Maximal and Maximum Independent Sets In Graphs With At Most r Cycles

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September 25, 2005

Key Words: cycle, ear decomposition, maximal and maximum independent set

AMS classification: Primary 05C35; Secondary 05C38, 05C69.

Abstract

Let $m(G)$ denote the number of maximal independent sets of vertices in a graph G and let $c(n,r)$ be the maximum value of $m(G)$ over all connected graphs with n vertices and at most r cycles. A theorem of Griggs, Grinstead, and Guichard gives a formula for $c(n,r)$ when r is large relative to n, while a theorem of Goh, Koh, Sagan, and Vatter does the same when r is small relative to n. We complete the determination of $c(n,r)$ for all n and r and characterize the extremal graphs. Problems for maximum independent sets are also completely resolved.

1 Introduction and preliminary lemmas

Let $G=(V,E)$ be a simple graph. A subset $I \subseteq V$ is *independent* if there is no edge of G between any two vertices of I. Also, I is *maximal* if it is not properly contained in any other independent set. We let $m(G)$ be the number of maximal independent sets of G. Several previous authors have been interested in the problem of maximizing $m(G)$ over different families of graphs.

*Partially supported by an award from DIMACS and an NSF VIGRE grant to the Rutgers University Department of Mathematics.

In [2] the authors studied two families of graphs: the family of all graphs with at most r cycles, and the family of all connected graphs with at most r cycles. For the family of all graphs, they were able to completely settle the problem, by using the result of Moon and Moser [7] (Theorem 1.1 below) when n is small relative to r and providing new arguments for all values of (n, r) to which the Moon-Moser Theorem does not apply (see Theorem 1.5 (I), also below).

For the family of connected graphs, [2] only characterizes the extremal graphs when $n \geq 3r$ (Theorem 1.5 (II)) while the connected analogue of the Moon-Moser Theorem (the Griggs-Grinstead-Guichard Theorem, Theorem 1.2 below) settles the problem for n small relative to r, leaving a gap between the values where these two theorems apply. This gap is filled in Section 2 by a careful analysis of the possible endblocks of extremal graphs.

In the later sections we turn our attention to maximum independent sets (independent sets of maximum cardinality) in these two families of graphs. Like with maximal independent sets, we start with the case where n is large relative to r in Section 3 and then consider the gap in Section 4.

For the remainder of this section we briefly recount the results we will need. These results appear in [2] (with the single exception of Proposition 1.8, which occurs here in a strengthened form), and we refer the reader to that paper for examples and proofs.

For any two graphs G and H, let $G \uplus H$ denote the disjoint union of G and H, and for any nonnegative integer t, let tG stand for the disjoint union of t copies of G.

Let

$$G(n) := \begin{cases} \frac{n}{3}K_3 & \text{if } n \equiv 0 \pmod 3, \\ 2K_2 \uplus \frac{n-4}{3}K_3 & \text{if } n \equiv 1 \pmod 3, \\ K_2 \uplus \frac{n-2}{3}K_3 & \text{if } n \equiv 2 \pmod 3. \end{cases}$$

Further, let

$$G'(n) := K_4 \uplus \tfrac{n-4}{3}K_3 \text{ if } n \equiv 1 \pmod 3.$$

Also define

$$g(n) := m(G(n)) = \begin{cases} 3^{\frac{n}{3}} & \text{if } n \equiv 0 \pmod 3, \\ 4 \cdot 3^{\frac{n-4}{3}} & \text{if } n \equiv 1 \pmod 3, \\ 2 \cdot 3^{\frac{n-2}{3}} & \text{if } n \equiv 2 \pmod 3. \end{cases}$$

Note that $m(G'(n)) = m(G(n))$ when $n \equiv 1 \pmod 3$.

Theorem 1.1 (Moon and Moser [7]) *Let G be a graph with $n \geq 2$ vertices. Then*

$$m(G) \leq g(n)$$

with equality if and only if $G \cong G(n)$ or, for $n \equiv 1 \pmod 3$, $G \cong G'(n)$. ■

The extremal connected graphs were found by Griggs, Grinstead, and Guichard. To define these graphs we need one more piece of notation. Let G be a graph all of whose components are complete and let K_m be a complete graph disjoint from G. Construct the graph $K_m * G$ by picking a vertex v_0 in K_m and connecting it to a single vertex in each component of G. If $n \geq 6$ then let

$$C(n) := \begin{cases} K_3 * \frac{n-3}{3} K_3 & \text{if } n \equiv 0 \pmod{3}, \\ K_4 * \frac{n-4}{3} K_3 & \text{if } n \equiv 1 \pmod{3}, \\ K_4 * \left(K_4 \uplus \frac{n-8}{3} K_3\right) & \text{if } n \equiv 2 \pmod{3}. \end{cases}$$

It can be calculated that

$$c(n) := m(C(n)) = \begin{cases} 2 \cdot 3^{\frac{n-3}{3}} + 2^{\frac{n-3}{3}} & \text{if } n \equiv 0 \pmod{3}, \\ 3^{\frac{n-1}{3}} + 2^{\frac{n-4}{3}} & \text{if } n \equiv 1 \pmod{3}, \\ 4 \cdot 3^{\frac{n-5}{3}} + 3 \cdot 2^{\frac{n-8}{3}} & \text{if } n \equiv 2 \pmod{3}. \end{cases}$$

Theorem 1.2 (Griggs, Grinstead, and Guichard [4]) *Let G be a connected graph with $n \geq 6$ vertices. Then*

$$m(G) \leq c(n)$$

with equality if and only if $G \cong C(n)$. ∎

The study of $m(G)$ for graphs with a restricted number of cycles began with Wilf. Let

$$t(n) := \begin{cases} 2^{\frac{n-2}{2}} + 1 & \text{if } n \text{ is even}, \\ 2^{\frac{n-1}{2}} & \text{if } n \text{ is odd}. \end{cases}$$

Theorem 1.3 (Wilf [9]) *If G is a tree with $n \geq 1$ vertices then $m(G) \leq t(n)$.* ∎

Sagan [8] gave another proof of this theorem in which he also characterized the extremal graphs, but we will not need them.

Now let

$$f(n) := 2^{\lfloor \frac{n}{2} \rfloor}.$$

From Theorem 1.3, one can easily solve the problem for forests.

Theorem 1.4 *If G is a forest with $n \geq 1$ vertices then $m(G) \leq f(n)$.* ∎

To move from trees to a bounded number of cycles, suppose that n, r are positive integers with $n \geq 3r$. Define

$$G(n,r) := \begin{cases} rK_3 \uplus \frac{n-3r}{2} K_2 & \text{if } n \equiv r \pmod{2}, \\ (r-1)K_3 \uplus \frac{n-3r+3}{2} K_2 & \text{if } n \not\equiv r \pmod{2}. \end{cases}$$

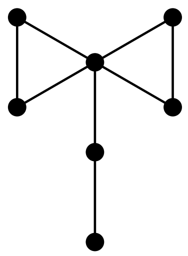


Figure 1: The exceptional graph E

Again, it can be computed that

$$g(n,r) := m(G(n,r)) = \begin{cases} 3^r \cdot 2^{\frac{n-3r}{2}} & \text{if } n \equiv r \pmod 2, \\ 3^{r-1} \cdot 2^{\frac{n-3r+3}{2}} & \text{if } n \not\equiv r \pmod 2. \end{cases}$$

It is also convenient to define $G(n,r) := G(n)$ and $g(n,r) := g(n)$ when $n < 3r$. The extremal connected graphs where $n \geq 3r$ are given by

$$C(n,r) := \begin{cases} K_3 * \left((r-1)K_3 \uplus \frac{n-3r}{2}K_2\right) & \text{if } n \equiv r \pmod 2, \\ K_1 * \left(rK_3 \uplus \frac{n-3r-1}{2}K_2\right) & \text{if } n \not\equiv r \pmod 2. \end{cases}$$

As usual, we let

$$c(n,r) := m(C(n,r)) = \begin{cases} 3^{r-1} \cdot 2^{\frac{n-3r+2}{2}} + 2^{r-1} & \text{if } n \equiv r \pmod 2, \\ 3^r \cdot 2^{\frac{n-3r-1}{2}} & \text{if } n \not\equiv r \pmod 2. \end{cases}$$

Theorem 1.5 ([2]) *Let G be a graph with n vertices and at most r cycles where $r \geq 1$.*

(I) If $n \geq 3r-1$ then $m(G) \leq g(n,r)$ with equality if and only if $G \cong G(n,r)$.

(II) If $n \geq 3r$ then for all such graphs that are connected we have $m(G) \leq c(n,r)$. Equality occurs if and only if $G \cong C(n,r)$, or if G is one of the exceptional cases listed in the following table.

n	r	*possible* $G \not\cong C(n,r)$
4	1	P_4
5	1	C_5
7	2	$C(7,1), E$

(Here P_4 and C_5 are the path and cycle on 4 and 5 vertices, respectively, and E is the graph shown in Figure 1.)

We have a list of inequalities that will be useful in our proofs. Here and elsewhere it will be convenient to let $g(n,0) = f(n)$ and $c(n,0) = t(n)$.

Lemma 1.6 ([2]) *We have the following monotonicity results.*

(1) If $r \geq 1$ and $n > m \geq 3r-1$ then

$$g(n,r) > g(m,r).$$

(2) If $r \geq 1$ and $n > m \geq 3r$ then

$$c(n,r) > c(m,r).$$

(3) If $r > q \geq 0$ and $n \geq 3r-1$ then

$$g(n,r) \geq g(n,q)$$

with equality if and only if n and r have different parity and $q = r-1$.

(4) If $r > q \geq 0$ and $n \geq 3r$ then

$$c(n,r) \geq c(n,q)$$

with equality if and only if $(n,r,q) = (4,1,0)$ or $(7,2,1)$.

We also need two results about $m(G)$ for general graphs G. In what follows, if $v \in V$ then the *open* and *closed neighborhoods of* v are $N(v) = \{u \in V \mid uv \in E\}$ and $N[v] = \{v\} \cup N(v)$, respectively. We also call a block an *endblock of* G if it has at most one cutvertex in the graph as a whole.

Proposition 1.7 *The invariant $m(G)$ satisfies the following.*

(1) If $v \in V$ then $m(G) \leq m(G-v) + m(G-N[v])$.

(2) If G has an endblock B that is isomorphic to a complete graph then

$$m(G) = \sum_{v \in V(B)} m(G - N[v]).$$

In fact, the same equality holds for any complete subgraph B having at least one vertex that is adjacent in G only to other vertices of B.

We will refer to the formulas in parts (1) and (2) of this proposition as the *m-bound* and *m-recursion*, respectively.

Using the fact that the blocks and cutvertices of a graph have a tree structure [1, Proposition 3.1.1], one obtains the following result.

Proposition 1.8 *Every graph has an endblock which intersects at most one non-endblock. Furthermore, if a graph is not 2-connected itself, then it contains at least two such endblocks.*

Note that any block with at least 3 vertices is 2-connected. Our analysis of the possible endblocks of the extremal graphs will rely upon Whitney's Ear Decomposition Theorem from [11].

Theorem 1.9 (Ear Decomposition Theorem) *A graph B is 2-connected if and only if there is a sequence*

$$B_0, B_1, \ldots, B_l = B$$

such that B_0 is a cycle and B_{i+1} is obtained by taking a nontrivial path and identifying its two endpoints with two distinct vertices of B_i. ■

Proofs of the Ear Decomposition Theorem may also be found in Diestel [1, Proposition 3.1.2] and West [10, Theorem 4.2.8].

2 Filling the gap

For any graph G, it will be convenient to let

$$r(G) = \text{number of cycles of } G.$$

For the family of all graphs on n vertices we have already seen the maximum value of $m(G)$ for all possible $r(G)$. Now consider the family of connected graphs. If $n \equiv 0 \pmod 3$ then Theorems 1.2 and 1.5 (II) characterize the maximum for all possible $r(G)$. So for the rest of this section we will concentrate on connected graphs with $n \equiv 1, 2 \pmod 3$.

Let

$$r_0 := \lfloor n/3 \rfloor = \text{the largest value of } r \text{ for which Theorem 1.5 (II) is valid.}$$

Also let

$$r_1 := r(C(n)) = \begin{cases} r_0 + 6 & \text{if } n \equiv 1 \pmod 3, \\ r_0 + 12 & \text{if } n \equiv 2 \pmod 3. \end{cases}$$

To characterize the extremal graphs in the gap $r_0 < r < r_1$ we will need an extension of the star operation. Let G and H be graphs all of whose components are complete and such that each component of H has at least 2 vertices. Construct $K_m * [G, H]$ by picking a vertex v_0 of K_m and connecting it to a single vertex in each component of G and to two vertices in each component of H. If $n = 3r_0 + 1$ then define

$$C(n, r) := \begin{cases} K_1 * [(r_0 - 1)K_3, K_3] & \text{if } r = r_0 + 2, \\ K_1 * [(r_0 - 2)K_3, 2K_3] & \text{if } r = r_0 + 4. \end{cases}$$

Note that $C(4, 5)$ is not well-defined because then $r_0 - 2 = -1 < 0$, and we will leave this graph undefined. We also need the exceptional graph

$$C(7, 3) := K_1 * [3K_2, \emptyset].$$

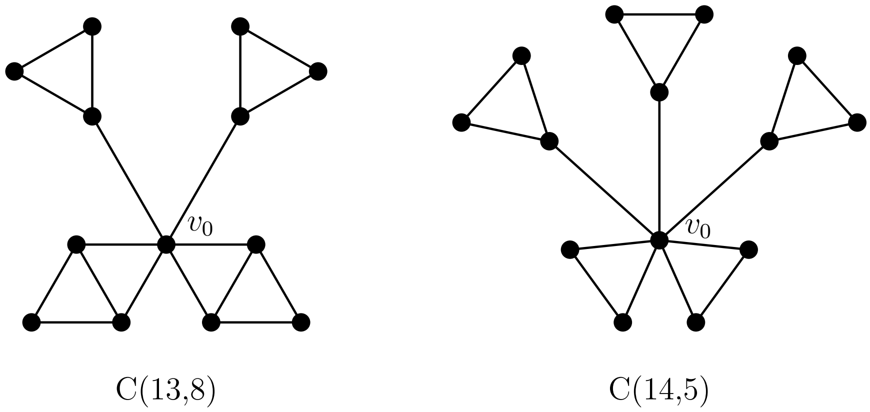


Figure 2: Examples of $C(n, r)$ for $n < 3r$

For the case $n = 3r_0 + 2$, let

$$C(n, r) := K_1 * [(r_0 - 1)K_3, 2K_2] \quad \text{if } r = r_0 + 1.$$

These will turn out to be the new extremal graphs in the gap. Two examples may be found in Figure 2.

Note that for all graphs just defined we have $m(C(n, r)) = m(C(n, r_0))$. We let

$$m_0 := m(C(n, r_0)) = \begin{cases} 3^{r_0} & \text{if } n = 3r_0 + 1, \\ 4 \cdot 3^{r_0 - 1} + 2^{r_0 - 1} & \text{if } n = 3r_0 + 2. \end{cases}$$

It will also be convenient to extend the domain of $c(n, r)$ to all n and r by defining

$$\begin{aligned} c(n, r) &= c(n) \quad \text{when } r \geq r(C(n)), \\ c(n, r) &= m_0 \quad \text{when } r_0 < r < r_1. \end{aligned}$$

We extend the definition of $C(n, r)$ similarly by defining $C(n, r) := C(n)$ when $r \geq r(C(n))$.

We will need a special case of the Moon-Moser transformation [4, 7] which, in conjunction with the Ear Decomposition Theorem, will prove useful in cutting down on the number of cases to consider. Suppose the graph G contains a path $tuvw$ such that

(a) $\deg u = \deg v = 2$,

(b) the edge uv lies on a cycle, and

(c) u is in at least as many maximal independent sets as v.

Then construct the (connected) graph $G_{u,v}$ where $V(G_{u,v}) = V(G)$ and

$$E(G_{u,v}) = E(G) \cup \{tv\} - \{vw\}.$$

The edge uv lies on a unique cycle (a 3-cycle) in $G_{u,v}$, and so $r(G_{u,v}) \leq r(G)$ by (b).

Lemma 2.1 *Suppose G contains a path $tuvw$ satisfying (a)–(c). Then $m(G_{u,v}) \geq m(G)$, with equality only if $N_G(w) - \{v\} \subseteq N_G(t)$.*

Proof: Let M be a maximal independent set in G. Then there are three mutually exclusive possibilities for M, namely $u, v \notin M$; $u \in M$ and $v \notin M$; or $v \in M$ and $u \notin M$. In the first two cases, M gives rise to distinct maximal independent set(s) in $G_{u,v}$ as in the following chart.

type of MIS M in G	corresponding MIS(s) in $G_{u,v}$
$u, v \notin M$	M
$u \in M, v \notin M$	$M, M \cup \{v\} - \{u\}$

Therefore by (c) we have $m(G_{u,v}) \geq m(G)$ without even considering the third case where $v \in M$ and $u \notin M$. In this case, if there is some vertex $x \in N_G(w) - N_G(t) - \{v\}$, then there is a maximal independent set M in G with $\{t, x, v\} \subseteq M$. Hence $M - \{v\}$ is a maximal independent set in $G_{u,v}$ so $m(G_{u,v}) > m(G)$, as desired. If such a vertex x does not exist, we have $N_G(w) - \{v\} \subseteq N_G(t)$. ∎

Given a graph G, we let $\mathcal{T}(G)$ denote the set of all graphs that can be obtained from G by applying a maximal sequence of these special Moon-Moser transformations. By Lemma 2.1, every graph in $\mathcal{T}(G)$ has at least as many cycles and at least as many maximal independent sets as G. Furthermore, since all of these graphs are formed by maximal sequences of transformations, if $H \in \mathcal{T}(G)$ then H cannot contain a path $tuvw$ satisfying (a)–(c) above. Another way to state this is that every $H \in \mathcal{T}(G)$ has the following property:

(Δ) If $uv \in E(H)$ lies on a cycle and $\deg u = \deg v = 2$, then uv lies on a 3-cycle.

Before closing the gap, we wish to mention a result which we will need to rule out some graphs from the list of possible extremals. To state this lemma, we say that a vertex $v \in V(G)$ is *duplicated* if there is a vertex $w \in V(G)$ such that v and w have the same neighbors, that is, $N(v) = N(w)$.

Lemma 2.2 *Let G be a graph with n vertices and a vertex v that is duplicated.*

(1) We have $m(G) = m(G - v)$.

(2) If $n \geq 7$, $n \equiv 1$ or $2 \pmod 3$, and G is connected with less than r_1 cycles then $m(G) < m_0$.

Proof: If u and v are duplicated vertices, then they lie in the same maximal independent sets and neither is a cutvertex. So $m(G) = m(G - v)$ and under the hypotheses of (2), $m(G - v) \leq c(n - 1) < m_0$. ∎

We now finish our characterization of the extremal graphs.

Theorem 2.3 *Let G be a connected graph with $n \geq 7$ vertices, $n \equiv 1$ or $2 \pmod 3$, and less than r_1 cycles. Then*

$$m(G) \leq m_0,$$

with equality if and only if $G \cong C(n, s)$ for some s with $r_0 \leq s < r_1$.

Proof: We prove the theorem by induction on n. The cases where $n \leq 10$ have been checked by computer, so let G be an extremal connected graph with $n > 10$ vertices and less than r_1 cycles.

First note that it suffices to prove the theorem for graphs that satisfy (Δ): If G satisfies the hypotheses of the theorem then every graph $H \in \mathcal{T}(G)$ also satisfies these hypotheses and satisfies (Δ). Since G is extremal, then in Lemma 2.1 we would always have equality and thus the given subset relation, when replacing G by $G_{u,v}$. Since none of our candidate extremal graphs can be generated by this transformation if such a condition is imposed, we must have that $\mathcal{T}(G) = \{G\}$ and so G satisfies (Δ).

Pick an endblock B of G satisfying the conclusion of Proposition 1.8 with $|V(B)|$ maximum among all such endblocks. If $G \neq B$ then we will use x to denote the cutvertex of G in B. The argument depends on the nature of B.

If $B \cong K_2$ then the argument used in the proof of Theorem 1.5 (II) can be easily adapted for use in this context. Let $V(B) = \{x, v\}$ so that $\deg v = 1$ and $\deg x \geq 2$. By the choice of B, $G - N[v]$ is the union of some number of K_1's and a connected graph with at most $n-2$ vertices and at most r cycles. Also, $G - N[x]$ has at most $n-3$ vertices and at most r cycles, so the m-recursion and monotonicity give, for $n \geq 11$,

$$m(G) \leq c(n-2, r) + g(n-3, r) \leq c(n-2) + g(n-3) \leq m_0,$$

with equality if and only if $n = 3r_0 + 2$ and $G \cong C(n, r_0)$.

All other possible endblocks must be 2-connected and so we will use the Ear Decomposition Theorem to organize the cases to consider based on l, the number of paths that are added to the initial cycle.

If $l = 0$ then (Δ) guarantees that $B \cong K_3$. Let $V(B) = \{x, v, w\}$ where x denotes the cutvertex. Let i denote the number of other K_3 endblocks containing x. If the graph consists entirely of K_3 endblocks which intersect at x, then

$$m(G) = 2^{\frac{n-1}{2}} + 1,$$

which shows that G is not extremal. Thus x is adjacent to at least one vertex which does not lie in a K_3 endblock. Since B was chosen with $|V(B)|$ maximal, it follows that $G - N[v] = G - N[w]$ has some number of trivial components, i components isomorphic to K_2, and at most one other component, H, with at most $n - 2i - 3$ vertices at at most $r - i - 1 \leq r_1 - i - 2$ cycles. Since x is adjacent to at least one vertex not in the K_3 endblocks, $G - N[x]$ has at most $n - 2i - 4$ vertices. This gives us the upper bound

$$m(G) \leq 2^{i+1} c(n - 2i - 3, r_1 - i - 2) + g(n - 2i - 4). \qquad (1)$$

To show that this bound is always at most m_0, we consider the two values $n = 3r_0 + 1$ and $n = 3r_0 + 2$ as well as the three possible congruence classes of i modulo 3 separately. So let $j = \lfloor i/3 \rfloor$. Considering the number of vertices in $G - N[x]$ gives $n - 2i - 4 \geq 0$ and translating this into a bound involving r_0 and j gives $r_0 \geq 2j + k_0$ where $1 \leq k_0 \leq 3$ depending on which of the six cases we are in. We now wish to show that the right-hand side of (1) is a strictly decreasing function of j for any fixed but sufficiently large n. This

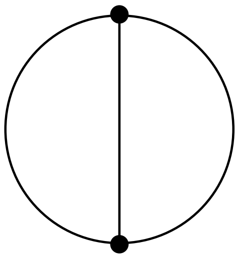


Figure 3: D, the multigraph for $l = 1$

is clearly true of the $g(n-2i-4)$ term, so let $f(r_0, j) = 2^{i+1}c(n-2i-3, r_1-i-2)$ where the right side has been converted to a function of r_0 and j. In all cases, we get that

$$f(r_0, j) = a(r_0)2^j + b(r_0)(8/9)^j$$

for certain functions $a(r_0), b(r_0)$. It follows that $f(r_0, j) - f(r_0, j+1) > 0$ if and only if $b(r_0)/a(r_0) > 9(3/2)^{2j}$. Solving for r_0 shows that we have a decreasing function of j for $r_0 \geq 2j + k_1$ where $4 \leq k_1 \leq 7$. So it suffices to check that the right-hand side of (1) is at most m_0 for $2j + k_0 \leq r_0 \leq 2j + k_1$. This is done by substituting each value of r_0 in turn to get a function of j alone, noting that this function is decreasing for all j sufficiently large to make $r_0 \geq 3$, and then verifying that this function is bounded by m_0 when j is at this minimum value. The only cases where we get equality are when $n = 3r_0 + 2$ and $G \cong C(n, r_0)$.

If $l = 1$, then B must be a subdivision of the multigraph D in Figure 3, i.e., it must be obtained from D by inserting vertices of degree 2 into the edges of D. By (Δ), we can insert at most one vertex into an edge, unless one of the inserted vertices is the cutvertex x in which case it is possible to insert a vertex before and after x as well. To turn this multigraph into a graph, it is necessary to subdivide at least two of the edges. If all three edges are subdivided, or two edges are subdivided and x is one of the original vertices of D, then G has a duplicated vertex and so is not extremal by Lemma 2.2. In the only remaining case, the following lemma applies.

Lemma 2.4 (Triangle Lemma) *Suppose G contains three vertices $\{u, v, w\}$ satisfying the following restrictions.*

(a) These vertices form a K_3 with $\deg u = 2$ and $\deg v, \deg w \geq 3$.

(b) The graph $G - \{u, v, w\}$ is connected.

Then $m(G) \leq m_0$ with equality only if $n = 3r_0 + 1$ and $G \cong C(n, s)$ for some s with $r_0 \leq s < r_1$.

Proof: Because of (a), the K_3 satisfies the alternative hypothesis in the m-recursion. Using induction to evaluate the c and g functions, we get

$$\begin{aligned} m(G) &= m(G - N[u]) + m(G - N[v]) + m(G - N[w]) \\ &\leq c(n-3, r-1) + 2g(n-4, r-1) \\ &\leq m_0. \end{aligned}$$

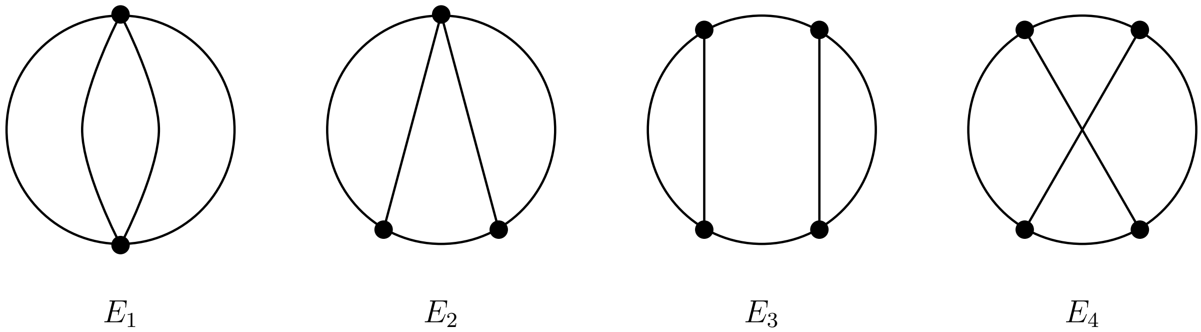


Figure 4: The multigraphs for $l = 2$

with equality only if $n = 3r_0+1$, $G-N[v] \cong G-N[w] \cong G(n-4)$, and $G-N[u] \cong C(n-3, s)$ for some $s \le r_0 + 4$. These easily imply the conclusion of the lemma. ■

For $l \ge 2$, we must consider the two congruence classes for n separately. First consider $n = 3r_0 + 1$. The following lemma will help eliminate many cases. In it, we use $r(v)$ to denote the number of cycles of G containing the vertex v.

Lemma 2.5 *Let $n = 3r_0 + 1$ and let v be a non-cutvertex with $\deg v \ge 3$ and $r(v) \ge 6$. Then G is not extremal.*

Proof: Using the m-bound, we have

$$\begin{aligned} m(G) &\le c(n-1, r-6) + g(n-4, r-6) \\ &\le c(3r_0, r_0 - 1) + g(3r_0 - 3, r_0 - 1) \\ &= 2 \cdot 3^{r_0-1} + 3^{r_0-1} \\ &= m_0. \end{aligned}$$

However, if we have equality then this forces us to have $G - v \cong C(3r_0, r_0 - 1)$ and $G - N[v] \cong G(3r_0 - 3, r_0 - 1)$. The only way this can happen is if $G \cong C(3r_0 + 1, r_0 + 2)$ where v is one of the degree 3 vertices in the 4-vertex block. But then v is in only 3 cycles, contradicting our hypothesis that $r(v) \ge 6$. ■

When $l = 2$, B must be a subdivision of one of the multigraphs in Figure 4. Lemma 2.5 shows that B cannot be a subdivision of E_1, E_4, or any block formed by a sequence of length $l \ge 3$ since in all these cases there are at least two vertices having degree at least 3 and lying in at least 6 cycles. So even if B has a cutvertex, there will still be a non-cutvertex in B satisfying the hypotheses of the lemma.

If B is formed by subdividing E_2, the same lemma shows that we need only consider the case where the vertex of degree 4 in E_2 is a cutvertex, x, of G. Also, since B can't have duplicated vertices and must satisfy (Δ), each pair of doubled edges has a vertex inserted in exactly one edge. This means there are only two possibilities for B, depending on whether the non-doubled edge is subdivided or not, and it is easy to check that in both cases G is not extremal by using the m-bound on the vertex x.

Finally, if B is a subdivision of E_3 then, because of the pair of disjoint doubled edges, there will always be one doubled edge which does not contain a cutvertex. In B that pair will give rise to either a duplicated vertex or a K_3 satisfying the hypotheses of the Triangle Lemma, and thus in either case we will be done. This ends the proof for $n = 3r_0 + 1$.

Now we look at the case where $n = 3r_0 + 2$. The analogue of Lemma 2.5 in this setting is as follows and since the proof is similar, we omit it.

Lemma 2.6 *Let $n = 3r_0 + 2$ and let v be a non-cutvertex that satisfies either*

(1) $\deg v \geq 3$ *and* $r(v) \geq 12$, *or*

(2) $\deg v \geq 4$ *and* $r(v) \geq 6$.

Then G is not extremal.

The ideas used to rule out E_3 for $n = 3r_0 + 1$ will be used many times in the current case, so we codify them in the lemma below.

Lemma 2.7 *Suppose $n = 3r_0 + 2$ and the block B is a subdivision of a multigraph having two disjoint submultigraphs each of which is of one of the following forms:*

(i) a doubled edge, or

(ii) a vertex v satisfying $\deg v \geq 3$ *and* $r(v) \geq 12$, *or*

(iii) a vertex v satisfying $\deg v \geq 4$ *and* $r(v) \geq 6$.

Then G is not extremal.

Proof: If any set of doubled edges has both edges subdivided exactly once, then G is not extremal by Lemma 2.2. Otherwise, since B has at most one cutvertex x in G, either the hypotheses of the Triangle Lemma or of the previous lemma will be satisfied. ∎

Finally, we will need a way to eliminate blocks that only have vertices of degree at most 3, but not sufficiently many cycles to satisfy Lemma 2.6 (1). One way would be to make sure that $G - N[v]$ is connected. Since a given multigraph M has many possible subdivisions, we also need a criterion on M that will guarantee that most of the subdivisions will have the desired connectivity.

Lemma 2.8 *Let $n = 3r_0 + 2$.*

(1) Suppose that G contains a non-cutvertex v such that $\deg v \geq 3$ *and* $r(v) \geq 6$. *Suppose further that $G - N[v]$ contains at most two nontrivial components and that if there are two, then one of them is a star (a complete bipartite graph of the form $K_{1,s}$). Then G is not extremal.*

(2) Suppose G comes from subdivision of a multigraph M that contains a vertex v with $\deg v = 3$, $r(v) \geq 6$, *and such that all vertices in $N_M[v]$ are non-cutvertices in M and $M - N_M[v]$ is connected. Suppose further that there are at most two edges of M between the elements of $N_M(v)$. Then G and v satisfy the hypotheses of (1).*

Proof: For (1), first assume that there is only one nontrivial component in $G - N[v]$. Then, using the induction hypothesis about the behavior of graphs in the gap,

$$\begin{aligned} m(G) &\leq c(n-1, r-6) + c(n-4, r-6) \\ &= c(3r_0+1, r_0+5) + c(3r_0-2, r_0+5) \\ &= c(3r_0+1, r_0) + c(3r_0-2) \\ &= 3^{r_0} + 3^{r_0-1} + 2^{r_0-2} \\ &< m_0. \end{aligned}$$

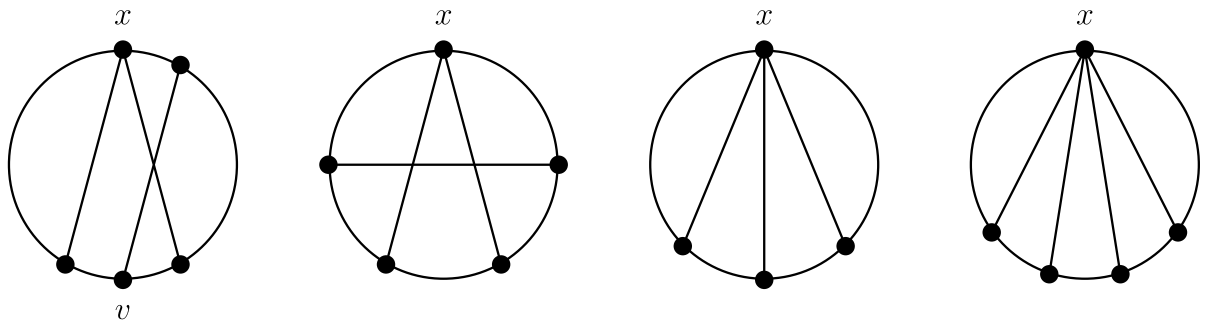
In the case with a star component, we use $m(G) \leq c(n-1, r-6) + 2c(n-6, r-6)$ to obtain the same result.

For (2) we will break the proof into several cases depending on how the edges of M at v are subdivided in G, noting that by our hypotheses each can be subdivided at most once and that the same is true of any edge between elements of $N_M(v)$. Let $N_M(v) = \{s, t, u\}$ and let H be the subdivision of $L = M - N_M[v]$ induced by G. Note that H is connected by assumption. If none of vs, vt, vu are subdivided in G, then $G - N_G[v]$ is just H together, possibly, with some vertices of degree one attached (if any edges from s, t, or u to L were subdivided) and some trivial components (if any edges between s, t, and u were subdivided). If exactly one of the three edges is subdivided, suppose it is vs. Then it is possible that s is in a different nontrivial component of $G - N_G[v]$ than H. But since there are at most two edges from s to t and u, the component of s is a star. Now suppose that vs and vt are subdivided, but not vu. At least one of s, t are connected to H otherwise u becomes a cutvertex. So again the only possibility for a nontrivial component other than H is a star containing either s or t, but not both. Finally, if all three edges are subdivided, then $G - N_G[v]$ is connected because v is not a cutvertex in M. ■

We need a little terminology before we handle the $n = 3r_0 + 2$ case. Let L and M be 2-connected multigraphs with no vertices of degree two. We say that M *is a child of* L if there is some sequence $B_0, B_1, \ldots, B_l$ formed as in the Ear Decomposition Theorem with $B_{l-1} = L$ and $B_l = M$. We will use words like "descendant," "parent," and so on in a similar manner.

We now pick up the proof for $n = 3r_0 + 2$ where we left off, namely with $l = 2$. Lemma 2.7 (iii) shows that B cannot be a subdivision of E_1 or any of its descendants. Also, if B is a subdivision of E_2, then by Lemma 2.6 (2) we need only consider the case where the vertex of degree 4 is a cutvertex x, and the same argument we used in the $3r_0+1$ case shows that such graphs are not extremal.

Next we consider the children of E_2. The only multigraphs not ruled out by Lemma 2.7 are the first three listed in Figure 5. As before, we need only consider the case where there is a cutvertex x at the vertex as indicated. It can be checked that F_1 can't lead to an extremal graph by using the vertex marked v in Lemma 2.8. For F_2, first note that if any of the edges containing x is subdivided, and it doesn't matter which one by symmetry, then taking v to be the other endpoint of that edge (after subdivision) in Lemma 2.8 shows that G is not extremal. If none of the edges containing x are subdivided then $b := |V(B)|$ satisfies $5 \leq b \leq 9$ because G has Property (Δ), and applying the m-bound to the vertex



F_1 F_2 F_3 G_1

Figure 5: Three children and one grandchild of E_2

x gives

$$m(G) \leq \max_{5 \leq b \leq 9} \{c(b-1,1)g(n-b) + g(n-b-1)\} < m_0.$$

Finally, F_3 is treated the same way as F_2, noting that the two pairs of doubled edges must both be subdivided in the same manner, the only edge containing x which can be subdivided further is the vertical one in the diagram, and the maximum is now taken over $6 \leq b \leq 9$.

The only grandchild of E_2 not thrown out by either Lemma 2.6 or Lemma 2.7 is the multigraph G_1 in Figure 5, a child of F_3. It is handled in the same way as F_2 and F_3 and the reader should be able to fill in the details at this point. It is easy to check that the children of G_1 are all eliminated, and so we have finished with the descendants of E_2.

Lemma 2.7 rules out subdivisions of E_3 directly as well as, in conjunction with Lemma 2.6, many of its children and all of its grandchildren. The only surviving multigraphs not previously considered are those children listed in Figure 6. In F_4, we are reduced in the usual manner to the case where the vertex of degree at least four is a cutvertex x. But then we can take v as indicated in Lemma 2.8 and so this child is not extremal. In F_5 we need only consider when there is a cutvertex x in the doubled edge. But then either v_1 or v_2 (depending on the placement of x) can be used in Lemma 2.8 to take care of this child. Similarly, in F_6 it is easy to see by symmetry that no matter where the cutvertex is placed, there is a v for Lemma 2.8.

Finally we come to $E_4 \cong K_4$. Lemma 2.8 shows that if B is a subdivision of E_4 then we need only consider when one of the degree 3 vertices is a cutvertex x. If one of the edges xv is subdivided then v satisfies Lemma 2.8. Therefore if there is a degree 2 vertex in B, it must be formed by subdividing an edge between two non-cutvertices of E_4, say u and v. Hence $G - \{u, v\}$ contains only one nontrivial connected component with $n-3$ vertices, and we get

$$\begin{aligned} m(G) &\leq m(G-u) + m(G-N[u]) \\ &\leq m(G-\{u,v\}) + m(G-u-N[v]) + m(G-N[u]) \\ &\leq c(n-3) + g(n-5) + g(n-4) \end{aligned}$$

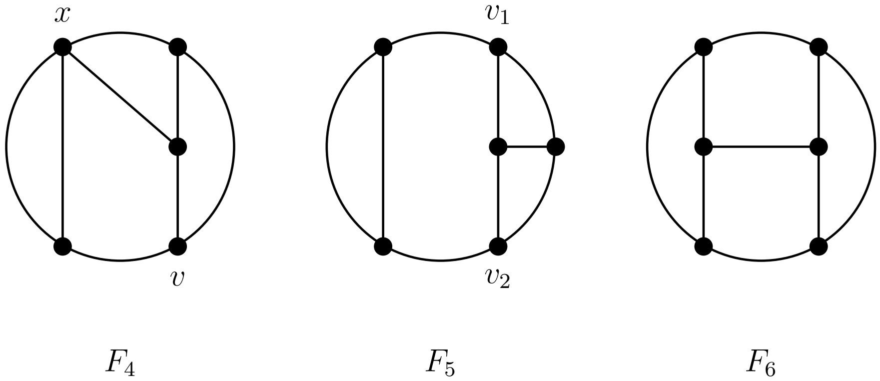


Figure 6: Three children of E_3

$$< \quad m_0,$$

so these graphs are not extremal.

We are reduced to considering the case where $B \cong K_4$, that is, when no edges are subdivided. By our choice of B and the cases we have disposed of so far, we can assume that all other endblocks containing the cutvertex x are isomorphic to K_2, K_3, or K_4 Assume that there are i copies of K_3 and j copies of K_4 other than B. If these are the only blocks of G, then $m(G) = 2^i 3^{j+1} + 1$ where $2i + 3j + 4 = n = 3r_0 + 2$. This quantity is maximized when $i = 2$ and $j = r_0 - 2$, giving $m(G) = 4 \cdot 3^{r_0 - 1} + 1 < m_0$.

Hence we may assume that G has other blocks. We subdivide this case into three subcases. First, if $j \geq 2$ (in other words, if x lies in at least three K_4 endblocks) then applying m-bound shows that

$$m(G) \leq 27g(n-10, r-21) + g(n-11, r-21) < m_0.$$

Now consider $j = 1$. Here our upper bound is

$$\begin{aligned} m(G) &\leq 9 \cdot 2^i c(n - 2i - 7, r - i - 14) + g(n - 2i - 8) \\ &\leq 9 \cdot 2^i c(3r_0 - 2i - 5, r_0 - i - 3) + g(3r_0 - 2i - 6). \end{aligned}$$

This is a decreasing function of i within each congruence class modulo 2, and using this fact it is routine to check that $m(G) < m_0$.

We are left with the case where $j = 0$. The m-bound gives

$$\begin{aligned} m(G) &\leq 3 \cdot 2^i c(n - 2i - 4, r - i - 7) + g(n - 2i - 5) \\ &\leq 3 \cdot 2^i c(3r_0 - 2i - 2, r_0 - i + 4) + g(3r_0 - 2i - 3). \end{aligned}$$

Note that $3r_0 - 2i - 2 \geq 3(r_0 - i + 4)$ only for $i \geq 14$. For these values of i, the upper bound is a decreasing function of i within each congruence class modulo 2, so we only need to verify that $m(G) < m_0$ for $i \leq 15$. These cases can all be routinely checked, although when $i \equiv 1, 2 \pmod 3$ the desired inequality will hold only for sufficiently large r_0, and

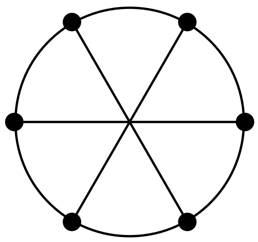


Figure 7: F_7, a child of E_4

one must note that these cases can arise only for such sufficiently large r_0. In particular, $n \geq 2i + 5$ by our assumptions, and this implies that $r_0 \geq 2i/3 + 1$.

The only child of E_4 that is not a child of any other E_k and is not ruled out by Lemma 2.6 is F_7, shown in Figure 7. One can verify by considering several cases that whether or not there is a cutvertex in B, there is a vertex v satisfying the hypotheses of Lemma 2.8. Finally, the grandchildren of E_4 all fall under the purview of Lemma 2.6.

We have now considered all the cases and so completed the proof of the theorem. ■

3 Maximum independent sets

We now turn to the consideration of maximum independent sets. An independent set I if G is *maximum* if it has maximum cardinality over all independent sets of G. We let $m'(G)$ denote the number of maximum independent sets of G. Since every maximum independent set is also maximal we have $m'(G) \leq m(G)$, so for any finite family of graphs,

$$\max_{G \in \mathcal{F}} m'(G) \leq \max_{G \in \mathcal{F}} m(G). \tag{2}$$

We say that G is *well covered* if every one of its maximal independent sets is also maximum. Then we have equality in (2) if and only if some graph with a maximum number of maximal independent sets is well covered. The graphs $G(n)$, $G'(n)$, $C(n)$, and $G(n,r)$ are well covered for all pairs n, r and $C(n,r)$ is well covered when $n \equiv r \pmod 2$ and $n \geq 3r$, so we immediately have the following result.

Theorem 3.1 *Let G be a graph with n vertices and at most $r \geq 1$ cycles.*

(I) For all such graphs,

$$m'(G) \leq g(n,r),$$

with equality if and only if $G \cong G(n,r)$.

(II) If $n \geq 3r$ and $n \equiv r \pmod 2$, or if $r \geq r(C(n))$, then for all such graphs that are connected,

$$m'(G) \leq c(n,r),$$

with equality if and only if $G \cong C(n,r)$.

This leaves only the case where G is connected and $n \not\equiv r \pmod 2$. To state and motivate the result in this case, we recall the work done on trees and graphs with at most one cycle.

When n is even, the tree $K_1 * (K_1 \uplus \frac{n-2}{2} K_2)$ has $t(n)$ maximum independent sets (in fact, by the upcoming Theorem 3.2, it is the only such tree). When n is odd there is only one extremal tree for the maximal independent set problem, and it is not well-covered. Define a family of trees by

$$T'(n) := \begin{cases} K_1 * (K_1 \uplus \frac{n-2}{2} K_2) & \text{if } n \equiv 0 \pmod 2, \\ K_1 * (2K_1 \uplus \frac{n-3}{2} K_2) & \text{if } n \equiv 1 \pmod 2, \end{cases}$$

and let

$$t'(n) := m'(T'(n)) = \begin{cases} 2^{\frac{n-2}{2}} + 1 & \text{if } n \equiv 0 \pmod 2, \\ 2^{\frac{n-3}{2}} & \text{if } n \equiv 1 \pmod 2. \end{cases}$$

Zito [12] proved the following result.

Theorem 3.2 (Zito [12]) *If T is a tree on $n \geq 2$ vertices then*

$$m'(T) \leq t'(n)$$

with equality if and only if $T \cong T'(n)$.

In [6], Jou and Chang gave a short proof of this theorem and considered graphs with at most one cycle.

Theorem 3.3 (Jou and Chang [6]) *Let G be a connected graph with at most one cycle and n vertices where $n \equiv 0 \pmod 2$. Then*

$$m'(G) \leq t(n)$$

with equality if and only if G is a tree, and thus, by Theorem 3.2, if and only if $G \cong T'(n)$.

The rest of this section will be devoted to the proof of the maximum independent set version of Theorem 1.5.

Theorem 3.4 *Let G be a connected graph with $n \geq 3r$ vertices and at most r cycles, where $r \geq 1$ and $n \not\equiv r \pmod 2$. Then*

$$m'(G) \leq c(n, r-1),$$

with equality if and only if G has precisely $r-1$ cycles, and thus, by Theorem 1.5, if and only if $G \cong C(n, r-1)$ or if G is isomorphic to one of the exceptional graphs listed there.

Before proving Theorem 3.4 we present two lemmas. The first is an analogue the m-bound and m-recursion that proved useful for maximal independent sets. Its proof is similar and so is omitted.

Lemma 3.5 *The invariant $m'(G)$ satisfies the following inequalities.*

(1) If $v \in V$ then

$$m'(G) \leq m'(G - v) + m'(G - N[v]).$$

(2) If G has a complete subgraph B with at least one vertex adjacent only to other vertices of B then

$$m'(G) \leq \sum_{v \in V(B)} m'(G - N[v]).$$

We will refer to parts (1) and (2) of this lemma as the m'-bound and m'-recursion, respectively.

Our second lemma will be useful for eliminating the cases where a vertex serves as a cutvertex for more than one endblock.

Lemma 3.6 *Suppose that the graph G contains a vertex x and a set of at least two other vertices U such that the induced graph $G[U]$ is not complete and for all $u \in U$,*

$$x \in N(u) \subseteq U \cup \{x\}. \tag{3}$$

Then $m'(G) = m'(G - x)$.

Proof: It suffices to show that no maximum independent set of G contains x. Suppose not, and let I be a maximum independent set with $x \in I$. Then by (3) we have $I \cap U = \emptyset$. Also, since $G[U]$ is not complete there is an independent set $A \subset U$ containing at least two vertices. But then $I \cup A - x$ is a larger independent set than I, a contradiction. ■

Proof (of Theorem 3.4): We will use induction on r. The base case of $r = 1$ is precisely Theorem 3.3, so we will assume $r \geq 2$. Note that since $n \geq 3r$ and $n \not\equiv r \pmod 2$ we have $n \geq 3r + 1$. Let G be an extremal graph satisfying the hypothesis of the theorem.

If G has less than r cycles then we are done by Theorem 1.5, so we may assume that G has exactly r cycles. Since we have chosen G to be extremal, we may assume that

$$m'(G) \geq c(n, r-1) = 3^{r-2} \cdot 2^{\frac{n-3r+5}{2}} + 2^{r-2}.$$

Let B be an endblock of G. First, if B has intersecting cycles, then the Ear Decomposition Theorem shows that B must contain a subdivision of the multigraph D (shown in Figure 3). This implies that B contains a non-cutvertex of degree at least 3 that lies in at least 3 cycles, from which m'-bound gives the contradiction

$$m'(G) \leq c(n-1, r-3) + g(n-4, r-3) < c(n, r-1).$$

Hence B is either K_2, K_3, or C_p for some $p \geq 4$. Since these possibilities have at most one cycle and we are assuming that G has $r \geq 2$ cycles, G cannot be a single block. Hence B must contain a cutvertex x of G.

First suppose that $B \cong C_p$ for some $p \geq 4$. Label the vertices of B as $x, u, v, w, \ldots$ so that they read one of the possible directions along the cycle. Since $G - v$ is connected with $n-1$ vertices and $r-1$ cycles, induction applies to give $m'(G-v) \leq c(n-1, r-2)$. Furthermore, $G-v$ has exactly $r-1$ cycles, so by induction we cannot have equality. Similarly, $G-N[v]$ has $n-3$ vertices and $r-1$ cycles, so $m'(G-N[v]) < c(n-3, r-2)$. An application of the m'-bound gives the contradiction

$$m'(G) \leq m'(G-v) + m'(G-N[v]) < c(n-1, r-2) + c(n-3, r-2) = c(n, r-1).$$

We now know that all endblocks of G must be copies of either K_2 or K_3. We claim that such endblocks must be disjoint. Suppose to the contrary that two endblocks share a vertex, which must therefore be the cutvertex x. Considering the two cases when at least one endblock is a K_2 (so that $G-x$ has an isolated vertex which must be in each of its maximum independent sets) or when both are copies of K_3, we can use Lemma 3.6 and the fact that $n \geq 3r+1$ to get

$$\begin{aligned} m'(G) &= m'(G-x) \\ &\leq \begin{cases} g(n-2, r) & \text{if one endblock is a } K_2, \\ g(n-1, r-2) & \text{if both endblocks are } K_3\text{'s} \end{cases} \\ &< c(n, r-1). \end{aligned}$$

This contradiction proves the claim.

Now let B be an endblock of G satisfying Proposition 1.8, so that it intersects at most one non-endblock. By what we have just shown, B intersects precisely one other block and that block is not an endblock. We claim that this block is isomorphic to K_2. Suppose not. Then the cutvertex x of B lies in at least one cycle not contained in B and is adjacent to at least two vertices not in B. Again, we consider the cases $B \cong K_2$ and $B \cong K_3$ separately to obtain

$$\begin{aligned} m'(G) &\leq m'(G-x) + m'(G-N[x]) \\ &\leq \begin{cases} c(n-2, r-1) + g(n-4, r-1) & \text{if } B \cong K_2, \\ 2c(n-3, r-2) + g(n-5, r-2) & \text{if } B \cong K_3 \end{cases} \\ &< c(n, r-1), \end{aligned}$$

proving our claim.

Since G is not itself a block, Proposition 1.8 shows that G contains at least two endblocks, say B and B', that each intersect at most one non-endblock. Let the cutvertices of these endblocks be labeled x and x', respectively. We have shown that B and B' are disjoint and that they each intersect precisely one other block, which must be isomorphic to K_2. We claim that there is vertex v_0 such that these two copies of K_2 have vertices $\{x, v_0\}$ and $\{x', v_0\}$. If this is not the case then

$$|(N(x) - B) \cup (N(x') - B')| = 2. \tag{4}$$

There are now three cases to consider depending on the nature of B and B'. First, suppose $B \cong B' \cong K_2$. Using the m'-recursion twice, we get

$$\begin{aligned} m'(G) &\leq m'(G-B) + m'(G-N[x]) \\ &\leq m'(G-B) + m'(G-N[x]-B') + m'(G-N[x]-N[x']). \end{aligned}$$

Since the three graphs in this last expression may still have r cycles, we will also have to use induction on n. We consider two cases, depending on whether or not $G - N[x] - N[x']$ has parameters lying in the range of Theorem 1.5. Clearly $G - B$ is connected with $n-2$ vertices and r cycles, so if $n \geq 3r+5$ then $n-6 \geq 3r-1$. Also, $n-2$ and r are of different parity with $n-2 \geq 3r$. By assumption (4), $G - N[x] - N[x']$ has $n-6$ vertices and at most r cycles yielding $m'(G - N[x] - N[x']) \leq g(n-6, r)$. Secondly, $G - N[x] - B'$ has $n-5$ vertices and at most r cycles so $m'(G - N[x] - B') \leq g(n-5, r)$. Finally, we can apply induction to conclude that $m'(G - B) \leq c(n-2, r-1)$. Putting everything together we get

$$\begin{aligned} m'(G) &\leq \begin{cases} c(n-2, r-1) + g(n-5, r) + g(n-6, r) & \text{if } n \geq 3r+5, \\ c(n-2, r) + g(n-5) + g(n-6) & \text{if } n = 3r+1 \text{ or } n = 3r+3 \end{cases} \\ &< c(n, r-1), \end{aligned}$$

a contradiction.

Now suppose that $B \cong K_2$ and $B' \cong K_3$. Proceeding in much the same manner as before gives

$$\begin{aligned} m'(G) &\leq m'(G-B) + 2m'(G - N[x] - B') + m'(G - N[x] - N[x']) \\ &\leq \begin{cases} c(n-2, r-1) + 2g(n-6, r-1) + g(n-7, r-1) & \text{if } n \geq 3r+3, \\ c(n-2, r) + 2g(n-6) + g(n-7) & \text{if } n = 3r+1 \end{cases} \\ &\leq c(n, r-1), \end{aligned}$$

and equality cannot occur because $G - B$ has exactly r cycles.

The third case is when $B \cong K_3$ and $B' \cong K_3$. Trying the same technique we obtain

$$\begin{aligned} m'(G) &\leq 2m'(G-B) + 2m'(G - N[x] - B') + m'(G - N[x] - N[x']) \\ &\leq 2c(n-3, r-2) + 2g(n-7, r-2) + g(n-8, r-2) \\ &\leq c(n, r-1). \end{aligned}$$

Again we cannot have equality throughout because $G - B$ has exactly $r-1$ cycles.

Now that we have established the existence of v_0, we are almost done. Observe that there is at most one block C other than the K_2's connecting v_0 to endblocks and those endblocks themselves. (If there were more than one such block, then since endblocks can't intersect this would force the existence of another K_2 and corresponding endblock which we hadn't considered.) So C, if it exists, must be an endblock containing v_0. By our characterization of endblocks, this leaves only three possibilities, namely $C \cong \emptyset, K_2, K_3$. It is easy to check the corresponding graphs G either do not exist because of parity considerations or satisfy $m'(G) < c(n, r-1)$. We have now shown that no graph with exactly n vertices and r cycles has as many maximum independent sets as $C(n, r-1)$, and thus finished the proof of Theorem 3.4. ■

4 The gap revisited

We now need to look at maximum independent sets in the gap. Consider first the case when G is connected with $n = 3r_0 + 2$ vertices and less than r_1 cycles. Then $n \equiv r_0 \pmod 2$

and by Theorem 2.3 we have $m'(G) \leq m(G) \leq c(n, r_0)$, with the second inequality reducing to an equality if and only if $G \cong C(n, r_0)$ or $C(n, r_0 + 1)$. Since the former graph is well covered but the latter is not, we have the following result.

Theorem 4.1 *Let G be a connected graph with n vertices, $n = 3r_0 + 2$ where $n \geq 7$, and less than r_1 cycles. Then*

$$m'(G) \leq c(n, r_0)$$

with equality if and only if $G \cong C(n, r_0)$.

For the $n = 3r_0 + 1$ case we need to adapt the proof of Theorem 2.3. The result mirrors the trend exhibited by Theorem 3.4.

Theorem 4.2 *Let G be a connected graph with n vertices, $n = 3r_0 + 1$ where $n \geq 7$, and less than r_1 cycles. Then*

$$m'(G) \leq c(n, r_0 - 1)$$

with equality if and only if $G \cong C(n, r_0 - 1)$.

Proof: We will use induction on n. The $n = 7$ case has been checked by computer and so we assume G is a graph satisfying the hypotheses of the theorem where $n \geq 10$, or equivalently, $r_0 \geq 3$. We will begin as in the proof of Theorem 2.3, considering possible endblocks produced by the inductive procedure in the Ear Decomposition Theorem.

Our first order of business will be to show that any endblock of an extremal G must be isomorphic to K_i for $2 \leq i \leq 4$ or the graph D_1 shown in Figure 8. Note that, unlike in the proof of Theorem 2.3, here we will consider all endblocks of G, not just those that satisfy the conclusion of Proposition 1.8. As the proof of Lemma 2.1 no longer holds for maximum independent sets, we need the following result to replace Property (Δ). In it, and in the future, it will be convenient to use the notation

$$m'_0 = c(n, r_0 - 1) = 8 \cdot 3^{r_0 - 2} + 2^{r_0 - 2}.$$

Lemma 4.3 (Path Lemma) *Suppose there is a path $P = v_1v_2v_3$ in a block of G satisfying the following three conditions.*

(1) $\deg v_1 \geq 3$, $\deg v_2 = 2$, *and* $v_1v_3 \notin E(G)$,

(2) $G - P$ *is connected,*

(3) One of the following two subconditions hold

- *(a)* $G - N[v_1]$ *is connected, or*
- *(b)* $G - v_1 - N[v_3] = G_1 \uplus G_2$ *where G_1 is connected and $|V(G_2)| \leq 2$.*

Then $m'(G) < m'_0$.

Proof: Using the m'-bound twice gives

$$m'(G) \leq m'(G - v_1 - v_3) + m'(G - v_1 - N[v_3]) + m'(G - N[v_1]). \tag{5}$$

Conditions (1) and (2) of the current lemma imply that $G - v_1 - v_3 = H \uplus \{v_2\}$ where H is connected. Furthermore, since P is in a block, v_2 must lie in at least one cycle of G. Hence H has $n-3$ vertices and less than $r_1 - 1$ cycles with these two parameters satisfying the hypotheses of the theorem. By induction,

$$m'(G - v_1 - v_3) \leq c(n-3, r_0 - 2).$$

Now suppose (3a) holds. Then $m'(G - N[v_1]) \leq c(n-4)$. Also $\deg v_3 \geq 2$ (since it is in a non-K_2 block) and $v_1 \notin N[v_3]$ by condition (1), so $m'(G - v_1 - N[v_3]) \leq g(n-4)$. Putting all these bounds into (5) gives

$$m'(G) \leq c(n-3, r_0 - 2) + c(n-4) + g(n-4) \leq m'_0.$$

Equality can only be achieved if $r_0 = 3$, $H \cong C(7,1)$, and $G - v_1 - N[v_3] \cong G(6)$. But $C(7,1)$ has only one cycle while $G(6)$ has two, contradicting the fact that $G - v_1 - N[v_3] \subset H$, so we must have $m'(G) < m'_0$ in this case.

Next we look at (3b). Considering the cases where $|V(G_2)| = 0$, 1, or 2 gives

$$m'(G - v_1 - N[v_3]) \leq \max\{c(n-4),\ c(n-5),\ 2c(n-6)\} = c(n-4),$$

while $m'(G - N[v_1]) \leq g(n-4)$. Thus we get the same bound on $m'(G)$ as in the case where (3a) held. Equality implies $H \cong C(7,1)$ and $G_1 \cong C(6)$ or K_4 since $c(6) = 2c(4)$, but then we have the same problem with cycles. This final contradiction ends the proof of Lemma 4.3. ■

In all of our applications of the Path Lemma we will set up the notation so that $v_1 = v$.

Lemma 4.4 *Let B be a block of G which comes from subdividing a multigraph M, and suppose that M and B satisfy either*

(1) M contains an edge uv such that

- *(a) v is a non-cutvertex in G,*
- *(b) $\deg_M v$ $(= \deg_B v) = 3$, and*
- *(c) uv is subdivided more than twice in B and none of these inserted vertices are cutvertices,*

or

(2) M contains a doubled edge where both edges are subdivided exactly once and neither of these inserted vertices is a cutvertex.

Then G is not extremal.

Proof: For (1), suppose such an edge vw is subdivided three or more times. Then the hypotheses of the Path Lemma are satisfied with condition (3b) and $v_1 = v$, so $m'(G) < m'_0$.

Part (2) of the lemma follows from the fact that extremal graphs cannot have duplicated vertices: if both edges are subdivided exactly once then we have a pair s, t of duplicated vertices, and Lemma 2.2 shows that

$$m'(G) \leq m(G) = m(G-s) \leq c(n-1) < m'_0,$$

another contradiction. ∎

We are now ready to begin restricting the type of endblocks an extremal graph may possess. Let B denote an endblock of G. First we consider the case where we have $B \cong C_p$. As $c(n,1) < m'_0$ for all $n \geq 10$, G may not itself be a block, and thus we may assume that there is a cutvertex, say x, in B. The $p = 4$ case is ruled out by Lemma 4.4 (2). If $p = 5$ then considering the cases where x is adjacent to exactly one or more than one vertex outside B gives

$$\begin{aligned} m'(G) &\leq m'(G-x) + m'(G-N[x]) \\ &\leq \begin{cases} 3c(n-5) + 2g(n-6) & \text{if } x \text{ has exactly one neighbor outside } B, \\ 3g(n-5) + 2g(n-7) & \text{if } x \text{ has more than one neighbor outside } B \end{cases} \\ &< m'_0 \end{aligned}$$

If $p \geq 6$, then label the non-cutvertices of B by $v_1, v_2, \ldots$ so that $xv_1v_2 \ldots v_{p-1}$ is a path. Using the m'-bound twice gives

$$\begin{aligned} m'(G) &\leq m'(G-v_2) + m'(G-N[v_2]) \\ &\leq m'(G-v_2-v_4) + m'(G-v_2-N[v_4]) + m'(G-N[v_2]). \end{aligned}$$

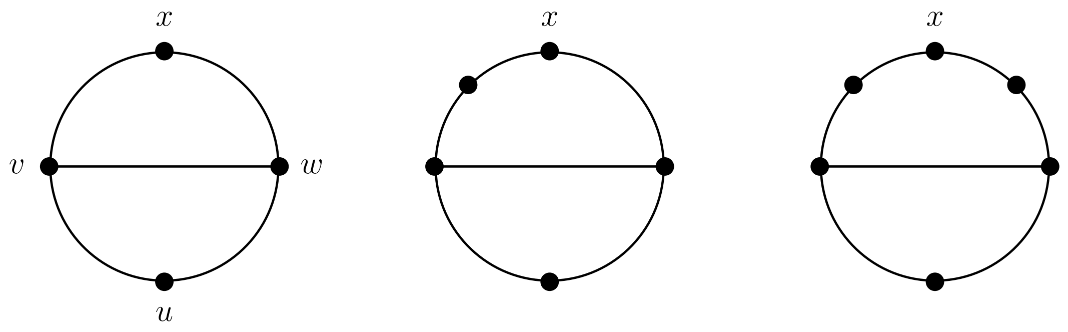
We can apply induction to $m'(G-v_2-v_4)$ since it consists of an isolated vertex together with a connected graph with $n-3$ vertices and one less cycle than G, and similarly to $m'(G-N[v_2])$. This gives

$$m'(G) \leq 2c(n-3, r_0-2) + c(n-4) \leq m'_0,$$

with equality if and only if $r_0 = 3$. But then $G - v_2 - N[v_4] \cong C(6)$, which contradicts the fact that the former graph has a vertex of degree 1 while the latter does not. So if B is a cycle in an extremal graph then $B \cong K_3$.

We now consider the case where B comes from subdividing the graph D in Figure 3. If G is itself a subdivision of D then Lemma 4.4 (1) shows that G has at most 8 vertices, which falls within the range of the computer calculations we have performed. Therefore B must contain a cutvertex x.

First suppose that x is a vertex of D and let v be the other vertex of degree 3. If all of the edges of D are subdivided then at most one of them can be subdivided once by Lemma 4.4 (2), and so G is not extremal because of option (3a) in the Path Lemma. If one of the edges is not subdivided, then the only two possibilities for the number of subdividing



D_1 D_2 D_3

Figure 8: Three special subdivisions of D

vertices in the three edges are $(2,1,0)$ and $(2,2,0)$ by Lemma 4.4 (1). It is then easy to check that

$$\begin{aligned} m'(G) &\leq m'(G-x) + m'(G-N[x]) \\ &\leq \begin{cases} 3g(n-5) + g(n-6) & \text{in the } (2,1,0) \text{ case} \\ g(n-6) + g(n-7) & \text{in the } (2,2,0) \text{ case,} \end{cases} \\ &< m'_0. \end{aligned}$$

This leaves the case where x is interior to an edge of D. Neither of the other edges of D may be subdivided more than twice by Lemma 4.4 (1), so if one of them is subdivided twice then G is not extremal by the Path Lemma (3b). Lemma 4.4 (2) rules out the case where both edges not containing x are subdivided once. Therefore we may assume that one of these edges is subdivided once and the other is not subdivided. Furthermore, the Path Lemma (3a) shows that the edge containing x may be subdivided at most once to either side of x. Thus we are reduced to considering the three endblocks D_1, D_2, and D_3 shown in Figure 8.

Both D_2 and D_3 can be eliminated by using the m'-bound on the vertex x:

$$\begin{aligned} m'(G) &\leq m'(G-x) + m'(G-N[x]) \\ &\leq \begin{cases} 2g(n-5) + 2g(n-6) & \text{if } B \cong D_2, \\ g(n-6) + 3g(n-7) & \text{if } B \cong D_3 \end{cases} \\ &< m'_0. \end{aligned}$$

This leaves us with the case $B \cong D_1$, which we postpone until later.

We now need to go through the E graphs from Figure 4. It will be useful to have an analogue of the large degree and large number of cycles results (Lemmas 2.5 and 2.6) used in proving the first Gap Theorem.

Lemma 4.5 *Suppose that $v \in V(G)$ is a non-cutvertex satisfying either*

(1) $\deg v \geq 3$ *and* $r(v) \geq 7$*, or*

(2) $\deg v \geq 4$ *and* $r(v) \geq 6$.

Then $m'(G) < m'_0$.

Proof: Since $r \leq r_0 + 5$, we have

$$\begin{array}{rcl} m'(G) & \leq & m'(G-v) + m'(G-N[v]), \\ & \leq & \left\{ \begin{array}{ll} c(n-1, r_0-2) + g(n-4, r_0-2) & \text{if } \deg v \geq 3 \text{ and } r(v) \geq 7, \\ c(n-1, r_0-1) + g(n-5) & \text{if } \deg v \geq 4 \text{ and } r(v) \geq 6 \end{array} \right. \\ & < & m'_0, \end{array}$$

proving the lemma. ■

No subdivision of E_1 can be an endblock in an extremal graph by part (2) of the previous lemma. The same reasoning shows that B may only be a subdivision of E_2 if the vertex of degree 4 in E_2 is the cutvertex x. Label the other two vertices v and w. Lemma 4.4 shows that the number of subdividing vertices for the pair of vx edges must be one of $(1,0)$, $(2,0)$, $(2,1)$, or $(2,2)$. The last two cases are eliminated by (3a) of the Path Lemma. The same can be said of the wx edges and that vw can be subdivided at most once by the Path Lemma (3b). So $5 \leq |V(B)| \leq 8$ and

$$\begin{array}{rcl} m'(G) & \leq & m'(G-x) + m'(G-N[x]) \\ & \leq & \max\limits_{5 \leq b \leq 8} \{t'(b-1)g(n-b)\} + g(n-6) \\ & < & m'_0, \end{array}$$

where we remind the reader that $t'(n)$ denotes the maximum number of maximum independent sets in a tree with n vertices.

To deal with E_3, we need an analogue of the Triangle Lemma in this setting.

Lemma 4.6 (Strict Triangle Lemma) *Suppose G contains three vertices $\{v_1, v_2, v_3\}$ satisfying the following two restrictions.*

(1) These vertices form a K_3 with $\deg v_2 = 2$ and $\deg v_1, \deg v_3 \geq 3$.

(2) $G - N[v_2]$ is connected and at least one of $G - N[v_1]$ or $G - N[v_3]$ is connected.

Then $m'(G) < m'_0$.

Proof: Since $G - N[v_2]$ has $n-3$ vertices and less than $r_1 - 1$ cycles, we can use induction to conclude $m'(G - N[v_2]) \leq c(n-3, r_0-2)$. Now using the m'-recursion

$$\begin{array}{rcl} m'(G) & \leq & m'(G-N[v_2]) + m'(G-N[v_1]) + m'(G-N[v_3]) \\ & \leq & c(n-3, r_0-2) + c(n-4) + g(n-4) \\ & \leq & m'_0. \end{array}$$

Equality forces $r_0 = 3$, $G - N[v_2] \cong C(7,1)$, and $G - N[v_1] \cong C(6)$ or $G(6)$. Considering numbers of cycles and containments gives a contradiction. ■

Now consider subdivisions of E_3. Let vw be one of the doubled edges. By symmetry, we can assume that the cutvertex of B in G (if there is one) is neither in one of the vw edges nor adjacent to v. If one of the vw edges is subdivided more than once then G is not extremal by (3b) of the Path Lemma and Lemma 4.4 (1). By Lemma 4.4 (2) the only other option is to have one edge subdivided once and the other not subdivided at all. But then the Strict Triangle Lemma shows that G is not extremal.

Finally we come to $E_4 \cong K_4$. First we claim that any edge vw of K_4 that does not contain a cutvertex of G cannot be subdivided. By the Path Lemma (3b) such an edge cannot be subdivided more than once. If it is subdivided exactly once, then we can use the m'-bound twice and induction to get

$$\begin{aligned} m'(G) &\leq m'(G-v-w) + m'(G-v-N[w]) + m'(G-N[v]) \\ &\leq c(n-3, r_0-2) + g(n-5) + g(n-4) \\ &< m'_0. \end{aligned}$$

So in order for G to have at least 10 vertices this K_4 must contain a cutvertex x of G. Suppose first that x is a vertex of K_4 (before subdivision). Let u, v, w be the other three vertices of K_4. Since none of the edges between these three vertices are subdivided, we can use (3a) of the Path Lemma to conclude that the edge vx is subdivided at most once. If vx is subdivided exactly once, then

$$\begin{aligned} m'(G) &\leq m'(G-v) + m'(G-N[v]) \\ &\leq c(n-1, r_0-1) + c(n-4, r_0-2) \\ &< m'_0. \end{aligned}$$

If x is interior to an edge of K_4, then taking v to be a vertex of K_4 which is not adjacent to x gives the same inequality, so G is not extremal in this case either. This shows that if B comes from subdividing K_4 then we must have $B \cong K_4$ and one of the vertices of B is a cutvertex. We will return to eliminate this case at the end of the proof.

Like in the $n \equiv 1 \pmod 3$ case of the first Gap Theorem, one can use Lemma 4.5 to rule out all descendants of the E graphs. So now we know that all endblocks are copies of K_i, $2 \leq i \leq 4$, or D_1.

The rest of our proof will parallel the last part of the demonstration of Theorem 3.4. There we were able to show that any two endblocks are disjoint. Here we will have to settle for showing that only copies of D_1 may intersect.

Suppse that the endlbocks B and B' both contain the cutvertex x. If $B \cong K_i$ and $B' \cong K_j$ where $2 \leq i, j \leq 4$ then Lemma 3.6 shows that

$$m'(G) = m'(G-x) \leq (i-1)(j-1)g(n-i-j+1) < m'_0$$

unless $i = j = 4$. But in that case removing B and B' destroys 14 of the at most $r_0 + 5$ cycles and we can use the bound $m'(G-x) \leq 9g(n-7, r_0-9) < m'_0$ instead.

Next consider the case where $B \cong K_i$, $2 \leq i \leq 4$, and $B' \cong D_1$ where D_1 is labeled as in Figure 8. Then using the m'-bound, Lemma 3.6, and induction, we have

$$m'(G) \leq m'(G-u) + m'(G-N[u])$$

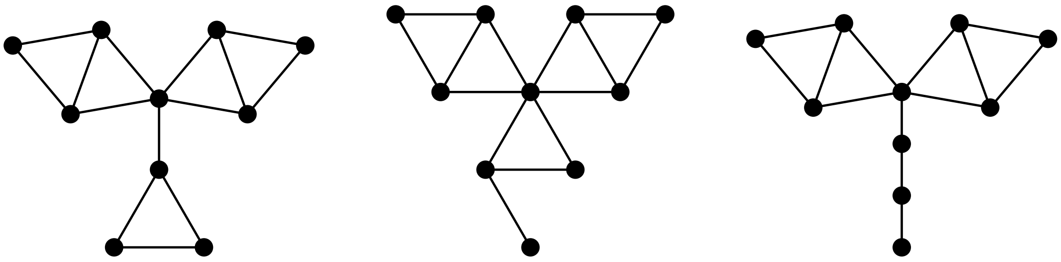


Figure 9: Three graphs to eliminate

$$
\begin{aligned}
&\le \begin{cases} 2g(n-5)+c(n-3,r_0-2) & \text{if } B\cong K_2,\\ 4g(n-6)+c(n-3,r_0-2) & \text{if } B\cong K_3,\\ 6g(n-7,r_0-5)+c(n-3,r_0-2) & \text{if } B\cong K_4,\end{cases}\\
&< m_0'.
\end{aligned}
$$

Hence we know that if two or more endblocks intersect at a cutvertex x then they must all be copies of D_1. Now choose B so that it intersects at most one other non-endblock (by Proposition 1.8). We can eliminate K_4 as a possibility for B by labeling the cutvertex x and counting cycles:

$$
\begin{aligned}
m'(G) &\le m'(G-x)+m'(G-N[x])\\
&\le 3c(n-4,r_0-2)+g(n-5,r_0-2)\\
&< m_0'.
\end{aligned}
$$

If $B\cong D_1$ doesn't intersect another D_1 endblock then the Strict Triangle Lemma can be used to show that G is not extremal. If $B\cong D_1$ intersects with i other endblocks isomorphic to D_1 then we can adapt the proof of the Strict Triangle Lemma to show that G is not extremal. Label the vertices of B as in Figure 8. Using the m'-recursion we get

$$m'(G)\le m'(G-N[u])+m'(G-N[v])+m'(G-N[w])$$

If $i\ge 2$, then both $m'(G-N[v])$ and $m'(G-N[w])$ lie in the range of Theorem 1.5, and by applying induction to $G-N[u]$ we have

$$m'(G)\le c(n-3,r_0-2)+2\cdot 3^i c(n-3i-4,r_0-3i+2).$$

This is a decreasing function of i, and it is strictly less that m_0' when $i=2$.

This leaves the $i=1$ case. Here we use the bound

$$m'(G-N[v])=m'(G-N[w])\le 3\cdot c(n-7)$$

and induction to get

$$m'(G)\le 20\cdot 3^{r_0-3}+7\cdot 2^{r_0-3}.$$

This quantity is strictly less than m'_0 for $r_0 \geq 4$. As we are assuming $r_0 \geq 3$, we have only to eliminate the case $r_0 = 3$. In this case G has 10 vertices, of which 7 are accounted for by the two intersecting copies of D_1. This leaves 3 vertices unaccounted for, and since we have limited the types of endblocks that can occur, there are only three possible graphs of this description. These graphs are depicted in Figure 9. It is easy to check that $m'(G_1) = 2$, $m'(G_2) = 10$, and $m'(G_3) = 11$, which are all less than $c(10, 2) = 26$.

Hence if B intersects precisely one non-endblock then B is isomorphic to either K_2 or K_3, and thus by our previous work B intersects precisely one other block, say A. We claim that $A \cong K_2$ in the other two cases. If not, then the cutvertex x is adjacent to two vertices of a cycle in A. Using the m'-bound as well as induction

$$\begin{aligned} m'(G) &\leq m'(G-x) + m'(G - N[x]) \\ &\leq \begin{cases} c(n-2) + g(n-4) & \text{if } B \cong K_2, \\ 2c(n-3, r_0-2) + g(n-5) & \text{if } B \cong K_3 \end{cases} \\ &< m'_0 \end{aligned}$$

for $r_0 \geq 3$, proving our claim that $A \cong K_2$. Thus if an endblock B intersects at most one non-endblock A then B is the only endblock intersecting A, $B \cong K_2$ or K_3, and $A \cong K_2$.

Suppose B and B' are endblocks of the type considered in the previous paragraph with cutvertices x and x', respectively. We now claim that the associated K_2 blocks must have vertex sets $\{x, v_0\}$ and $\{x', v_0\}$ for some v_0. Suppose not and consider first the case $B \cong B' \cong K_2$. The same argument as in Theorem 3.4 shows that

$$\begin{aligned} m'(G) &\leq m'(G-B) + m'(G - N[x] - B') + m'(G - N[x] - N[x']) \\ &\leq c(n-2) + g(n-5) + g(n-6) \\ &< m'_0 \end{aligned}$$

for $r_0 \geq 3$, so such graphs are not extremal. Now suppose that $B \cong K_3$ and $B' \cong K_2$. In order to apply induction, it is important to use the m'-recursion first on B and then on B' to get

$$\begin{aligned} m'(G) &\leq 2m'(G-B) + m'(G - N[x] - B') + m'(G - N[x] - N[x']) \\ &\leq 2c(n-3, r_0-2) + g(n-6) + g(n-7) \\ &< m'_0, \end{aligned}$$

again resulting in a non-extremal graph. Finally, if $B \cong B' \cong K_3$ then

$$\begin{aligned} m'(G) &\leq 2m'(G-B) + 2m'(G - N[x] - B') + m'(G - N[x] - N[x']) \\ &\leq 2c(n-3, r_0-2) + 2g(n-7) + g(n-8) \\ &= m'_0. \end{aligned}$$

Equality can only be achieved if $G - B \cong C(n-3, r_0-2)$. But then $r(G-B) = r_0 - 2$ and so $r(G) = r_0 - 1$, and then Theorem 3.1 implies that $G \cong C(n, r_0 - 1)$ as desired.

Now that v_0 must exist, the possibilities for other blocks in G are severely limited: G can have no other blocks, or a K_2, K_3, or K_4 endblock containing v_0, or any number of D_1 endblocks which intersect at v_0. Checking the cases where G contains either a complete block or no other block gives us either a graph which is either not extremal or isomorphic to $C(n, r_0 - 1)$ if that block is isomorphic to K_3. Now suppose that G contains one or more copies of D_1 endblocks intersecting at v_0. Let $u_1, u_2, \ldots, u_i$ denote the vertices of these blocks that are not adjacent to v_0. Every maximum independent set in G must contain $\{v_0, u_1, u_2, \ldots, u_i\}$, and from this it is easy to see that such graphs are not extremal. This completes the proof of Theorem 4.1. ■

Acknowledgment. We are indebted to Jason Tedor for suggesting the use of the Ear Decomposition Theorem, to Herbert Wilf for suggesting that we look at the maximum independent set problem once we had done the maximal one, and to the anonymous referees for their numerous insightful remarks and corrections.

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