{E}^{(d)}_{dm}. \tag{20}$$

The special case $d=1$ reflects the duality between the elementary and complete homogeneous functions. It also can be used to prove a known symmetric function identity as we will see shortly. So, we will state this special case of the previous theorem.

Theorem 4.5. *For $n \geq 0$, we consider Ω_n.*

(a) The dual Whitney homology of Ω_n is given, for $0 \le k \le n-1$, by

$$\operatorname{ch} WH_k^*(\Omega_n) = \sum_{\alpha \in \mathcal{C}(m,k+1)} h_\alpha,$$

and in the top dimension by

$$\operatorname{ch} WH_m^*(\Omega_n) = e_n.$$

In particular, it is a permutation module in all but the top dimension.

(b) The Whitney homology of Ω_n is given, for $0 \le k \le m-1$, by

$$\operatorname{ch} WH_{m-k}(\Omega_n) = \sum_{\alpha \in \mathcal{C}(m,k+1)} e_\alpha,$$

and in the bottom dimension by

$$\operatorname{ch} WH_0(\Omega_{dm}^{(d)}) = h_n.$$

(c) As $\mathfrak{S}_n$-modules, the Whitney homology and the dual Whitney homology are related via the sign representation sgn_n of $\mathfrak{S}_n$ by the equation

$$WH_{n-k}(\Omega_n) = \operatorname{sgn}_n \otimes WH_k^*(\Omega_n). \qquad \square$$

From part (a) of the previous theorem and the dual version of Proposition 4.3 we immediately have

$$e_n = \sum_{k=1}^{n} (-1)^{n-k} \sum_{\alpha \in \mathcal{C}(n,k)} h_\alpha \tag{21}$$

We can rewrite this as an $n \times n$ determinant

$$e_n = \det(h_{1-i+j}). \tag{22}$$

Indeed, when expanding the determinant about the first row, the term in the sum corresponding to h_j gives all the h_α whose first factor is h_j. But this last equation is the special case of the Jacobi-Trudi determinant for $e_n = s_{(1^n)}$ where s_λ is the Schur function corresponding to the partition $\lambda = (1^n)$. Similarly, part (b) of the previous result gives the dual Jacobi-Trudi determinant for $h_n = s_n$. Alternatively, it can be derived from (22) by applying the standard involution on symmetric functions See the texts of Macdonald [Mac95, Chapter 1], Sagan [Sag01, Chapter 4], or Stanley [Sta99, Chapter 7] for more information. Combinatorial proofs of equation (22) can be given, for example, by using lattice paths and the Lindström-Gessel-Viennot method [GV85, Lin73], or by using brick tabloids as done by Eğecioğlu and Remmel in [ER91].

Our conventions for Ferrers diagrams of integer partitions and standard Young tableaux follow [Mac95, Sag01, Sta99], increasing left to right along rows, and increasing top to bottom down the columns. A rim hook (or border strip) [Mac95, Sta99] is a skew shape whose Ferrers diagram has the property that consecutive rows share exactly one column. Generalizing the previous observations, we show that the symmetric function $\beta_{dm}^{(d)}$ is the Schur function indexed by a rim hook.

Theorem 4.6. *Fix $m \geq 0$ and $d \geq 1$. Let $\lambda^{m,d}$ and $\mu^{m,d}$ be the partitions whose parts are defined, for $1 \leq i \leq m$, by*

$$\lambda_i^{m,d} = (d-1)(m-i+1)+1,$$
$$\mu_i^{m,d} = (d-1)(m-i),$$

so that the skew shape $\lambda^{m,d}/\mu^{m,d}$ is the rim hook consisting of m rows of length d. Then

$$\beta_{dm}^{(d)} = s_{\lambda^{m,d}/\mu^{m,d}}.$$

In particular,

$$\dim \tilde{H}_{m-2}(\Omega_{dm}^{(d)}) = (-1)^m \mathcal{E}_{dm}^{(d)}.$$

Proof. By the same argument used to establish the equivalence of (21) and (22), we see that

$$\det(h_{d(1-i+j)}) = \sum_{j=1}^{m} (-1)^{m-j} \sum_{\alpha \in \mathcal{C}(m,j)} h_{d\alpha}.$$

By the Jacobi-Trudi identity again, however, we have

$$s_{\lambda^{m,d}/\mu^{m,d}} = \det(h_{\lambda_i^{m,d}-i-\mu_j^{m,d}+j}).$$

A simple calculation shows that

$$\lambda_i^{m,d} - i - \mu_j^{m,d} + j = d(1-i+j).$$

and hence

$$s_{\lambda^{m,d}/\mu^{m,d}} = \det(h_{d(1-i+j)}).$$

The first claim of the theorem now follows from (16).

For the dimension result, note that the number of standard Young tableaux of shape $\lambda^{m,d}/\mu^{m,d}$ is the number of permutations in $\pi \in \mathfrak{S}_{dm}$ with descent set $\{d, 2d, \ldots, (m-1)d\}$. By Theorem 3.1 (e) in [Sag25], this number is $(-1)^m \mathcal{E}_{dm}^{(d)}$. The dimension formula now follows from the expression for $\beta_{dm}^{(d)}$. □

Note that the dimension formula just proved gives another demonstration of Theorem 3.4 (b).

Next we construct the ordinary generating function for the $\beta_{dm}^{(d)}$. Recall the exponential generating function used to define the $\mathcal{E}_{dm}^{(d)}$ in (9). One can see that it is precisely the exponential specialization in the sense of Stanley [Sta99, Proposition 7.8.4] of the generating function below.

Corollary 4.7. *Fix $d \geq 1$. Then we have the generating function*

$$\sum_{m \geq 0} (-1)^m \beta_{dm}^{(d)} x^{dm} = \frac{1}{1 + h_d x^d + h_{2d} x^{2d} + \cdots}. \tag{23}$$

In particular, the symmetric functions $\{\beta_{dm}^{(d)} \mid m \geq 1\}$ form an algebraically independent set of generators for the ring of symmetric functions generated by $\{h_{dm} \mid m \geq 1\}$.

Proof. For any formal power series $1 + c_1x + c_2x^2 + \cdots$ we have

$$\begin{aligned}\frac{1}{1 + c_1x + c_2x^2 + \cdots} &= \frac{1}{1 - (-c_1x - c_2x^2 - \cdots)} \\ &= \sum_{k\geq 0} (-c_1x - c_2x^2 - \cdots)^k \\ &= \sum_{k\geq 0} (-1)^k \sum_{m\geq 0} x^m \sum_{\alpha\in\mathcal{C}(m,k)} c_{\alpha_1} \cdots c_{\alpha_k},\end{aligned}$$

where the last equality comes from the fact that a term involving x^m in the expansion of $(c_1x + c_2x^2 + \cdots)^k$ comes picking the term $c_{\alpha_i}x^{\alpha_i}$ from the ith factor for $1 \leq i \leq k$. We now get the desired generating function by substituting x^d for x, $c_i = h_{di}$ for $i \geq 1$, and using equation (16). $\square$

There is a striking resemblance between the $\mathfrak{S}_{dm}$-action on the top homology of $\Omega^{(d)}_{dm}$ and the $\mathfrak{S}_{md-1}$-action on the top homology of the (unordered) d-divisible partition lattice Π^d_{dm}. In particular, compare (23) above with (24) below. The next result was derived in [Sun94b, Example 1.6(ii), Proposition 5.2]. And (24) was originally due to [CHR86], implicitly and in a different form. Let π^d_m denote the Frobenius characteristic of the $\mathfrak{S}_{dm}$-action on the top homology of the d-divisible lattice Π^d_{dm}.

Proposition 4.8 ([[CHR86], [Sun94b]])**.** *We have the plethystic identity*

$$\sum_{m\geq 0} (-1)^m \pi^d_m = \sum_{i\geq 1} (-1)^i \pi_i \left[\sum_{j\geq 1} h_{dj} \right].$$

We have the generating function

$$\sum_{m\geq 0} (-1)^m (\pi^d_m) \downarrow_{\mathfrak{S}_{dm-1}} = \frac{\sum_{j\geq 1} h_{dj-1}}{\sum_{j\geq 0} h_{dj}}, \tag{24}$$

where $(\pi^d_m) \downarrow_{\mathfrak{S}_{dm-1}}$ denotes restriction of the representation π^d_m to $\mathfrak{S}_{dm-1}$. Moreover, the $\mathfrak{S}_{md-1}$-representation $(\pi^d_m)\downarrow_{\mathfrak{S}_{md-1}}$ is the skew Schur function indexed by the rim hook whose top row has length $d-1$, and whose remaining $m-1$ rows have length d. $\square$

There is also a connection with rank selection in the Boolean lattice $\mathcal{B}_{dm}$. Define $\mathcal{B}^{(d)}_{dm}$ to be the rank-selected subposet of $\mathcal{B}_{dm}$ consisting of sets of cardinality divisible by d. Results of Solomon [Sol68, Section 6] implicitly and Stanley [Sta82, Theorem 4.3] explicitly imply that the representation of $\mathfrak{S}_{dm}$ on the homology of $\mathcal{B}^{(d)}_{dm}$ is given precisely by the Specht module for the skew shape $\lambda^{m,d}/\mu^{m,d}$ of Theorem 4.6. We discuss rank-selection in more detail later in the next section, but first we elucidate this connection.

Consider the order complex $\Delta(\mathcal{B}^{(d)}_{dm})$. As in [Sta82, Ex. 3.18.9], we have a bijection mapping a chain $X_1 \subset X_2 \subset \cdots \subset X_k$ in $\Delta(\mathcal{B}^{(d)}_{dm})$ to the ordered partition

$$\omega = (X_1, X_2 \setminus X_1, X_3 \setminus X_2, \ldots, [dm] \setminus X_k).$$

Since every cardinality $|X_i|$ is a multiple of d we have that $\omega \in \Omega_{dm}^{(d)}$. This map is also order reversing and so the face poset of $\Delta(\mathcal{B}_{dm}^{(d)})$ is isomorphic to ${\Omega_{dm}^{(d)}}^*$ with its maximal element removed. This poset isomorphism clearly commutes with the action of $\mathfrak{S}_{dm}$. More precisely, because the order complex of the face lattice of $\Delta(P)$ is the barycentric subdivision $\mathrm{sd}\,\Delta(P)$ of $\Delta(P)$ (see, e.g., [Sta82, §8]), there is an $\mathfrak{S}_{dm}$-equivariant homeomorphism

$$\mathrm{sd}\,\Delta(B_{dm}^{(d)}) \simeq \Delta({\Omega_{dm}^{(d)}}^*) \tag{25}$$

and hence we have an $\mathfrak{S}_{dm}$-isomorphism of homology modules

$$\tilde{H}(B_{dm}^{(d)}) \cong \tilde{H}(\Omega_{dm}^{(d)}).$$

This gives, in effect, another proof of Theorem 4.6. It also establishes that $\Omega_{dm}^{(d)}$ is Cohen-Macaulay, because $B_{dm}^{(d)}$ is Cohen-Macaulay, and hence the homotopy type of $\Delta(\Omega_{dm}^{(d)})$ is a wedge of spheres in the top dimension.

An examination of the $\mathfrak{S}_{dm}$-action on the maximal chains leads to the following.

Proposition 4.9. *The action of $\mathfrak{S}_{dm}$ on the maximal chains of $\Omega_{dm}^{(d)}$ has Frobenius characteristic*

$$(m-1)!\, h_d^m.$$

In particular, the number of maximal chains is

$$(m-1)! \binom{dm}{d,\ldots,d}.$$

Proof. Let $\omega = (B_1, \ldots, B_m)$ be an atom in $\Omega_{dm}^{(d)}$. Since ω has m blocks each of size d, its stabilizer is the Young subgroup

$$G_\omega := \mathfrak{S}_{B_1} \times \cdots \times \mathfrak{S}_{B_m} \cong \mathfrak{S}_d^m.$$

Also, because of the isomorphism in Theorem 3.1 (d), there is a bijection between the maximal chains from ω to $\hat{1}$ and the maximal chains from $\hat{0}$ to $\hat{1}$ in the Boolean algebra $\mathcal{B}_{m-1}$. It follows that there are $(m-1)!$ such chains.

The action of $\mathfrak{S}_{dm}$ permutes the chains from an atom to $\hat{1}$ in $\Omega_{dm}^{(d)}$. Also, because the blocks are ordered, all chains corresponding to a given chain in $\mathcal{B}_{m-1}$ are in one orbit of $\mathfrak{S}_{dm}$. The vector space of maximal chains therefore decomposes into a direct sum of $(m-1)!$ orbits, and each orbit is isomorphic to the induced module $1 \uparrow_{G_\omega}^{\mathfrak{S}_{dm}}$. This immediately gives the desired Frobenius characteristic and chain count. □

5 Rank-selection

Now we turn to rank-selection. For any bounded and graded poset P of rank r, the *trivial ranks* are rank 0 and rank r. The same terminology applies to coranks. For any subset S of the nontrivial ranks $\{1, \ldots, r-1\}$, we define the rank-selected subposet P_S to be the bounded and graded poset

$$P_S = \{x \in P : \mathrm{rk}(x) \in S\} \cup \{\hat{0}, \hat{1}\}.$$

It is known that rank-selection preserves the Cohen-Macaulay property [Bac80, Theorem 6.4]. For a subset of ranks $S \subseteq [m-1]$, write $\beta_{dm}^{(d)}(S)$ for the Frobenius characteristic of $\tilde{H}(\Omega_{dm}^{(d)}(S))$.

Let P be an arbitrary Cohen-Macaulay poset of rank r and let S be a subset of the ranks $\{1, \ldots, r-1\}$. Let G be a group of automorphisms of P, and let $\alpha_P(S)$ and $\beta_P(S)$ denote, respectively, the G-modules arising from the action of G on the maximal chains of $P(S)$ and on the top homology of $P(S)$. Then from [Sta82] one has, as virtual G-modules,

$$\beta_P(S) = \sum_{T \subseteq S} (-1)^{|T|-|S|} \alpha_P(T). \tag{26}$$

Let $\mathrm{sd}(P)$ denote the poset of chains in $\overline{P}$, including the empty chain. We append an artificial top element to make $\mathrm{sd}(P)$ bounded. Then the order complex of the poset $\mathrm{sd}(P)$ is the barycentric subdivision of the order complex of P, and hence the two order complexes are homeomorphic. Thus $\mathrm{sd}(P)$ inherits the Cohen-Macaulay property from P, and any group of automorphisms G of P acts also on $\mathrm{sd}(P)$.

Stanley [Sta82, §8] gave an elegant formula for the action of G on the rank-selected homology of $\mathrm{sd}(P)$, in terms of the action on P. The relevance of this result comes from the observation (25) that the order complex of $\Omega_{dm}^{(d)}$ is the barycentric subdivision of the order complex of the d-divisible Boolean lattice $\mathcal{B}_{dm}^{(d)}$.

In order to state Stanley's theorem we require some results about the Boolean algebra. The following result of Solomon [Sol68] and Stanley [Sta82] completely determines the $\mathfrak{S}_n$-action on the rank-selected homology in the Boolean lattice $\mathcal{B}_n$. If $T = \{1 \leq t_1 < \cdots < t_r \leq n-1\}$ is a nonempty subset of the nontrivial ranks of $\mathcal{B}_n$, define the following.

- ρ_T is the rim hook (border strip) of size n whose rows, top to bottom, have lengths $n - t_r, t_r - t_{r-1}, t_{r-1} - t_{r-2}, \ldots, t_2 - t_1, t_1$, and
- f^{ρ_T} is the number of standard Young tableaux of the skew shape ρ_T. This is also the number of permutations in $\mathfrak{S}_n$ with descent set equal to $\{t_1, t_2, \ldots, t_r\}$.

Recall that the rank of an element in the Boolean lattice is its cardinality as a set. The following explicit description is equivalent to the formulation of Stanley [Sta82, Theorem 4.3].

Theorem 5.1. *Let $T = \{1 \leq t_1 < \cdots < t_r \leq n-1\}$ be a subset of the nontrivial ranks of $\mathcal{B}_n$. The representation of $\mathfrak{S}_n$ on the homology of the rank-selected subposet $\mathcal{B}_n(T)$ is isomorphic to the Specht module indexed by the rim hook ρ_T, and hence has dimension f^{ρ_T}.* □

For example, when the rank set is $\{1, 2, \ldots, j\}$, $1 \leq j \leq n-1$, the rank-selected homology representation for the Boolean lattice $\mathcal{B}_n$ is the irreducible indexed by the hook $(n-j, 1^j)$. For the rank set $\{k, k+1, \ldots, n-1\}$, $1 \leq k \leq n-1$, the representation given by the previous theorem is the one indexed by the skew partition $(k^{n-k+1})/((k-1)^{n-k})$. This can be shown to be the same as the irreducible representation corresponding to the partition $(k, 1^{n-k})$.

Now let P be a Cohen-Macaulay poset of rank r and G a group of automorphisms of P. Following [Sta82], for each $i = 0, 1, \ldots, r-1$, define $\eta_i(P)$ to be the G-module

$$\eta_i(P) = \bigoplus_{\substack{T \subseteq [r-1] \\ |T| = i}} \beta_P(T).$$

Also define, for each $i = 0, 1, \ldots, r-1$ and each subset T of $[r-1]$, the nonnegative integer $c_{i,r}(T)$ to be

$$c_{i,r}(T) = \#\{\sigma \in \mathfrak{S}_r : \sigma(r) = r - i \text{ and } \operatorname{Des}\sigma = T\}.$$

Stanley's elegant formula is the following.

Theorem 5.2 ([Sta82, Theorem 8.1])**.** *Let P be a Cohen-Macaulay poset of rank r and let $T \subseteq [r-1]$. The rank-selected homology representation of the barycentric subdivision* $\mathrm{sd}(P)$ *is given by the formula*

$$\beta_{\mathrm{sd}(P)}(T) = \bigoplus_{i=0}^{r-1} c_{i,r}(T)\, \eta_i(P). \qquad \square$$

When $P = \mathcal{B}_n$, $\eta_i(\mathcal{B}_n)$ is the *Foulkes* representation, namely, the direct sum of the Specht modules indexed by ρ_T for all subsets of ranks T of cardinality i, and the Frobenius characteristic $\mathrm{ch}\, \eta_i(\mathcal{B}_n)$ is the sum of the Schur functions indexed by ribbons ρ_T for all T of cardinality i.

Stanley's theorem and the homeomorphism (25) now imply the following result. It can be used to give another proof of Theorem 4.6. When $P = \Omega_{dm}^{(d)}$ we write

$$\beta_P(T) = \beta_{dm}^{(d)}(T).$$

Corollary 5.3. *Let $T \subseteq [m-1]$. The rank-selected homology representation of $\Omega_{dm}^{(d)}$ is given by the formula*

$$\beta_{dm}^{(d)}(T) = \bigoplus_{i=0}^{m-1} c_{i,m}(T)\, \eta_i(\mathcal{B}_{dm}^{(d))}). \qquad \square$$

Using the structure of the Whitney homology determined in the previous section, we also derive a recurrence for the homology representations of $\Omega_{dm}^{(d)}$. Equation (28) below is the analogue of the general rank-selection formula for the unordered partition lattice in [Sun94b, Theorem 2.13]. The equivalence with Corollary 5.3 is not immediately obvious.

As in the preceding sections, when G is the symmetric group, for ease of notation we will write $\alpha_P(S)$ and $\beta_P(S)$ for both the modules and their Frobenius characteristics. The context will make the distinctions clear.

Recall that the corank of an element $\tau \in \Omega_{dm}^{(d)}$ is $\mathrm{crk}(\tau) = m - \mathrm{rk}(\tau)$. In order to give less encumbered formulas for the homology representation, we describe our rank sets in $\Omega_{dm}^{(d)}$ using the corank rather than the rank.

If τ has corank t, $0 \leq t \leq m-1$, then it has $t+1$ blocks, and the interval $[\tau, \hat{1}]$ is isomorphic to a Boolean lattice $\mathcal{B}_t$ by Theorem 3.1 (d). The important fact is that the isomorphism (as described in the proof of Theorem 2.1 (d) for the case $d = 1$ and which generalizes easily to all d) maps coranks in $[\tau, \hat{1}]$ to ranks in $\mathcal{B}_t$. It will be clearest to state our result in terms of coranks. We will use the same notation for corank selection as for rank selection. Context will also make it clear which of the two is meant.

Theorem 5.4. *Let $T = \{1 \leq t_1 < \cdots < t_r \leq m-1\}$ be a nonempty subset of the* coranks *$[m-1]$ of $\Omega^{(dmd)}$. Consider the action of the symmetric group $\mathfrak{S}_{dm}$ on the corank selected poset $\Omega_{dm}^{(d)}(T)$.*

(a) *The Frobenius characteristic of the action on the maximal chains of* $\Omega_{dm}^{(d)}(T)$ *is*

$$\alpha_{dm}^{(d)}(T) = a_T \sum_{\alpha \in \mathcal{C}(m, t_r+1)} h_{d\alpha} \tag{27}$$

where

$$a_T = \frac{t_r!}{t_1!(t_2-t_1)! \cdots (t_r - t_{r-1})!}.$$

So, it is a polynomial in $\{h_{di} \mid 1 \le i \le m\}$ *with nonnegative integer coefficients.*

(b) *The Frobenius characteristic of the action on the top homology of* $\Omega_{dm}^{(d)}(T)$ *satisfies*

$$\beta_{dm}^{(d)}(T) + \beta_{dm}^{(d)}(T \setminus \{t_r\}) = \delta(T) \sum_{\alpha \in \mathcal{C}(m, t_r+1)} h_{d\alpha}, \tag{28}$$

where

$$\delta(T) = \#\{\sigma \in \mathfrak{S}_{t_r} \mid \operatorname{Des}\sigma = \{t_1, \ldots, t_{r-1}\}\}.$$

So, it is a polynomial in $\{h_{di} : 1 \le i \le m\}$ *with integer coefficients.*

Proof. (a) For the single corank $T = \{t_1\}$, $1 \le t_1 \le m-1$, it is easy to see that the action on the maximal chains of $\Omega_{dm}^{(d)}(T)$ is given by

$$\alpha_{dm}^{(d)}(\{t_1\}) = \sum_{\alpha \in \mathcal{C}(m, t_1+1)} h_{d\alpha}.$$

Let $T = \{1 \le t_1 < \cdots < t_r \le m-1\}$ be a nonempty set of coranks. Let $\tau \in \Omega_{dm}^{(d)}$ be an element at corank t_r; then τ has $t_r + 1$ blocks and composition type α of length $t_r + 1$. We examine upper intervals $[\tau, \hat{1}]_T$ in the corank-selected subposet $\Omega_{dm}^{(d)}(T)$. From Theorem 3.1 (d), one sees that each such upper interval is isomorphic to a *rank*-selected subposet of a Boolean lattice, consisting of the subsets of $[t_r]$ of sizes $t_1 < t_2 < \cdots < t_{r-1}$, that is,

$$[\tau, \hat{1}]_T \cong \mathcal{B}_{t_r}(\{t_1, \ldots, t_{r-1}\}). \tag{29}$$

The stabiliser of τ is the Young subgroup isomorphic to $\times_{i=1}^{t_r+1} \mathfrak{S}_{\alpha_i}$, and acts trivially on the chains in the interval $[\tau, \hat{1}]_T$. It follows by collecting orbits that the action on the maximal chains of $\Omega_{dm}^{(d)}(T)$ has Frobenius characteristic

$$\alpha_{dm}^{(d)}(T) = a_T \sum_{\alpha \in \mathcal{C}(m, t_1+1)} h_{d\alpha},$$

where a_T is the number of maximal chains in the rank-selected Boolean lattice $\mathcal{B}_{t_r}(\{t_1, \ldots, t_{r-1}\})$, and thus

$$a_T = \frac{t_r!}{t_1!(t_2-t_1)! \cdots (t_r - t_{r-1})!}.$$

(b) Again, in the case of a single corank $T = \{t_1\}$, $1 \le t_1 \le m-1$, it is easy to see that the action on the homology of $\Omega_{dm}^{(d)}(T)$ is given by

$$\beta_{dm}^{(d)}(\{t_1\}) = \sum_{\alpha \in \mathcal{C}(m, t_1+1)} h_{d\alpha} - h_{dm}. \tag{30}$$

Noting that $\tilde{H}(\Omega_{dm}^{(d)}(T))$ is the trivial $\mathfrak{S}_{dm}$-module when $T = \emptyset$ and that the identity permutation is the only one with empty descent set, we have verified the recurrence in this case.

We now claim that

$$\beta_{dm}^{(d)}(T) + \beta_{dm}^{(d)}(T \setminus \{t_r\}) = \operatorname{ch} \bigoplus_{\operatorname{crk}(\tau)=t_r} \tilde{H}(\tau, \hat{1})_T.$$

The argument is similar to (a), except that now we exploit the implication of the poset isomorphism (29) for the homology

$$\tilde{H}([\tau, \hat{1}]_T) \cong \tilde{H}(\mathcal{B}_{t_r}(\{t_1, \ldots, t_{r-1}\})).$$

Since the stabiliser of τ fixes every element in the interval $[\tau, \hat{1}]_T$, the action on the homology is just the trivial action, but with multiplicity equal to its dimension. Theorem 5.1 tells us that the homology of $\mathcal{B}_{t_r}(\{t_1, \ldots, t_{r-1}\})$ is the Specht module for $\mathfrak{S}_{t_r}$ indexed by the rim hook ρ_T with t_r cells, whose rows, top to bottom, have lengths

$$t_r - t_{r-1}, t_{r-1} - t_{r-2}, \ldots, t_2 - t_1, t_1.$$

Its dimension f^{ρ_T} is precisely the number $\delta(T)$. Again, by collecting elements into orbits, we see that the Frobenius characteristic of $\bigoplus_{\operatorname{crk}(\tau)=t_r} \tilde{H}(\tau, \hat{1})_T$ is the right-hand side of (28), and we are done. □

We note that this proof can be adapted to give a proof of Stanley's formula in Theorem 5.2.

We have the following pleasing consequence of this theorem and its proof. Define the ring homomorphism

$$\Psi_d : h_k \mapsto h_{dk}$$

in the ring of polynomials $\mathbb{Z}[h_1, h_2, \ldots]$ using the formal variables $\{h_k\}_{k \geq 0}$.

Corollary 5.5. *Let f and g be the Frobenius characteristics of the $\mathfrak{S}_m$-action on the homology and on the maximal chains, respectively, of a corank-selected subposet $\Omega_m(T)$ of Ω_m. Write f and g as polynomials in the complete homogeneous symmetric functions $\{h_1, h_2, \ldots\}$. Then for any $d \geq 1$, the Frobenius characteristics of the $\mathfrak{S}_{dm}$-action on the corresponding modules in $\Omega_{dm}^{(d)}$ are given by $\Psi_d(f)$ and $\Psi_d(g)$, respectively.* □

We end this section with a brief digression regarding the case $d = 1$.

Proposition 5.6. *Let $S \subseteq [m-1]$ be a subset of ranks of Ω_m, and let $\overline{S}$ be its complement in $[m-1]$. Then*

$$\tilde{H}(\Omega_m(S)) \cong \tilde{H}(\Omega_m(\overline{S})) \operatorname{sgn}_{\mathfrak{S}_m}.$$

Proof. By equation (7) we know that Ω_m is Eulerian. Also this poset is Cohen-Macaulay by Theorem 2.5. So its order complex is homotopy equivalent to a sphere $\mathbb{S}^{m-2}$. The result now follows from Alexander duality in a sphere, as in [Sta82, Theorem 2.4]. □

6 Multiplicity of trivial representation

In this section we examine two enumerative invariants that arise in rank-selection for Ω_m, namely, the multiplicities of the trivial representation for the actions of $\mathfrak{S}_m$ and $\mathfrak{S}_{m-1}$. Here we are viewing $\mathfrak{S}_{m-1}$ as the subgroup of $\mathfrak{S}_m$ which fixes m. Our motivation is the case of the analogous numbers in the unordered partition lattice Π_n, which are known to refine the Euler numbers [Sta82], [Sun94b]. A systematic study was undertaken in [Sun94b], giving results and conjectures about their positivity. Some of these were subsequently resolved by Hanlon and Hersh in [HH03], using a partitioning of the quotient complex and spectral sequences. A complete list of currently known results for Π_n appears in [Sun16, Theorem 2.12].

For Π_n, the question of determining the multiplicity of the trivial representation in the rank-selected homology is a difficult one. Stanley had originally raised this question in [Sta82]. The first vanishing result is due to Hanlon [Han83] and additional results were given by Sundaram, e.g. [Sun94b, Proposition 3.4, Theorems 4.2-4.3 and 4.7]. By contrast, Theorem 6.1 below determines this multiplicity completely for the $\mathfrak{S}_m$-action on corank-selected subposets of Ω_m and Ω_{dm}.

Inspired by Stanley's question, the paper [Sun94b] also examined the multiplicity of the trivial representation for the action of $\mathfrak{S}_{n-1}$ on the rank-selected homology of Π_n, and showed that it exhibits interesting enumerative properties. Again in contrast to the situation for Π_n, for Ω_m and Ω_{dm} we are able to give a complete determination of this restricted multiplicity in Theorem 6.3.

Consider $\Omega_{dm}^{(d)}$ and a subset of coranks $T \subseteq [m-1]$ as in Theorem 5.4. Let $b_m(T)$ denote the multiplicity of the trivial representation of $\mathfrak{S}_{dm}$ in the homology of the corank-selected subposet $\Omega_{dm}^{(d)}(T)$. From (28) and the fact that $\#\mathcal{C}(m, i+1) = \binom{m-1}{i}$, one sees that these numbers satisfy the recurrence

$$b_m(T) + b_m(T \setminus \{t_r\}) = \delta(T)\binom{m-1}{t_r} \tag{31}$$

with initial condition $b_m(\emptyset) = 1$. Moreover, since the quantities $\delta(T)$ depend only on the subset T of $[m-1]$, the invariants $b_m(T)$ are independent of d. We therefore assume without loss of generality that $d = 1$.

From (26) and Proposition 4.9, we see that the numbers $\{b_m(T) : T \subseteq [m-1]\}$ refine the factorials $(m-1)!$

$$\sum_{T \subseteq [m-1]} b_m(T) = (m-1)!. \tag{32}$$

We have the following combinatorial description of the multiplicities $b_m(T)$. Note that part (b) can be expressed as in part (a) since the number of $\sigma \in \mathfrak{S}_{m-1}$ with $m-1$ in its descent set is zero. These results have a topological explanation that we discuss at the end of this section.

Theorem 6.1. *Let T be a subset of the nontrivial coranks of Ω_m.*

(a) If $m-1 \notin T$, then

$$b_m(T) = \#\{\sigma \in \mathfrak{S}_{m-1} : \operatorname{Des}\sigma = T\}.$$

Hence $b_m(T) \geq 1$ in this case.

(b) If $m-1 \in T$, then $b_m(T) = 0$.

Proof. Both parts are obtained from Corollary 5.3, by putting $d = 1$. This gives, for Ω_m,

$$\beta_{\Omega_m}(T) = \bigoplus_{i=0}^{m-1} c_{i,m}(T)\, \eta_i(\mathcal{B}_m).$$

Recall from Theorem 5.1 that $\eta_i(\mathcal{B}_m)$ is the sum of $\mathfrak{S}_m$-irreducibles indexed by ribbons ρ_T for all T of size i. The multiplicity of the trivial representation in the representation indexed by the ribbon ρ_T is nonzero if and only if ρ_T consists of a single row, and hence the nonzero contribution comes from the term $i = 0$ in the above sum. Thus we have

$$b_m(T) = c_{0,m}(T) = \#\{\sigma \in \mathfrak{S}_m : \sigma(m) = m \text{ and } \operatorname{Des}\sigma = T\}. \tag{33}$$

Now for (a), observe that for any subset $T = \{i_1 < \cdots < i_k\} \subseteq [m-2]$, it is easy to exhibit a permutation $\sigma \in \mathfrak{S}_m$ with $\sigma(m) = m$ and descent set T. For example, let $\sigma(i_j) = m - j$, $1 \le j \le k$, and, in one-line notation, fill the remaining slots of σ with the letters in $[m-1] \setminus T$ in increasing order from left to right. As far as part (b), note that if $m-1 \in T$ then $m-1$ would have to be a descent of σ. But this is impossible since $\sigma(m) = m$. □

It is interesting to compare the symmetry result below with the sign-twisted symmetry of the homology representations resulting from Alexander duality, Proposition 5.6. In particular, $b_m(T)$ is also the multiplicity of the *sign* representation in $\beta_m([m-1] \setminus T)$.

Corollary 6.2. *Let T be a subset of the nontrivial coranks of Ω_m. The multiplicity of the trivial representation in $\tilde{H}(\Omega_m(T))$ is nonzero if and only if $m-1 \notin T$. Moreover, if $T \subseteq [m-2]$ then one then has the following symmetry:*

$$b_m(T) = b_m([m-2] \setminus T).$$

Proof. The first statement is immediate from the two parts of Theorem 6.1.

For the second statement, given $\sigma \in \mathfrak{S}_{m-1}$, let τ be the permutation defined by letting $\tau(i) = m - \sigma(i)$. The map $\sigma \mapsto \tau$ is clearly a bijection. It also has the property that $\operatorname{Des}\tau = [m-2] \setminus \operatorname{Des}\sigma$. The result now follows from Part (a) of Theorem 6.1. □

Next we consider the action of $\mathfrak{S}_{dm-1}$ on $\Omega^{(d)}_{dm}$. We follow the notation in [Sun94b]. Let $b'_m(T)$ denote the multiplicity of the trivial representation of $\mathfrak{S}_{dm-1}$ on the corank-selected homology $\tilde{H}(\Omega^{(d)}_{dm}(T))$, for every subset T of $[m-1]$.

From (28) one sees that these numbers satisfy the initial condition $b'_m(\emptyset) = 1$ and the recursion

$$b'_m(T) + b'_m(T \setminus \{t_r\}) = \delta(T)(t_r + 1)\binom{m-1}{t_r}, \tag{34}$$

since the restriction of $h_{d\alpha}$ to $\mathfrak{S}_{dm-1}$ is $\sum_{i=1}^{\ell(\alpha)} h_{d\alpha_i - 1}\left(\prod_{j \ne i} h_{d\alpha_j}\right)$, and the inner product of each term in the previous sum is 1. Again these numbers are independent of d, so that $b'_m(T)$ is also the multiplicity of the trivial representation of $\mathfrak{S}_{m-1}$ on the corank-selected homology

$\tilde{H}(\Omega_m(T))$, for every subset T of $[m-1]$. Thus, as in the case of $b_m(T)$, it suffices to analyse the case $d=1$.

From (26) and Proposition 4.9, we see that the numbers $\{b'_m(T) : T \subseteq [m-1]\}$ give the following refinement of the factorials $m!$

$$\sum_{T\subseteq[m-1]} b'_m(T) = m! \tag{35}$$

Let $\gamma_m(T) = b'_m(T) - b_m(T)$. By Frobenius reciprocity, $\gamma_m(T)$ is the multiplicity of the irreducible indexed by $(m-1,1)$ in $\tilde{H}(\Omega_m(T))$, and hence it is nonnegative. In other words, $b'_m(T) \geq b_m(T)$. From (31) and (34) we have the recurrence

$$\gamma_m(T) + \gamma_m(T \setminus \{t_r\}) = \delta(T)\, t_r \binom{m-1}{t_r}, \tag{36}$$

with $\gamma_m(\emptyset) = 0$. Since $t_r\binom{m-1}{t_r} = (m-1)\binom{m-2}{t_r-1}$, this shows that $\gamma_m(T)$ is divisible by $(m-1)$ for all T.

In [Sta82] and [Sun94b, §4], it was shown that for the rank-selected homology of the unordered partition lattice Π_n, the multiplicities of the trivial representation of $\mathfrak{S}_n$ and $\mathfrak{S}_{n-1}$ refine the Euler numbers E_{n-1} and E_n, respectively, as sums over subsets of $[n-2]$. Thus equations (32) and (35) are the corresponding analogues for Ω_m, respectively refining the factorials $(m-1)!$ and $m!$ by subsets of $[m-1]$.

For Π_n it was conjectured in [Sun94b, Conjectures, p. 289], and proved in [HH03, Theorems 2.1, 2.2], that the restricted multiplicity $b'_n(S)$ is always positive. Table 1 shows that this is not true of the restricted multiplicities $b'_m(T)$ for Ω_m. In fact, the data suggests that $b'_m(T) = 0$ if and only if T contains both the coranks $m-2, m-1$ and this is the case. Indeed, we have the following theorem.

Theorem 6.3. *Let T be a set of coranks, $T \subseteq [m-1]$. Then $b'_m(T) = 0$ if and only if T contains both the coranks $m-2, m-1$. More precisely, we have the following:*

(a) If $\{m-2, m-1\} \subseteq T$, then $b'_m(T) = 0$.

(b) If $m-1 \in T$, $m-2 \notin T$, then $b'_m(T) \geq 1$. In fact

$$b'_m(T) = \gamma_m(T) = (m-1)b_{m-1}(T \setminus \{m-1\}).$$

(c) If $m-1 \notin T$, then $b'_m(T) \geq 1$. In fact,

$$b'_m(T) = mb_m(T) - (m-1)b_{m-1}(T).$$

Proof. Since

$$b'_m(T) = \gamma_m(T) + b_m(T) \tag{37}$$

we wish to determine $\gamma_m(T)$. This is the multiplicity of the irreducible indexed by $(m-1,1)$ in the right-hand side of Corollary 5.3 when $d=1$

$$\beta_{\Omega_m}(T) = \bigoplus_{i=0}^{m-1} c_{i,m}(T)\, \eta_i(\mathcal{B}_m). \tag{38}$$

m, T $T \subseteq [m-1]$	$\mathbf{b_m(T)}$	$\mathbf{b'_m(T)}$
3, $\emptyset$	1	1
3, {1}	1	3
3, {2}	0	2
3, {1,2}	0	0
4, $\emptyset$	1	1
4, {1}	2	5
4, {2}	2	8
4, {3}	0	3
4, {1,2}	1	4
4, {1,3}	0	3
4, {2,3}	0	0
4, {1,2,3}	0	0
5, $\emptyset$	1	1
5, {1}	3	7
5, {2}	5	17
5, {3}	3	15
5, {4}	0	4
5, {1,2}	3	11
5, {1,3}	5	25
5, {1,4}	0	8
5, {2,3}	3	15
5, {2,4}	0	8
5, {3,4}	0	0
5, {1,2,3}	1	5
5, {1,2,4}	0	4
5, {1,3,4}	0	0
5, {2,3,4}	0	0
5, {1,2,3,4}	0	0

m, T $T \subseteq [m-1]$	$\mathbf{b_m(T)}$	$\mathbf{b'_m(T)}$
6, $\emptyset$	1	1
6, {1}	4	9
6, {2}	9	29
6, {3}	9	39
6, {4}	4	24
6, {5}	0	5
6, {1,2}	6	21
6, {1,3}	16	71
6, {1,4}	11	66
6, {1,5}	0	15
6, {2,3}	11	51
6, {2,4}	16	96
6, {2,5}	0	25
6, {3,4}	6	36
6, {3,5}	0	15
6, {4,5}	0	0
6, {1,2,3}	4	19
6, {1,2,4}	9	54
6, {1,2,5}	0	15
6, {1,3,4}	9	54
6, {1,3,5}	0	25
6, {1,4,5}	0	0
6, {2,3,4}	4	24
6, {2,3,5}	0	15
6, {2,4,5}	0	0
6, {3,4,5}	0	0
6, {1,2,3,4}	1	6
6, {1,2,3,5}	0	5
6, {1,2,4,5}	0	0
6, {1,3,4,5}	0	0
6, {2,3,4,5}	0	0
6, {1,2,3,4,5}	0	0

Table 1: The multiplicities $b_m(T)$ and $b'_m(T)$ by corank subsets T, for $3 \leq m \leq 6$. The second column adds up to $(m-1)!$ and the third column to $m!$.

The multiplicity of $(m-1,1)$ in the irreducible indexed by a ribbon $\rho(T)$ is nonzero if and only if the ribbon has only two rows, in which case it is 1. There are $m-1$ such ribbons of size m, and hence the multiplicity of $(m-1,1)$ in (38) equals

$$(m-1)\, c_{1,m}(T) = (m-1)\#\{\sigma \in \mathfrak{S}_m : \sigma(m) = m-1 \text{ and } \operatorname{Des}\sigma = T\},$$

and

$$b'_m(T) = b_m(T) + (m-1)\, c_{1,m}(T). \tag{39}$$

If $m-1 \in T$ then, by Theorem 6.1 (b), we have $b_m(T) = 0$. We also see that when $m-1 \in T$ and $\sigma \in \mathfrak{S}_m$ we have

$$\sigma(m) = m-1 \text{ and } \operatorname{Des}\sigma = T \iff \sigma(m) = m-1, \sigma(m-1) = m \text{ and } \operatorname{Des}\sigma = T.$$

Hence

$$c_{1,m} = \#\{\sigma \in \mathfrak{S}_{m-2} : \operatorname{Des}\sigma = T \setminus \{m-1\}\}.$$

So, $c_{1,m} = 0$ if and only if $m-2 \in T$. Substituting all this information into (37) proves both (a) and (b).

For (c), suppose $m-1 \notin T$. It follows that $T \subseteq [m-2]$. We claim that

$$c_{1,m}(T) + b_{m-1}(T) = b_m(T). \tag{40}$$

First note that if σ is counted by $c_{1,m}$ then $\sigma(m) = m-1 > \sigma(m-1)$ since $m-1 \notin T$. So, by definition, $c_{1,m}$ counts the number of bijections $\tau : \{1, \ldots, m-2\} \to \{1, \ldots, m\} \setminus \{\sigma(m-1), m-1\}$ such that $\tau(j) > \tau(j+1)$ if and only if $j \in T$.

On the other hand, by equation (33), $b_m(T)$ counts the number of permutations σ in $\mathfrak{S}_m$ with $\sigma(m) = m$ and $\operatorname{Des}\sigma = T$. This set can be decomposed according to the image of $m-1$. By the preceding paragraph, the permutations with $\sigma(m-1) < m-1$ are counted by $c_{1,m}(T)$. And if $\sigma(m-1) = m-1$ then the permutations are counted by b_{m-1}. This completes the proof of (40).

Combining (39) and (40) gives the equation in part (c) of the theorem. As far as the inequality, we have $b'_m(T) \geq b_m(T)$ from (39). And, by Theorem 6.1 (a), we have $b_m(T) \geq 1$ in this case. Combining the two inequalities finishes the proof. □

We now give the promised topological explanation for Theorem 6.1. The reader may have noticed that the multiplicities $b_m(T)$, $T \subseteq [m-1]$, coincide with the rank-selected Betti numbers for the Boolean lattice $\mathcal{B}_{m-1}$. This is no accident, as we now explain. We refer to [HH03] for some background on quotient complexes and to [Sta12, §3.13] for flag f- and h-vectors.

Recall that we write P^* for the dual of the poset P. If P has a $\hat{0}$ and a $\hat{1}$ and G is a group of automorphisms of P, the quotient complex $\Delta(P)/G$ consists of the G-orbits of the faces of $\Delta(P)$, i.e., the G-orbits of the chains of the proper part $\overline{P}$. See, e.g., [HH03, p. 522]. Furthermore, the multiplicity of the trivial representation of G in the rank-selected homology of $P(T)$ for a rank-set T is given by the flag h-vector $h_T(\Delta(P)/G)$ of the quotient complex [HH03, p. 523].

Define the orbit poset P/G as follows. Its elements are the G-orbits $\mathcal{O}_x$ of the elements x of P, with order relation $\mathcal{O}_x < \mathcal{O}_y$ in P/G if there exist $x' \in \mathcal{O}_x$ and $y' \in \mathcal{O}_y$ such

$$\{1\} \text{ —— } \{1,2\} \text{ —— } \{2\}$$

Figure 2: The quotient complex $\Delta(\Omega_3)/\mathfrak{S}_3$

that $x' < y'$ in P. The quotient complex $\Delta(P)/G$ does not usually coincide with the order complex $\Delta(P/G)$ of the orbit poset P/G. In general the quotient complex may not even be a simplicial complex; an example is the ordinary partition lattice Π_n with the $\mathfrak{S}_n$-action [HH03, p. 522]. But when $P = \Omega_{dm}$ the quotient complex is not only a simplicial complex, but also equals the order complex of the orbit poset, as we show in Proposition 6.4 below.

Let $[\mathcal{B}_{m-1}]$ denote the face lattice of the $(m-2)$-dimensional ball on $m-1$ vertices, or equivalently the $(m-2)$-dimensional simplex on $(m-1)$ vertices, with an artificially appended $\hat{1}$. Hence, as a poset, $[\mathcal{B}_{m-1}]$ consists of all 2^{m-1} subsets of $[m-1]$, with an additional $\hat{1}$ appended.

The order complex of $[\mathcal{B}_{m-1}]$ is the barycentric subdivision of the $(m-2)$-dimensional simplex. It is therefore contractible. We record the following observations.

1. $[\mathcal{B}_{m-1}]$ is a ranked Cohen-Macaulay poset, of rank m.

2. The proper part of $[\mathcal{B}_{m-1}]$ then has a unique maximal element, the set $[m-1]$ of all $m-1$ elements, and hence the order complex of $[\mathcal{B}_{m-1}]$ is contractible.

3. For the same reason, the order complex of any rank-selected subposet $[\mathcal{B}_{m-1}](T)$ where $m-1 \in T$ is also contractible.

4. The rank-selected subposet $[\mathcal{B}_{m-1}](T)$ for a rank-set T *not* containing rank $m-1$ coincides with the rank-selected subposet $\mathcal{B}_{m-1}(T)$ of $\mathcal{B}_{m-1}$. In particular, when $T = [m-2]$, $[\mathcal{B}_{m-1}](T)$ coincides with the Boolean algebra $\mathcal{B}_{m-1}$.

As an example we compute the quotient complex of $\Delta(\Omega_3)$ by the action of $\mathfrak{S}_3$. Figure 1 illustrates Ω_3. The 6 ordered partitions at corank 1 are of the form $(\{a,b\},c)$ or $(a,\{b,c\})$ where $\{a,b,c\} = [3]$, and hence fall into two $\mathfrak{S}_3$-orbits that we label $\{1\}$ and $\{2\}$ respectively. The atoms at corank 2 constitute a single orbit that we label $\{1,2\}$. The twelve chains between coranks 1 and 2 also fall into two orbits, corresponding to the two edges in Figure 2. Hence the quotient complex looks like the one-dimensional simplicial complex shown in Figure 2. This is precisely the order complex of the (dual of the) poset $[\mathcal{B}_2]$. Note that coranks in Ω_3 correspond to ranks in $[\mathcal{B}_2]$.

The proof of the next result highlights the precise mapping between $\mathfrak{S}_{dm}$-orbits of chains in the dual of Ω_{dm}, and chains in $[\mathcal{B}_{m-1}]$. It will be convenient to use compositions $\alpha = (\alpha_1, \ldots, \alpha_r)$ of m. Let C_m denote the set of compositions of m. For two compositions $\alpha, \beta \in C_m$, we say β is a refinement of $\alpha = (\alpha_1, \ldots, \alpha_r)$ if β is obtained from α by replacing each α_i with a composition of α_i. For example, the composition $(1,2,1,1,3,2)$ is a refinement of the composition $(1,4,5)$. With respect to this order, C_m is a poset with minimal element (1^m), where (1^m) is the composition of all 1's, and maximal element (m). There is a well-known bijection (see [Sag20, Theorem 1.7.1]) mapping compositions with r parts to subsets of $[m-1]$ of size $r-1$, namely

$$\alpha = (\alpha_1, \dots, \alpha_r) \mapsto \{\alpha_1, \alpha_1 + \alpha_2, \dots, \sum_{j=1}^{r-1} \alpha_j\}. \tag{41}$$

This makes the poset C_m isomorphic to the dual of the Boolean algebra $\mathcal{B}_{m-1}$, with the minimal composition (1^m) mapping to the maximal subset $[m-1]$, and the maximal composition (m) mapping to the empty set.

Finally, define $[C_m]$ to be the poset C_m augmented with an additional $\hat{0}$ appended, i.e.,

$$[C_m] = C_m \cup \{\hat{0}\}.$$

Thus $[C_m]$ is poset-isomorphic to the dual of $[\mathcal{B}_{m-1}]$ as defined above.

Proposition 6.4. *For $m \geq 0$ and $d \geq 1$ we have the following.*

(a) There is an order-preserving isomorphism between the faces of the quotient complex $\Delta(\Omega_{dm})/\mathfrak{S}_{dm}$ and the faces of the order complex of the augmented composition poset $[C_m]$.

(b) The quotient complex $\Delta(\Omega^*_{dm})/\mathfrak{S}_{dm}$ is isomorphic to the order complex of $[\mathcal{B}_{m-1}]$. In particular, it is contractible.

(c) The quotient complex $\Delta(\Omega_{dm})/\mathfrak{S}_{dm}$ coincides with the order complex $\Delta(\Omega_{dm}/\mathfrak{S}_{dm})$ of the orbit poset.

Proof. (a) We will show that the chains of the quotient complex $\Delta(\Omega_{dm})/\mathfrak{S}_{dm}$ are in order-preserving bijection with the chains in $C_m \setminus \{[m]\}$, the proper part of $[C_m]$. We will do this when $d = 1$. The proof carries over to arbitrary d almost verbatim, since compositions of dm with all parts divisible by d are in bijection with compositions of m.

Let $c = \omega_1 < \omega_2 < \cdots < \omega_r$ be a chain in the proper part of Ω_m, so $\omega_r \neq [m]$. Let $\omega_i = (B^i_1, \dots, B^i_{t_i})$, with block sizes $b^i_1, \dots, b^i_{t_i}$. Then the composition type $\beta^i = (b^i_1, \dots, b^i_{t_i})$ of ω_i is a composition of m with $\beta^i \neq [m]$ for all i, i.e., an element of $C_m \setminus \{[m]\}$. Moreover the definition of the order relation in Ω_m implies that $\beta^i < \beta^{i+1}$ in the poset of compositions C_m. Note that if ω_1 is an atom, then its composition type is $\beta^1 = (1^m)$. Now recall that the order complex of a poset consists of simplices which are chains in the *proper* part of the poset. Hence we have a surjection $f : \Delta(\Omega_m) \longrightarrow \Delta([C_m])$. Note that f can be viewed as an extension of the bijection in the proof of Theorem 2.1 (d) to chains, where B_{m-1} takes the place of C_m.

This is clearly an order-preserving (and rank-preserving) surjection, and the equivalence classes under this surjection are precisely the $\mathfrak{S}_m$-orbits of $\Delta(\Omega_m)$. This can be seen by observing that an orbit representative of a chain c is uniquely determined, for example, by writing the elements in each block of the bottom element of the chain in increasing order.

It is best to illustrate this with an example. In Ω_9, consider the chain

$$c = (1, 2, 345, 6, 7, 89) < (12, 3456, 789) < (123456, 789).$$

It maps to the following chain of compositions in $[C_9]$

$$f(c) = (1,1,3,1,1,2) < (2,4,3) < (6,3)$$

Any other chain in the preimage of the chain $f(c)$ has the form

$$(a_1, a_2, a_3a_4a_5, a_6, a_7, a_8a_9) < (a_1a_2, a_3a_4a_5, a_6, a_7a_8a_9) < (a_1a_2a_3a_4a_5a_6, a_7a_8a_9).$$

Hence the permutation σ defined by $\sigma(i) = a_i$ takes the chain c to the chain c' in $\Delta(\Omega_9)$.

We have shown that the preimage of the chain $f(c)$ is in fact the $\mathfrak{S}_m$-orbit of the chain c. It follows that the quotient complex $\Delta(\Omega_m)/\mathfrak{S}_m$ is homotopy equivalent to $\Delta([C_m])$.

(b) Combining (a) with the bijection (41) between C_m and $\mathcal{B}_{m-1}$, it follows that coranks in Ω_m correspond to ranks in $[\mathcal{B}_{m-1}]$. Hence we have the poset isomorphism

$$\Delta(\Omega_m^*)/\mathfrak{S}_m \cong \Delta([\mathcal{B}_{m-1})].$$

(c) The demonstration of (a) shows that the orbit poset $\Omega_{dm}/\mathfrak{S}_{dm}$ coincides with the composition poset $[C_m]$. This finishes the proof of the proposition. $\square$

By the discussion preceding the proposition, this implies that the multiplicities $b_m(T)$, $T \subseteq [m-1]$, of the trivial representation in the homology of the *corank*-selected subposet $\Omega_{dm}(T)$ coincide with the flag h-vector (indexed by *ranks*) of $\Delta([\mathcal{B}_{m-1}])$. But, from Theorem 5.1, when $m-1 \notin T$, the latter are precisely the rank-selected invariants for the Boolean lattice $\mathcal{B}_{m-1}$. Again, see [Sta12, §3.13] for details. Similarly, when $m-1 \in T$, the multiplicity $b_m(T)$ is zero because the corresponding rank-selected subposet in $[\mathcal{B}_{m-1}]$ is contractible. This concludes our topological explanation of Theorem 6.1.

Finally we remark that Theorem 6.3 determines the flag h-vector for the quotient complex $\Delta(\Omega_m)/(\mathfrak{S}_{m-1} \times \mathfrak{S}_1)$. We have not investigated the structure of this quotient complex and this would be interesting to do. See [HH03, §4] for the analogous analysis for Π_n.

7 Block sizes with remainder 1

Rather than just considering ordered set partitions where all blocks have size divisible by d, one could look at those where each block size has remainder 1 when divided by d. Such ordered partitions of n are partially ordered as a subset of Ω_n. Adding a $\hat{0}$, we get the poset

$$\breve{\Omega}_n^{(d)} = \{\hat{0}\} \uplus \{\omega = (B_1, \ldots, B_k) \models [n] \mid \#B_i \equiv 1 \pmod d \text{ for all } 1 \le i \le k\}.$$

Note that $\breve{\Omega}_n^{(1)} = \Omega_n$.

In this section, we will analyze this poset. To do this, we need a generalization of the Boolean algebra. A *run* in a set $S \subseteq [n]$ is a maximum sequence of consecutive integers. For example, the runs in $S = \{2,3,4,6,8,9\}$ are $2,3,4$; 6; and $8,9$. Let

$$\mathcal{B}_n^{(d)} = \{S \subseteq [n] \mid \text{every run of } S \text{ has length divisible by } d\},$$

ordered by inclusion of sets.

Theorem 7.1. *The poset $\breve{\Omega}^{(d)}_{dm+1}$ satisfies the following.*

(a) It has $\hat{1} = ([dm+1])$.

(b) Its atoms are the ω with $\#B = 1$ for all blocks B of ω.

(c) Every $\omega \in \breve{\Omega}^{(d)}_{dm+1}$ has $dk+1$ blocks for some $k \geq 0$.

(d) It is ranked. The rank and corank of $\omega = (B_1, \ldots, B_{dk+1})$ are

$$\operatorname{rk}\omega = m - k + 1 \quad \textit{and} \quad \operatorname{crk}\omega = k.$$

In particular

$$\operatorname{rk}\breve{\Omega}^{(d)}_{dm+1} = m + 1.$$

(e) For any two ordered partitions $\psi, \omega \in \breve{\Omega}^{(d)}_{dm+1}$ we have

$$[\psi, \omega] \cong \mathcal{B}^{(d)}_{\operatorname{rk}(\psi,\omega)},$$

(f) The poset $\breve{\Omega}^{(d)}_{dm+1}$ is not a lattice for $d \geq 2$.

Proof. (a)–(e) These proofs follow the same lines as in the demonstrations of Theorems 2.1 when $d = 1$ and so are omitted.

(f) We provide a counterexample, using the notation in (1). Consider the elements of rank 2 given by

$$(1, [2, d+2], d+3, \ldots, dm+1) \quad \text{and} \quad (1, 2, [3, d+3], d+4, \ldots, dm+1).$$

Then these are covered by both

$$([1, 2d+1], 2d+2, \ldots, dm+1) \quad \text{and} \quad (1, [2, 2d+2], 2d+3, \ldots, dm+1)$$

so they have no join. □

We will now show that the Möbius function on intervals in $\breve{\Omega}^{(d)}_{dm+1}$ not containing $\hat{0}$ is given, up to sign, by a k-Catalan number as defined below. This follows from Lemma 3A about lattice paths in Chapter 1 in Narayana's book [Nar79] for which he indicates two proofs, one by inclusion-exclusion and one using determinants. In Stanley's text [Sta15], he asks for a proof of the same formula for an isomorphic poset in Problem A18. This problem was solved using algebraic manipulations by Kim and Stanton [KS25]. We will give a combinatorial proof using a sign-reversing involution on lattice paths. For simplicity, we will begin with the case $d = 2$. We will need the following lemma.

Lemma 7.2. *Let $a \in \breve{\Omega}^{(2)}_{2m+1}$ be an atom. Then*

$$\#\{\omega \in [a, \hat{1}] \mid \operatorname{crk}\omega = k\} = \binom{m+k}{2k}.$$

Proof. If $\omega \in \breve{\Omega}^{(2)}_{2m+1}$ then every block of ω has odd size. And we know from part (d) of the previous theorem that ω has an odd number of blocks. We will first look at the number of compositions which could be the type of such an ω.

Fix k and m and consider the quantity

$$A_{m,k} := \#\{\alpha = (\alpha_1, \ldots, \alpha_{2k+1}) \mid \alpha \models 2m+1 \text{ and all } \alpha_i \text{ are odd}\}.$$

Letting $\beta_i = \alpha_i + 1$ for all i we see that

$$A_{m,k} = \#\{\beta = (\beta_1, \ldots, \beta_{2k+1}) \mid \beta \models 2m+2k+2 \text{ and all } \beta_i \text{ are even}\}.$$

Now defining $\gamma_i = \beta_i/2$ for $1 \leq i \leq 2k+1$ gives

$$A_{m,k} = \#\{\gamma = (\gamma_1, \ldots, \gamma_{2k+1}) \mid \gamma \models m+k+1\} = \binom{m+k}{2k}. \tag{42}$$

We now return to the problem of counting the number of $\omega \in [a, \hat{1}]$ of corank k. By Theorem 7.1 we know that ω has $2k+1$ blocks. So its type is one of the compositions in the set defining $A_{m,k}$ above. But each such type corresponds to a unique ω in the interval because once the type is specified, the blocks must be filled consistent with the fact that $\omega \geq a$. It follows that the binomial coefficient in equation (42) also enumerates the ordered set partitions in question. $\square$

We now review the necessary facts about sign-reversing involutions and lattice paths. Let S be a finite set and let $\iota : S \to S$ be an involution, i.e., a bijection such that $\iota^2 = \iota$. Considering ι as a permutation of S, it can be decomposed into cycles. And ι is an involution if and only if each of these cycles has length 1 (a fixed point), or length 2. Now suppose that S is signed in that one is given a function $\operatorname{sgn} : S \to \{+1, -1\}$. We say that ι is *sign reversing* if

(i1) For every fixed point (s) we have $\operatorname{sgn} s = 1$, and

(i2) For every 2-cycle (s, t) we have

$$\operatorname{sgn} s = -\operatorname{sgn} t.$$

It follows that

$$\sum_{s \in S} \operatorname{sgn} s = \# \operatorname{Fix} \iota \tag{43}$$

where $\operatorname{Fix} \iota$ is the set of fixed points of ι. This is because, by (i1), all fixed points have positive sign. And, by (i2), the summands corresponding to elements in 2-cycles cancel in pairs.

A *lattice path* is a sequence of points $P : p_0, p_1, \ldots, p_k$ in the integer lattice $\mathbb{Z}^2$. These points are connected by line segments called *steps*. The *length* of P, $\ell(P)$, is the number of steps. We will use the following steps, called *north* (N), *east* (E), and *diagonal* (D), which go from $p = (x, y)$ to $(x, y+1)$, $(x+1, y)$, and $(x+1, y+1)$, respectively. A *Motzkin path* is a lattice path, M, satisfying

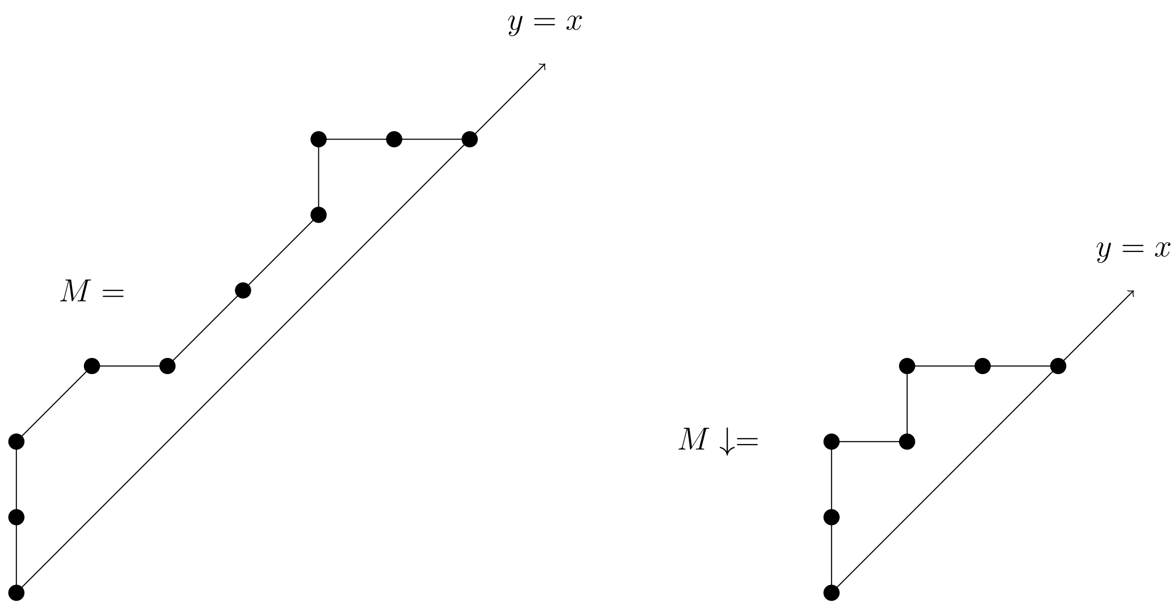

Figure 3: The restriction of a Motzkin path

(m1) M starts at $(0,0)$ and ends at (n,n) for some n, using steps N, E, and D, and

(m2) M never goes below the line $y = x$.

We will often specify a Motzkin path by listing its sequence of steps. In such a sequence, a consecutive pair of steps of the form NE will be called a *corner*. To illustrate, the Motzkin path M of Figure 3 has a unique corner formed by the seventh and eighth steps. A *Dyck path* is a Motzkin path with no diagonal steps. The *restriction* of a Motzkin path M is the Dyck path $M\downarrow$ obtained by removing all the diagonal steps of M and concatenating what remains. An example can be found in Figure 3 where $M = NNDEDDNEE$ and $M\downarrow= NNENEE$.

The last bit of notation we need is that for the *Catalan numbers*

$$C_n = \frac{1}{n+1}\binom{2n}{n}.$$

It is well known that C_n is the number of Dyck paths P ending at (n,n), equivalently, those with $\ell(P) = 2n$.

Theorem 7.3. *For any ordered set partitions $\psi \le \omega$ in $\breve{\Omega}^{(2)}_{2m+1}$ with $\operatorname{rk}(\psi,\omega) = k$ we have*

$$\mu(\psi,\omega) = (-1)^k C_k$$

Proof. By Theorem 7.1, it suffices to prove the result for intervals of the form $[\psi, \hat{1}]$. We will induct on $\operatorname{crk}\psi$, where the base case is easy. By induction, we can assume that ψ is an atom. Now $\mu(\psi,\hat{1})$ is uniquely defined as the solution to the equation

$$\sum_{\omega\in[\psi,\hat{1}]} \mu(\omega,\hat{1}) = 0$$

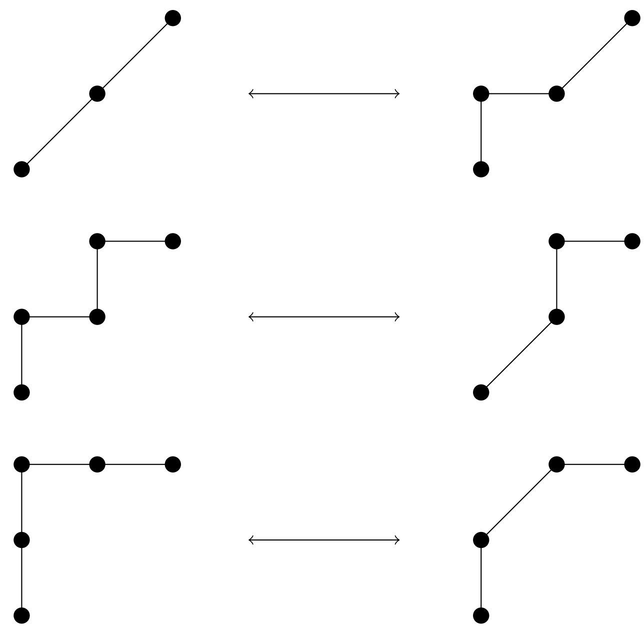

Figure 4: The involution ι of Theorem 7.3 when $m = 2$

where the values $\mu(\omega, \hat{1})$ for $\omega > \psi$ are already known by induction. Since the solution is unique, it suffices to prove that

$$\sum_{\omega \in [\psi, \hat{1}]} (-1)^{\operatorname{crk}\omega} \; C_{\operatorname{crk}\omega} = 0. \tag{44}$$

But using Lemma 7.2 we obtain

$$\sum_{\omega \in [\psi, \hat{1}]} (-1)^{\operatorname{crk}\omega} \; C_{\operatorname{crk}\omega} = \sum_{k=0}^{m} \sum_{\substack{\omega \in [\psi, \hat{1}] \\ \operatorname{crk}\omega = k}} (-1)^k C_k = \sum_{k=0}^{m} (-1)^k \binom{m+k}{2k} C_k.$$

The (unsigned) kth term in this last sum is just the number of Motzkin paths M ending at (m, m) with $m + k$ steps such that $M \downarrow$ has length $2k$. Indeed, start with the Motzkin path M_0 consisting of $m + k$ diagonal steps. Now choose $2k$ of them to be replaced by N and E steps. Finally, given one of the C_k Dyck paths P, there is a unique Motzkin path M such that $M \downarrow = P$ and the orthogonal projection of the D steps of M onto the line $y = x$ are the ones not chosen to be replaced in M_0. Also, note that the sign associated to M is $(-1)^{\ell(P)/2}$.

By (43) we will be done if we can find a sign-reversing involution, ι, without fixed points. Scan one of the Motzkin paths, M, in question from left to right until one finds the first occurrence of either a diagonal step or a pair of steps forming a corner. Define $\iota(M) = M'$ where M' is formed by switching this first occurrence from a diagonal step to a corner or vice-versa. See Figure 4 for an example where the paths of positive sign are on the left and those of negative sign are on the right. By the definition of the map, it is clearly an involution. And it is also sign-reversing since the lengths of $M \downarrow$ and $M' \downarrow$ differ by 2. Since any nontrivial Motzkin path has either a diagonal step or a corner, ι has no fixed points. This completes the proof. $\square$

To describe the Möbius function of intervals in $\breve{\Omega}_{dm+1}^{(d)}$ not containing $\hat{0}$ for all d, we need a generalization of the Catalan numbers. For $n \geq 0$ ad $k \geq 1$, the *k-Catalan numbers* are

$$C_{n,k} = \frac{1}{(k-1)n+1}\binom{kn}{n}.$$

Note that $C_{n,1} = 1$ and $C_{n,2} = C_n$. The associated generating function

$$C_k(x) = \sum_{n \geq 0} C_{n,k} x^n$$

satisfies the implicit relation

$$C_k(x) = 1 + x(C_k(x))^k.$$

To describe the associated lattice paths, define a *k-diagonal* step, $D^{(k)}$, to be one which goes from a lattice point (x, y) to $(x+1, y+k-1)$. So $D^{(2)} = D$. A *k-Motzkin path* is defined by

(M1) M starts at $(0,0)$ and ends at $(n,(k-1)n)$ for some n, using steps N, E, and $D^{(k)}$, and

(M2) M never goes below the line $y = (k-1)x$.

A k-corner in such a path is $k-1$ steps N followed by 1 step E. A *k-Dyck path* is a k-Motzkin path with no k-diagonal steps. It is well known that the number of k-Dyck paths ending at $(n,(k-1)n)$ is $C_{n,k}$. The *restriction* of a k-Motzkin path is defined similarly to the case $k = 2$.

Theorem 7.4. *For all $d \geq 1$ and ordered set partitions $\psi \leq \omega$ in $\breve{\Omega}_{dm+1}^{(d)}$ with $\mathrm{rk}(\psi, \omega) = k$*

$$\mu(\psi, \omega) = (-1)^k C_{k,d}. \tag{45}$$

Proof. The case $d = 1$ is equation (7), so we assume that $d \geq 2$. Much of the demonstration parallels that of the case when $d = 2$, so we will only provide details for the differences. Again, we need to prove equation (44) where now ψ lies in $\breve{\Omega}_{dm+1}^{(d)}$. Using a d-analogue of Lemma 7.2, we see that this is equivalent to

$$\sum_{k=0}^{m} (-1)^k \binom{(d-1)k+m}{dk} C_{k,d} = 0.$$

Now the terms in the sum count d-Motzkin paths ending at $(m,(d-1)m)$ with $(d-1)k+m$ steps such that $M\downarrow$ has dk steps. The involution switches a d-diagonal step with a d-corner in the same way as when $d = 2$. The reader should now be able to fill in the details. □

We will now discuss the Möbius values $\mu(\hat{0}, \omega)$ in $\breve{\Omega}_{dm+1}^{(d)}$. It is easy to see that if $\omega = (B_1, \ldots, B_l)$ then we have the reduced product

$$[\hat{0}, \omega] \cong \breve{\Omega}_{\#B_1}^{(d)} \dot{\times} \cdots \dot{\times} \breve{\Omega}_{\#B_1}^{(d)}. \tag{46}$$

So, by Theorem 3.3, it suffices to compute $\mu(\breve{\Omega}_{dm+1}^{(d)})$. Motivated by Theorem 3.4 (b), let us define the *remainder* 1 *Euler numbers* by

$$\breve{\mathcal{E}}_n^{(d)} = \mu(\breve{\Omega}_n^{(d)}) \tag{47}$$

where μ is zero if $n \not\equiv 1 \pmod d$. This notation is extended to ordered set partitions $\omega = (B_1, \ldots, B_l)$ by letting

$$\breve{\mathcal{E}}_\omega^{(d)} = \breve{\mathcal{E}}_{\#B_1}^{(d)} \cdots \breve{\mathcal{E}}_{\#B_l}^{(d)}.$$

On the generating function level, we let

$$\breve{\mathcal{E}}_d(x) = \sum_{n \geq 1} \breve{\mathcal{E}}_n^{(d)} \ \frac{x^n}{n!}.$$

We will also need the series

$$F_d(x) = \sum_{m \geq 0} \frac{x^{dm+1}}{(dm+1)!}.$$

Other notation includes

$$\omega \breve{\models}_d [n]$$

to indicate that ω is an ordered set partition of $[n]$ where all parts have size congruent to 1 modulo d. Finally we will need the round up and round down functions $\lceil \ell/d \rceil$ and $\lfloor \ell/d \rfloor$, respectively. Here are some of the properties of the $\breve{\mathcal{E}}_n^{(d)}$.

Theorem 7.5. *Suppose $d, n \geq 1$ and $n \equiv 1 \pmod d$.*

(a) We have

$$\breve{\mathcal{E}}_n^{(d)} = \sum_{\omega \breve{\models}_d [n]} (-1)^{\lceil \ell/d \rceil} \ C_{\lfloor \ell/d \rfloor, d}$$

where $\ell = \ell(\omega)$.

(b) We have

$$\breve{\mathcal{E}}_n^{(d)} = -1 + \sum_{\substack{\omega \breve{\models}_d [n] \\ \omega \neq ([n])}} (-1)^\ell \ \breve{\mathcal{E}}_\omega^{(d)}$$

where $\ell = \ell(\omega)$.

(c) We have

$$\breve{\mathcal{E}}_d(x) = -F_d(x) \cdot C_d(-F_d(x)^d).$$

(d) We have

$$\breve{\mathcal{E}}_d(x) = -F_d(x) \cdot (1 - (-1)^d \breve{\mathcal{E}}_d(x)^d).$$

Proof. (a) Recall definitions (47) and (5), equation (45), as well as the fact that in $\breve{\Omega}_n^{(d)}$ we have $\operatorname{crk}\omega = \lfloor \ell/d \rfloor$ where $\ell = \ell(\omega)$. Putting these together gives

$$\begin{aligned}
\breve{\mathcal{E}}_n^{(d)} &= \mu(\breve{\Omega}_n^{(d)}) \\
&= -\sum_{\omega \breve{\models}_d [n]} \mu(\omega, \hat{1}) \\
&= -\sum_{\omega \breve{\models}_d [n]} (-1)^{\lfloor \ell/d \rfloor}\, C_{\lfloor \ell/d \rfloor, d}
\end{aligned}$$

and bringing the negative sign inside the sum completes the proof of this part.

(b) Using definition (47), equation (46), and Theorem 3.3 we see that if an ordered set partition has $\operatorname{type}(\omega) = (\alpha_1, \ldots, \alpha_\ell)$ then

$$\mu(\hat{0}, \omega) = (-1)^{\ell-1} \mu(\breve{\Omega}_{\alpha_1}^{(d)}) \cdots \mu(\breve{\Omega}_{\alpha_\ell}^{(d)}) = (-1)^{\ell-1} \breve{\mathcal{E}}_\omega^{(d)}.$$

Noting that this also applies to $\omega = ([n])$ and that $\mu(\hat{0}, \hat{0}) = 1$, we can bring all the terms in the desired equality over to the left side and write it as

$$\sum_{x \in \breve{\Omega}_n^{(d)}} \mu(\hat{0}, x) = 0.$$

where x runs over both the ordered set partitions and $\hat{0}$ in $\breve{\Omega}_n^{(d)}$. But this is true by definition (4).

(c) Similar to the proof of Corollary 4.7 we have, for any exponential generating function of the form $E(x) = c_1 x/1! + c_2 x^2/2! + \cdots$,

$$\begin{aligned}
E(x)^\ell &= \sum_{n \geq \ell} \left(\sum_{\alpha = (\alpha_1, \ldots \alpha_\ell) \models n} \frac{c_{\alpha_1} x^{\alpha_1}}{\alpha_1!} \cdots \frac{c_{\alpha_\ell} x^{\alpha_\ell}}{\alpha_\ell!} \right) \\
&= \sum_{n \geq \ell} \left(\sum_{\alpha \models n} \binom{n}{\alpha} c_{\alpha_1} \cdots c_{\alpha_\ell} \right) \frac{x^n}{n!} \\
&= \sum_{n \geq \ell} \left(\sum_{\omega = (B_1, \ldots, B_\ell) \models [n]} c_{\# B_1} \cdots c_{\# B_\ell} \right) \frac{x^n}{n!} \qquad (48)
\end{aligned}$$

where the last equality comes from the fact that the number of ordered set partitions ω with $\operatorname{type}\omega = \alpha$ is the multinomial coefficent.

Write $n = dm + 1$ and note that if $\omega \in \breve{\Omega}_n^{(d)}$ then $\ell(\omega)$ and the size of its blocks are congruent to 1 modulo d. Combining these facts with part (a) and the computations in the previous paragraph gives

$$\breve{\mathcal{E}}_d(x) = -C_{0,d} F_d(x) + C_{1,d}(x) F_d(x)^{d+1} - C_{2,d} F_d(x)^{2d+1} - \cdots.$$

Factoring out $-F_d(x)$ and simplifying completes the proof of part (c).

(d) From part (b) we have

$$0 = -1 + \sum_{\omega \breve{\models}_d [n]} (-1)^\ell \, \breve{\mathcal{E}}_\omega^{(d)}$$

Let $n = dm + 1$, multiply the previous displayed equation by $x^{dm+1}/(dm+1)!$, sum over m, and use (48) to get

$$0 = -F_d(x) + \sum_{m \geq 0} (-1)^{dm+1} \breve{\mathcal{E}}_d(x)^{dm+1} = -F_d(x) - \frac{\breve{\mathcal{E}}_d(x)}{1 - (-1)^d \breve{\mathcal{E}}_d(x)^d}.$$

Solving for $\breve{\mathcal{E}}_d(x)$ finishes the demonstration of (d) and of the theorem. □

One could hope that $\breve{\Omega}_n^{(d)}$ has an RAO. Unfortunately, this is not the case, at least if one uses the same lexicographic order employed for $\Omega_{dn}^{(d)}$ in the proof of Theorem 3.5. Consider what happens when $n = 7$ and $d = 2$. By Theorem 7.1 (f), $\breve{\Omega}_n^{(d)}$ need not be a lattice, much less semimodular. So we need to show that condition (R1) in the definition of an RAO holds for the full poset $\breve{\Omega}_7^{(2)}$. Consider the atom $a = (4, 1, 2, 3, 5, 6, 7)$. Then the interval $[a, \hat{1}]$ contains the atoms and $\psi = (4, 1, 2, 653, 7)$ and $\omega = (421, 3, 5, 6, 7)$. Note that lexicographically $\psi <_l \omega$ since $4 <_l 421$. But a is the smallest atom of $\breve{\Omega}_7^{(2)}$ below ψ since all such atoms are of the form $(4, 1, 2, a, b, c, 7)$ where abc is a permutation of $3, 5, 6$, And 356 as it appears in a is the lexicographically smallest such permutation. On the other hand, ω covers $(1, 2, 4, 3, 5, 6, 7)$ which is lexicographically smaller than a. So (R1) is violated.

It would be natural to consider posets derived from ordered set partitions of n where all block sizes are congruent to r modulo d for $r > 1$ and with a $\hat{0}$ added. Unfortunately, these posets do not seem to be well behaved. In fact, they are not even graded. For example, take $r = 2$, $d = 3$, and $n = 14$. Then, using the notation in (1), one maximal chain in this poset is

$$\hat{0} < ([1, 5],\ [6, 10],\ [11, 12],\ [13, 14]) < \hat{1}$$

which is of length 2. On the other hand, the maximal chains containing the atom

$$([1, 2],\ [3, 4],\ [5, 6],\ [7, 8],\ [9, 10],\ [11, 12],\ [13, 14])$$

all have length 3.

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