A new notion of transitivity for groups and sets of permutations

WILLIAM J. MARTIN and BRUCE E. SAGAN

[1] Department of Mathematical Sciences
and Department of Computer Science
Worcester Polytechnic Institute
martin@wpi.edu
[2] Department of Mathematics
Michigan State University
sagan@math.msu.edu

Abstract. Let $\Omega = \{1, 2, \ldots, n\}$ where $n \geq 2$. The *shape* of an ordered set partition $P = (P_1, \ldots, P_k)$ of Ω is the integer partition $\lambda = (\lambda_1, \ldots, \lambda_k)$ defined by $\lambda_i = |P_i|$. Let G be a group of permutations acting on Ω. For a fixed partition λ of n, we say that G is λ-*transitive* if G has only one orbit when acting on partitions P of shape λ. A corresponding definition can also be given when G is just a set. For example, if $\lambda = (n - t, 1, \ldots, 1)$, then a λ-transitive group is the same as a t-transitive permutation group and if $\lambda = (n - t, t)$, then we recover the t-homogeneous permutation groups.
We use the character theory of the symmetric group S_n to establish some structural results regarding λ-transitive groups and sets. In particular, we are able to generalize a celebrated result of Livingstone and Wagner [11] about t-homogeneous groups. We survey the relevant examples coming from groups. While it is known that a finite group of permutations can be at most 5-transitive unless it contains the alternating group, we show that it is possible to construct a nontrivial t-transitive set of permutations for each positive integer t. We also show how these ideas lead to a combinatorial basis for the Bose-Mesner algebra of the association scheme of the symmetric group and a design system attached to this association scheme.

1 Introduction

Throughout, let $n \geq 2$ be an integer and let $\Omega = \{1, \ldots, n\}$. Suppose that G is a subgroup of the symmetric group S_n. Recall that G is t-*transitive* if it is transitive on ordered t-tuples of distinct elements of Ω. It is trivial to see that if G is t-transitive, then it is $(t-1)$-transitive. Similarly, G is t-*homogeneous* if it is transitive on t-element subsets of Ω. The following is one of the main theorems in a paper of Livingstone and Wagner.

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Theorem 1 ([11]). *Let $G \leq S_n$ be t-homogeneous where $t \leq n/2$. Then G is also $(t-1)$-homogeneous.* $\square$

Our goal is to define a more general notion of transitivity so that both the t-transitive and t-homogeneous results will be special cases. To this end, let an *ordered set partition of Ω* be an ordered tuple

$$P = (P_1, \dots, P_k)$$

of pairwise disjoint nonempty subsets of Ω whose union is Ω. These partitions are also called *tabloids* and play an important role in the representation theory of S_n [8, 15]. We will only need to consider the case where $|P_i| \geq |P_{i+1}|$ for all $1 \leq i < k$. On the other hand, an *(integer) partition λ of n*, written $|\lambda| = n$, is a sequence

$$\lambda = (\lambda_1, \dots, \lambda_k)$$

of positive integers whose sum is n and satisfying the monotonicity condition

$$\lambda_1 \geq \lambda_2 \geq \cdots.$$

The λ_i are called the *parts* of λ. The *shape* of the set partition P is the integer partition

$$|P| = (|P_1|, \dots, |P_k|).$$

Let λ be a fixed partition of n. A subgroup G of $S_n := \mathsf{Sym}(\Omega)$ is said to be *λ-transitive* if it is transitive on partitions P of shape λ. Equivalently, for any two set partitions P and Q with $|P| = |Q| = \lambda$, there exists $g \in G$ taking P to Q in the sense that

$$gP_i = Q_i$$

(as a set) for each i, $1 \leq i \leq k$. This definition includes that of a t-transitive permutation group (use $\lambda = (n-t, 1, \dots, 1)$) and that of a t-homogeneous permutation group (use $\lambda = (n-t, t)$).

We extend this definition to arbitrary sets of permutations as follows. For $D \subseteq S_n$ ($D \neq \emptyset$) and λ a partition of n, we say that D is *λ-transitive* if there is a constant r_λ such that, for any two set partitions P and Q of shape λ, D contains exactly r_λ permutations g taking P to Q. We will also use the terminology "t-transitive," respectively "t-homogeneous," for sets of permutations which are $(n-t, 1, \dots, 1)$-transitive, respectively $(n-t, t)$-transitive.

The following two propositions follow directly from the definitions.

Proposition 1. *If $G \leq S_n$ is a λ-transitive group of permutations acting on Ω, then G is also λ-transitive as a set.* $\square$

Proposition 2. *If $D \subseteq S_n$ is a λ-transitive set of permutations acting on Ω and $G = \langle D \rangle$ is the subgroup generated by D, then G is a λ-transitive group.* $\square$

Each set partition P determines a Young subgroup $Y = Y_P$ which consists of the set of permutations of Ω which fix each subset P_i of P setwise. We say Y has shape λ where $\lambda = |P|$. There is a bijection between all set partitions of shape λ and all left cosets of Y_P that preserves the action of S_n. Using this observation, it is easy to prove the following characterization of λ-transitivity.

Lemma 1. *Suppose $|\lambda| = n$ and $D \subseteq S_n$. Then D is λ-transitive if and only if there is a constant r_λ such that for any coset gY_P of any Young subgroup of shape λ we have $|gY_P \cap D| = r_\lambda$.* □

The last definition that will be necessary to understand our generalization of the Livingstone-Wagner Theorem is that of the dominance order on integer partitions. If $|\lambda| = |\mu| = n$ then λ *is dominated by* μ, written $\lambda \trianglelefteq \mu$, if and only if

$$\sum_{i=1}^{j} \lambda_i \leq \sum_{i=1}^{j} \mu_i \qquad \text{(for all } j\text{)}$$

where we define $\lambda_i = 0$ if i is greater than the number of parts in λ and similarly for μ. We will give two proofs that if $\mu \trianglerighteq \lambda$ then λ-transitivity implies μ-transitivity, from which Theorem 1 follows immediately. (But see also [14].) The first, which is just for groups but also includes a stronger result about orbits, is essentially a generalization of the one given by Livingstone and Wagner themselves. The second proof holds for all λ-transitive sets D. But if D is not a group then it does not have orbits and so a combinatorial approach is needed in this case.

2 The group case

Let the permutation group G act on two sets S and T with characters χ and ψ, respectively. The inequality $\chi \geq \psi$ means that $\chi = \psi + \phi$ where ϕ is some character of G. Equivalently, if $\chi^1, \ldots, \chi^k$ are the irreducible characters of G and we write $\chi = \sum_i c_i \chi^i, \psi = \sum_i d_i \chi^i$ for nonnegative integers c_i, d_i then $\chi \geq \psi$ if and only if $c_i \geq d_i$ for all i. We will also need the notation

$$\mathrm{orb}(G, S) = \text{the number of orbits of } G \text{ acting on } S.$$

Lemma 2. *Let G be a permutation group acting on sets S and T with characters χ and ψ, respectively. If $\chi \geq \psi$ then*

$$\mathrm{orb}(G, S) \geq \mathrm{orb}(G, T).$$

Proof. Taking inner products with the trivial character and using $\chi = \psi + \phi$, we have

$$\mathrm{orb}(G, S) = \langle \chi, 1 \rangle = \langle \psi + \phi, 1 \rangle = \langle \psi, 1 \rangle + \langle \phi, 1 \rangle \geq \langle \psi, 1 \rangle = \mathrm{orb}(G, T). \square$$

We now apply this lemma to the situation at hand. Let P be any set partition of shape λ and let S_λ denote the set of all left cosets of the Young subgroup Y_P. Of course, S_λ depends on P and not just λ, but that fact will not play a role in the following theorem.

Theorem 2. *If $G \leq S_n$ then*

$$\lambda \trianglelefteq \mu \Longrightarrow \mathrm{orb}(G, S_\lambda) \geq \mathrm{orb}(G, S_\mu).$$

Proof. Let χ and ψ be the characters of G acting on S_λ and S_μ, respectively. Then by the previous lemma, it suffices to show that $\chi \geq \psi$. Let P and Q be ordered set partitions giving rise to the sets S_λ and S_μ, respectively. Then χ and ψ are the restrictions to G of the characters $\chi' = 1_{Y_P} \uparrow^{S_n}$ and $\psi' = 1_{Y_Q} \uparrow^{S_n}$, respectively, so we are reduced to showing $\chi' \geq \psi'$. But then by Young's Rule

$$\chi' = \sum_{\nu \trianglerighteq \lambda} K_{\nu\lambda} \chi^\nu \quad \text{and} \quad \psi' = \sum_{\nu \trianglerighteq \mu} K_{\nu\mu} \chi^\nu$$

where χ^ν is the irreducible character of S_n corresponding to ν and $K_{\nu\lambda}$ is a Kostka number. However, it is known [10, 17] that if $\lambda \trianglelefteq \mu$ then $K_{\nu\lambda} \geq K_{\nu\mu}$ for all ν. This finishes the proof. □

As an immediate corollary, we get our first generalization of the Livingstone-Wagner Theorem.

Theorem 3. *If $G \leq S_n$ is λ-transitive and $\mu \trianglerighteq \lambda$ then G is μ-transitive.* □

3 The center of the group algebra and the inner distribution

Our main tool for extracting information about λ-transitive sets will be the center of the group algebra of the symmetric group S_n, specifically in the form arising from the left regular representation $g \mapsto A(g)$ (over the reals). We will denote this commutative algebra of symmetric matrices by $\mathbb{A}$ and its dimension by $p(n)$, the number of partitions of the integer n. One may learn quite a bit about this algebra via the character theory of the symmetric group; see references [1, 8, 15]. We identify two distinguished bases of $\mathbb{A}$ as follows. The first basis contains one matrix for each conjugacy class $\mathcal{C}_\alpha$, namely

$$A_\alpha = \sum_{g \in \mathcal{C}_\alpha} A(g).$$

The second distinguished basis is the basis of primitive idempotents for the center. This basis contains one element, E_μ say, for each irreducible representation of S_n. The two bases are related by the equations

$$E_\mu = \frac{f_\mu}{n!} \sum_\alpha \chi^\mu_\alpha A_\alpha$$

where f_μ is the degree of the irreducible representation ρ_μ and χ^μ_α is the value taken by the corresponding irreducible character χ^μ on the conjugacy class $\mathcal{C}_\alpha$ [1, Theorem II.7.2].

The matrices in $\mathbb{A}$ are clearly simultaneously diagonalizable. The *eigenspace of* $\mathbb{A}$ *indexed by* μ is the column space of E_μ:

$$V_\mu = \mathsf{colsp}(E_\mu).$$

The dimension of V_μ is f_μ^2.

Let D be a nonempty subset of S_n and let $x = x_D$ denote the indicator vector of D. This is a 01-vector of length $n!$ with $x(g) = 1$ if and only if $g \in D$. The *inner distribution* of D is the rational vector of length $p(n)$ with α entry equal to

$$a_\alpha = \frac{1}{|D|} x^\top A_\alpha x.$$

The *dual distribution* of D is the vector b with entries indexed by irreducible representations and defined by

$$b_\mu = \frac{n!}{|D|} x^\top E_\mu x. \tag{1}$$

Since each E_μ is positive semidefinite, we have $b_\mu \geq 0$ and $b_\mu = 0$ if and only if $E_\mu x = 0$. And since E_μ is the matrix representing orthogonal projection onto V_μ we have another equivalent condition, namely $b_\mu = 0$ if and only if $x \perp V_\mu$.

For arbitrary nonempty D, the quantities a_α and b_μ have alternative expressions as follows:

$$a_\alpha = \frac{1}{|D|} \left|\{(g,h) \in D \times D : gh^{-1} \in \mathcal{C}_\alpha\}\right|;$$

and

$$b_\mu = \frac{f_\mu}{|D|} \sum_{g,h \in D} \chi^\mu(gh^{-1}). \tag{2}$$

When D is a group, these simplify to

$$a_\alpha = |D \cap \mathcal{C}_\alpha|$$

and

$$b_\mu = f_\mu \sum_{g \in D} \chi^\mu(g). \tag{3}$$

4 The set case

Lemma 3. *Suppose D is a coset of a Young subgroup of shape λ and b is the dual distribution of D. Then $b_\mu \neq 0$ if and only if $\mu \trianglerighteq \lambda$.*

Proof. Because of equation (2), we may assume D is a Young subgroup. In this case, by Young's rule and (3), the number b_μ is a nonzero multiple of the Kostka number $K_{\mu\lambda}$. □

Let N_λ denote the incidence matrix of S_n versus cosets of Young subgroups of shape λ. The $n!$ rows of N_λ are indexed by permutations and the columns are indexed by all left cosets hY_P of all Young subgroups of shape λ. In the entry indexed (g, hY_P), we have a 1 if $g \in hY_P$ and a 0 otherwise.

Corollary 1. *Let W_λ be the column space of N_λ. Then*

$$W_\lambda = \bigoplus_{\mu \trianglerighteq \lambda} V_\mu.$$

Proof. This space is invariant under both the left and right actions of the group algebra. So, in particular, it is $\mathbb{A}$ invariant. It is well-known that each eigenspace of a Schurian association scheme belonging to a group G is irreducible as a $(G \times G)$-module. So both W_λ and V_μ are $(S_n \times S_n)$-modules and, for each μ, the latter is irreducible. Now the result follows from the previous lemma. □

Theorem 4. *Let D be a nonempty set of permutations in S_n and λ a partition of n. Then D is λ-transitive if and only if for every μ with $\lambda \trianglelefteq \mu \triangleleft (n)$ we have $b_\mu = 0$.*

Proof. We prove the forward implication noting that the converse follows by reversing the steps of the proof.

From Corollary 1, the eigenspaces which appear in W_λ are precisely those V_μ for which $\mu \trianglerighteq \lambda$. Now if x denotes the indicator vector of D, then by Lemma 1 we have $x^\top N_\lambda = r_\lambda \mathbf{1}$ where $\mathbf{1}$ stands for the all ones vector of appropriate length. Let x_0 denote the projection of x onto W_λ. Note that x, and thus x_0, has the same inner products with columns of N_λ as does the vector $\frac{r_\lambda}{|Y_P|}\mathbf{1}$ where P is of shape λ. So the same will be true for products with the vectors from any orthonormal basis B for W_λ. It follows that $x_0 = \frac{r_\lambda}{|Y_P|}\mathbf{1}$. Since $x_0 \in V_{(n)}$, we have $x \perp V_\mu$ for all μ such that $\lambda \trianglelefteq \mu \triangleleft (n)$ and hence the theorem. □

In the case of t-transitive permutation groups, a result of this flavor was obtain by Conder and Godsil in [4] using a polynomial space rather than an association scheme. An immediate corollary of this result is our second generalization of the Livingstone-Wagner Theorem.

Theorem 5. *If the set D is λ-transitive and $\mu \trianglerighteq \lambda$, then D is μ-transitive.* □

We get a second corollary of Theorem 4 by combining it with equation (2).

Theorem 6. *The set D is λ-transitive if and only if*

$$\sum_{g,h \in D} \chi^\mu(gh^{-1}) = 0$$

for each μ such that $\lambda \trianglelefteq \mu \triangleleft (n)$. □

5 Examples

From Theorem 3, we know that a λ-transitive group of permutations is t-homogeneous where $t = \sum_{i=2}^{k} \lambda_i$. Thus we are severely restricted in our search for λ-transitive groups. Other than alternating and symmetric groups, there are no t-homogeneous groups for $t \geq 6$. (This is a consequence of the classification of finite simple groups; see [7] or [3].) There are precisely two nontrivial 5-transitive groups: the Mathieu groups M_{12} and M_{24}. Of course, a t-transitive group is λ-transitive for any partition λ refined by $(n-t, 1, \ldots, 1)$[1]. So we seek

[1] A set partition P *refines* a set partition Q if each P_i is a subset of some Q_j. A partition λ of integer n refines an integer partition μ of n if some P with $|P| = \lambda$ refines some Q with $|Q| = \mu$.

t-homogeneous groups which fail to be t-transitive. Livingstone and Wagner [11] showed that there are no such groups for $t \geq 5$. The balance of the following result is due to Kantor [9] who dealt with the cases $t = 2, 3, 4$.

Theorem 7 (Kantor, Livingstone/Wagner [7]). *Suppose G is a permutation group acting t-homogeneously on n points ($t \leq n/2$), but not t-transitively. Then one of the following is the case:*

- $t = 2$, $ASL_1(q) \leq G \leq A\Sigma L_1(q)$, $q \equiv 3 \bmod 4$;
- $t = 3$, $PSL_2(q) \leq G \leq P\Sigma L_2(q)$, $q \equiv 3 \bmod 4$;
- $t = 3$, $G = AGL_1(8)$, $A\Gamma L_1(8)$, $A\Gamma L_1(32)$;
- $t = 4$, $G = PGL_2(8)$, $P\Gamma L_2(8)$, $P\Gamma L_2(32)$.

In all cases, the group acts $(t-1)$-transitively.

For each group appearing in this theorem, we wish to determine its full level of transitivity. (For $t = 2$, there is no partition between $(n-2, 2)$ and $(n-2, 1, 1)$ to consider.)

- For $PSL_2(q)$ acting on the projective line ($q+1$ points), we find the stabilizer of a 3-set has order three. Since the action is faithful, we conclude that $PSL_2(q)$ acts $(n-3, 2, 1)$-transitively on the projective line. It follows that this is also the full degree of transitivity of $P\Sigma L_2(q)$ and any group between the two.
- $AGL_1(8)$ has order 56, so this group acts sharply 3-homogeneously on $\mathbb{F}_8$.
- $A\Gamma L_1(8)$ acts faithfully on $\mathbb{F}_8$ and the stabilizer of any 3-set has order three. So this action is $(5, 2, 1)$-transitive.
- $A\Gamma L_1(32)$ has order 4960, so this group acts sharply 3-homogeneously on $\mathbb{F}_{32}$.
- Finally, we consider the groups with $t = 4$. First, $PGL_2(8)$ has 504 elements and acts faithfully on $PG(1, 8)$ with the stabilizer of any 4-set being isomorphic to the Klein four-group, so it is $(5, 3, 1)$-transitive in this action.
- $P\Gamma L_2(8)$ has 1512 elements and acts 4-homogeneously on nine points. The stabilizer of a 4-set has size 12, and thus is isomorphic to A_4. Hence this group is $(5, 2, 1, 1)$-transitive.
- For $P\Gamma L_2(32)$, the stabilizer of a 4-set is isomorphic to the Klein four-group, so the action is $(29, 3, 1)$-transitive.

We note that, in 1975, Peter Neumann [13] gave a classification scheme for t-homogeneous groups which are not fully t-transitive in terms of the isomorphism type of the stabilizer of a t-set. In Neumann's language, a group which acts *almost generously 3-transitively* is one in which the stabilizer of any 4-set contains A_4. (He does not presume a 4-homogeneous action in this definition.) Clearly a group acting 4-homogeneously has this property exactly when it is $(n-4, 2, 1, 1)$-transitive. Neumann calls a permutation group *a little generously 3-transitive* if the stabilizer of any 4-set contains the Klein four-group. Observe that a group acting 4-homogeneously and having this property will be $(n-4, 3, 1)$-transitive.

Since there are no 4-homogeneous groups with cyclic stabilizer of a 4-set, the converse is true as well. But this observation hinges on Kantor's result.

We now construct a family of λ-transitive sets of permutations for partitions λ having the property that any λ-transitive group must be strictly larger than the set we construct.

Example 1. [2] For q an odd prime power, the affine group $AGL_1(q)$ acts 2-transitively on $\mathbb{F}_q$. For $q > 3$, there can be no 2-homogeneous group of size $\binom{q}{2}$ but there is such a set of permutations. Choose a subset S of $\mathbb{F}_q - \{0\}$ such that S is disjoint from $-S$ and $S \cup (-S) = \mathbb{F}_q - \{0\}$. (Since q is odd, this is always possible.) Let

$$D = \{x \mapsto ax + b : a \in S, b \in \mathbb{F}_q\}.$$

We claim that this set is $(q-2, 2)$-transitive. For if the mapping $x \mapsto ax + b$ takes u to y and v to z, then the mapping which takes u to z and v to y (in $AGL_1(q)$) is $x \mapsto (-a)x + b'$ for some b'. Thus no two elements of D take an unordered pair $\{u, v\}$ to the same image. □

6 A recursive construction

In this section, we show that, given any partition λ of a positive integer, m say, if we are permitted to increase the largest part (thus obtaining a partition λ' of some integer $n \geq m$), we can construct a λ'-transitive set of permutations D which is "small" in comparison to $|S_n|$.

Suppose the following three structures are given:

- a t-(n, k, ν) block design on point set $\Omega = \{1, \ldots, n\}$ with block set B;
- a t-transitive set of permutations $D_1 \subseteq S_k$ acting on $\Omega_k = \{1, \ldots, k\}$;
- a t-transitive set of permutations $D_2 \subseteq S_{n-k}$.

From these, we will construct a t-transitive set D of permutations inside S_n.

We first define a number of bijections. Let

$$\tau : \{k+1, \ldots, n\} \longrightarrow \{1, \ldots, n-k\}$$

via $j \mapsto j - k$. For each block $b \in B$, choose bijections

$$\phi_b : \{1, \ldots, k\} \longrightarrow b$$

and

$$\psi_b : \{1, \ldots, n-k\} \longrightarrow \Omega - b,$$

which will remain fixed.

Lemma 4. *With the above notation, the set of functions*

$$D = \{(\phi_b \circ \pi) \cup (\psi_b \circ \sigma \circ \tau) : b \in B, \pi \in D_1, \sigma \in D_2\}$$

is a t-transitive set of permutations in S_n.

Example 2. Suppose $t = 2$, $n = 7$, $k = 3$ and the blocks are those of the Fano plane

$$B = \{124, 235, 346, 457, 561, 672, 713\},$$

$D_1 = S_3$ and $D_2 = A_4$. We then obtain a 2-transitive set of $7.6.12 = 504$ permutations in S_7. For instance, if $b = \{3, 4, 6\}$ and $\Omega - b = \{5, 7, 1, 2\}$ (listed in the order induced by ϕ_b and ψ_b) and $\pi = (12)$, $\sigma = (13)(24)$, the resulting permutation in D is

$$(\phi_b \circ \pi) \cup (\psi_b \circ \sigma \circ \tau) = (14)(2365). \qquad \square$$

The following notation will be required in the proof of this lemma. Suppose I and O are disjoint subsets of Ω. We would like to count blocks $b \in B$ which contain all of the points in I yet none of the points in O. If $|I| = i$ and $|O| = j$ and $i+j \leq t$, then there exists a constant $\nu_{i,j}$ such that there exactly $\nu_{i,j}$ blocks b such that $I \subseteq b$ and $O \cap b = \emptyset$ [3, Remark 4.17].

Suppose that $h \leq t$. If $a_1, \ldots, a_h$ are distinct elements of Ω_k and $b_1, \ldots, b_h$ are distinct elements of Ω_k, then there are exactly c_h permutations $\pi \in D_1$ such that $\pi a_j = b_j$ for all $1 \leq j \leq h$ for some constant c_h. Similarly, there exist constants $d_0, \ldots, d_t$ associated to D_2.

Proof of Lemma 4. Let $a_1 < \cdots < a_t$ be distinct elements of Ω and let $b_1, \ldots, b_t$ be distinct elements of Ω. If $a_1 > k$, set $i = 0$; otherwise, let i be the largest integer such that $a_i \leq k$ and set $j = t - i$. There are $\nu_{i,j}$ blocks b of the design (Ω, B) for which $b_1, \ldots, b_i \in b$ and $b_{i+1}, \ldots, b_t \notin b$. There are c_i permutations $\pi \in D_1$ taking a_ℓ to $\phi_b^{-1} b_\ell$ for $\ell \leq i$. There are d_j permutations σ in D_2 taking $a_\ell - k$ to $\psi_b^{-1} b_\ell$ for $i < \ell \leq t$. Thus there are

$$r = c_i d_j \nu_{i,j}$$

permutations $\rho \in D$ satisfying $\rho a_\ell = b_\ell$ for all $1 \leq \ell \leq t$. To finish the proof, we simply must show that this quantity r does not depend on i. But we have

$$\nu_{i+1,j}(n - i - j) = \nu_{i,j}(k - i)$$

and

$$\nu_{i,j+1}(n - i - j) = \nu_{i,j}(n - k - j),$$

giving us

$$\nu_{i+1,j-1}(n - k - j + 1) = \nu_{i,j}(k - i).$$

Moreover, it is obvious that $c_i = c_{i+1}(k-i)$ and $d_{j-1} = d_j(n-k-j+1)$. Putting these identities together, we find that

$$\nu_{i,j} c_i d_j = \nu_{i+1,j-1} c_{i+1} d_{j-1}$$

which completes our proof. $\square$

We will now make use of the following result:

Theorem 8 (Teirlinck [16]). *Given integers t and n with*

$$n \equiv t \pmod{(t+1)!^{2t+1}}$$

and $n \geq t+1 > 0$, there exists a t-$(n, t+1, (t+1)!^{2t+1})$ design without repeated blocks. □

Using these designs in the previous construction, we obtain the following asymptotic existence theorem.

Theorem 9. *Let $\lambda_2 \geq \lambda_3 \geq \cdots \geq \lambda_t$ be a nonincreasing sequence of positive integers and let $\epsilon > 0$. Then there exist infinitely many values for $\lambda_1 \geq \lambda_2$ such that there exists a λ-transitive set D of permutations inside S_n ($n = \lambda_1 + \cdots + \lambda_t$) having $|D|/n! < \epsilon$.*

Proof. It suffices to prove this for $\lambda_2 = \cdots = \lambda_t = 1$. Let $k = t+1$ and choose $n \geq 2t+1$ to satisfy Teirlinck's theorem. Then there exists a block design B on n points with each t-set appearing in exactly $(t+1)!^{2t+1}$ blocks. Using S_{t+1} and S_{n-t-1} as our component t-transitive permutation sets, we obtain a t-transitive set D of permutations in S_n with

$$|D| = |B| \cdot (t+1)! \cdot (n-t-1)! = \frac{(t+1)!^{2t+1}}{t+1} \cdot \binom{n}{t} \cdot (t+1)! \cdot (n-t-1)!,$$

that is, $|D| = K \cdot n!/(n-t)$ where $K = (t+1)!^{2t+1}$. With t fixed, we see that

$$\lim_{n\to\infty} \frac{|D|}{n!} = K \cdot \lim_{n\to\infty} \frac{1}{n-t} = 0. \qquad \square$$

7 A combinatorial basis for the association scheme of the symmetric group

We now turn our attention to the association scheme of the symmetric group. The center $\mathbb{A}$ of the group algebra of S_n is a vector space of symmetric matrices closed under both ordinary matrix multiplication and under Schur (entrywise) multiplication and containing both I and the all ones matrix J. In other words, $\mathbb{A}$ is a *Bose-Mesner algebra*. The underlying association scheme has S_n as its vertex set and contains one graph Γ_α for each conjugacy class $\mathcal{C}_\alpha$: two permutations g and h are adjacent in Γ_α if $gh^{-1} \in \mathcal{C}_\alpha$. The adjacency matrix of Γ_α is the matrix A_α introduced in Section 3. The basis of primitive central idempotents E_μ was introduced earlier as well.

In the study of association schemes, one often chooses a base point from the vertex set and considers the *trivial module* $\mathbb{A}\mathbf{e}$ where $\mathbf{e}$ is the elementary basis vector associated to the base point. If we choose the identity permutation as our base point and use the corresponding elementary basis vector $\mathbf{e}_{(1)}$, then the trivial module $\mathbb{A}\mathbf{e}_{(1)}$ for $\mathbb{A}$ acting on $\mathbb{R}^{n!}$ is simply the vector space of class functions. The trivial module has two distinguished bases: the indicator functions $A_\alpha\mathbf{e}_{(1)}$

for the conjugacy classes $\mathcal{C}_\alpha$ and the irreducible characters $\chi_\mu = \frac{n!}{f_\mu} E_\mu \mathbf{e}_{(1)}$. We found earlier that this module alone was sufficient for the study of λ-transitive groups. Yet, to extend the results to arbitrary sets of permutations, we found it necessary to appeal to the coding theorist's inner distribution and dual distribution. This is, in effect, a smoothing technique in which we average over all base points $g \in S_n$ in order to account for the possible asymmetry of a set D.

Consider the set $\mathcal{P}$ of all cosets of Young subgroups of S_n partially ordered by inclusion. We partition the elements of this poset into fibers corresponding to the shape of the coset gY_P. The matrix N_λ introduced in Section 4 is then the incidence matrix between the minimal elements (singletons) of this poset and the elements in the fiber indexed by λ. We are interested in the matrices $C_\lambda = N_\lambda N_\lambda^\top$ where λ is a partition of n.

Theorem 10. *The set $\{C_\lambda : |\lambda| = n\}$ is a basis for $\mathbb{A}$. The change-of-basis equations with respect to the A_α have the form*

$$C_\lambda = \sum_{\alpha \trianglelefteq \lambda} m_{\lambda,\alpha} A_\alpha$$

with $m_{\lambda,\lambda} > 0$. The change-of-basis equations with respect to the E_μ have the form

$$C_\lambda = \sum_{\mu \trianglerighteq \lambda} n_{\lambda,\mu} E_\mu$$

with $n_{\lambda,\mu} = \dfrac{n! K_{\mu,\lambda}}{f_\mu \cdot \prod_i m_i!}$ *where* $\lambda = 1^{m_1} 2^{m_2} \cdots$.

Proof. Since the entry of C_λ in row a column b is equal to the number of cosets hY_P of Young subgroups of shape λ containing both a and b, we clearly have $A(g)^{-1} C_\lambda A(g) = C_\lambda$ for any $g \in S_n$ showing that C_λ lies in $\mathbb{A}$. So $C_\lambda E_\mu = n_{\lambda,\mu} E_\mu$ for some $n_{\lambda,\mu}$. For the next calculation, let the first row and column of C_λ and E_μ be the ones indexed by the identity permutation and let the first column of N_λ be indexed by some Young subgroup Y_P of shape λ. Then the $(1,1)$ entry of E_μ is $f_\mu^2/n!$ and, by Frobenius Receprocity and Young's Rule, the $(1,1)$ entry of $N_\lambda^\top E_\mu$ is $\frac{f_\mu}{n!} \sum_{g \in Y_P} \chi^\mu(g) = \frac{f_\mu}{n!} |Y_P| K_{\mu,\lambda}$. Since the previous equality is true for any Y_P of shape λ, the $(1,1)$ entry of $C_\lambda E_\mu = N_\lambda N_\lambda^\top E_\mu$ is $\binom{n}{\lambda} \frac{f_\mu}{n!} f_\mu |Y_P| K_{\mu,\lambda}$ where $\binom{n}{\lambda}$ is the number of set partitions P of Ω_n having shape λ. Dividing this by $f_\mu^2/n!$ gives us our expression for $n_{\lambda,\mu}$. Next, note that $n_{\lambda,\mu} \neq 0$ if $\mu \trianglerighteq \lambda$ and $n_{\lambda,\mu} = 0$ otherwise since the same is true of $K_{\mu,\lambda}$. So the matrices C_λ indeed form a basis for $\mathbb{A}$.

Now consider the coefficients $m_{\lambda,\alpha}$. The (a,b)-entry of C_λ is nonzero if and only if there is some coset of some Young subgroup of shape λ which contains both a and b. Equivalently, $a^{-1}b$ is an element of that Young subgroup. But the conjugacy classes $\mathcal{C}_\alpha$ which intersect a Young subgroup of shape λ are precisely those for which α is a refinement of λ. So by definition of A_α, $m_{\lambda,\alpha} \geq 0$ with strict inequality precisely when α refines λ. The Theorem now follows since refinement implies $\alpha \trianglelefteq \lambda$ and because λ refines itself. ∎

In analogy with Delsarte's third basis for the Bose-Mesner algebra of the Johnson association scheme in [6, Sec. 4.2.1], we call the basis $\{C_\lambda : \lambda\}$ the *combinatorial basis* for the association scheme of S_n. Bases of this sort are known for many classical families of P- and Q-polynomial association schemes. We remark, however, that the typical linear ordering on relations and primitive idempotents is replaced in this case by the partial order $\trianglelefteq$.

In the language of association schemes, a λ-transitive set D is the same as a Delsarte $\mathcal{T}$-design [6, p32] where $\mathcal{T} = \{\mu : \mu \trianglerighteq \lambda\}$. This provides another viewpoint for our investigation and suggests an avenue for finding future nonexistence results. We remark that Delsarte's linear programming bound — the most natural technique to consider in this regard — is mostly ineffective in this particular setting, being overshadowed by the following more elementary divisibility conditions on λ-transitive sets.

Proposition 3. *If $D \subseteq S_n$ is a λ-transitive set of permutations, then $|D|$ is divisible by $\binom{n}{\mu} := \binom{n}{\mu_1,\ldots,\mu_k}$ for any partition $\mu \trianglerighteq \lambda$.*

Proof. Since D must also be μ-transitive by Theorem 5, each coset of each Young subgroup of shape μ contains r_μ permutations from D for some integer r_μ. Given a fixed Y_P of shape μ, it has $\binom{n}{\mu}$ cosets which partition S_n and thus D. So $|D| = r_\mu\binom{n}{\mu}$. □

8 Open questions

We list here some open questions raised by our work in the hopes that the reader will be tempted to work on them.

(1) An interesting construction is given in [5] for a set of 13!/7! permutation in S_{13} which Conway calls M_{13}. While this set is six-transitive in the sense of [5], it is not $(7,1,1,1,1,1,1)$-transitive in the sense we are using in this paper. We were unable to compute the full degree of transitivity of this set due to its size. We also ask if M_{13} can be combined with some small collection of its translates to yield a 6-transitive set of permutations in S_{13}.

(2) Is there an analogue of λ-transitive groups and sets for other Coxeter groups? In particular, it would be interesting to investigate the hyperoctahedral group. One would hope that there is a generalization of Theorem 5 in this setting.

(3) In view of Proposition 3, no λ-transitive subset of S_n can contain less than $\binom{n}{\lambda}$ permutations. Trivial examples where this lower bound is achieved are the cyclic, alternating and symmetric groups. At the end of Section 5, we presented examples achieving this lower bound for $\lambda = (q-2, 2)$ where q is an odd prime power. We ask if there are any other infinite families of examples achieving this lower bound.

(4) The *Krein parameters* $q^\nu_{\lambda,\mu}$ of the association scheme of the symmetric group are defined by the equations

$$E_\lambda \circ E_\mu = \frac{1}{n!}\sum_\nu q^\nu_{\lambda,\mu} E_\nu$$

where $\circ$ denotes the entrywise product of matrices. In the study of association schemes, one is often interested in determining which Krein parameters vanish for a given family of schemes. While this question is known to be hard in the case of the symmetric group, we identify one interesting result along these lines. Using Theorem 2.9.22 on p99 of the book by James and Kerber [8], we find that the Krein parameter $q^{\nu}_{\lambda,\mu}$ vanishes if $\mathsf{d}(\nu) > \mathsf{d}(\lambda)+\mathsf{d}(\mu)$ where $\mathsf{d}(\nu) = \nu_2+\nu_3+\cdots$. In this paper, we have constructed a "design system" attached to the association scheme of the symmetric group. Design systems are introduced by Martin in [12] and it is shown there how knowledge of vanishing Krein parameters can lead to lower bounds on the size of Delsarte $\mathcal{T}$-designs.

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