A Littlewood-Richardson Rule for factorial Schur functions

Alexander I. Molev
Centre for Mathematics and its Applications
Australian National University
Canberra, ACT 0200
AUSTRALIA
molev@pell.anu.edu.au

Bruce E. Sagan
Department of Mathematics
Michigan State University
East Lansing, MI 48824-1027
U.S.A.
sagan@math.msu.edu

June 27, 1997

Key Words: Capelli operator, factorial Schur function, Littlewood-Richardson Rule, quantum immanant, Young tableau

AMS subject classification (1991): Primary 05E05; Secondary 05E10, 17B10, 17B35, 20C30.

Proposed running head:

A factorial L-R Rule

Send proofs to:

Bruce E. Sagan
Department of Mathematics
Michigan State University
East Lansing, MI 48824-1027

Tel.: 517-355-8329
FAX: 517-432-1562
Email: sagan@math.msu.edu

Abstract

We give a combinatorial rule for calculating the coefficients in the expansion of a product of two factorial Schur functions. It is a special case of a more general rule which also gives the coefficients in the expansion of a skew factorial Schur function. Applications to Capelli operators and quantum immanants are also given.

1 Introduction

As λ runs over all partitions with length $l(\lambda) \leq n$, the Schur polynomials $s_\lambda(x)$ form a distinguished basis in the algebra of symmetric polynomials in the independent variables $x = (x_1, \ldots, x_n)$. By definition,

$$s_\lambda(x) = \frac{\det(x_j^{\lambda_i+n-i})_{1\leq i,j\leq n}}{\prod_{i<j}(x_i - x_j)}.$$

Equivalently, these polynomials can be defined by the combinatorial formula

$$s_\lambda(x) = \sum_T \prod_{\alpha\in\lambda} x_{T(\alpha)}, \tag{1}$$

summed over semistandard tableaux T of shape λ with entries in the set $\{1, \ldots, n\}$, where $T(\alpha)$ is the entry of T in the cell α.

Any product $s_\lambda(x)s_\mu(x)$ can be expanded as a linear combination of Schur polynomials:

$$s_\lambda(x)s_\mu(x) = \sum_\nu c_{\lambda\mu}^\nu\, s_\nu(x). \tag{2}$$

The classical Littlewood-Richardson rule [LR] gives a method for computing the coefficients $c_{\lambda\mu}^\nu$. These same coefficients appear in the expansion of a skew Schur function

$$s_{\nu/\lambda}(x) = \sum_\mu c_{\lambda\mu}^\nu s_\mu(x).$$

A number of different proofs and variations of this rule can be found in the literature; see, e.g. [M1, S1], and the references therein.

To state the rule, we introduce the following notation. If T is a tableau then let $\mathrm{cw}(T)$ be the (reverse) *column word* of T, namely the sequence obtained by reading the entries of T from top to bottom in successive columns starting from the right-most column. We will call the associated total order on the cells of T *column order* and write $\alpha < \beta$ if cell α comes before cell β in this order. A word $w = a_1 \cdots a_N$ in the symbols $1, \ldots, n$ is a *lattice permutation* if for $1 \leq r \leq N$ and $1 \leq i < n$ the number of occurrences of i in $a_1 \cdots a_r$ is at least as large as the number of occurrences of $i+1$.

The Littlewood-Richardson rule says that the coefficient $c_{\lambda\mu}^\nu$ is equal to the number of semistandard tableaux T of the shape ν/μ and weight λ such that $\mathrm{cw}(T)$ is a lattice permutation. (One usually uses *row words* in the formulation of the rule. However, it is known that these two versions are equivalent [FG].) In particular, $c_{\lambda\mu}^\nu$ is zero unless $\lambda, \mu \subseteq \nu$ and $|\nu| = |\lambda| + |\mu|$.

We will now state an equivalent formulation of the Littlewood-Richardson rule [JP, Z, KR] and establish some notation to be used in Section 3. Let R denote a sequence of diagrams

$$\mu = \rho^{(0)} \to \rho^{(1)} \to \ldots \to \rho^{(l-1)} \to \rho^{(l)} = \nu,$$

where $\rho \to \sigma$ means that $\rho \subset \sigma$ with $|\sigma/\rho| = 1$. Let r_i denote the row number of $\rho^{(i)}/\rho^{(i-1)}$. Then the sequence $r_1 \ldots r_l$ is called the *Yamanouchi symbol* of R. Equivalently, R corresponds to a standard tableau T of shape ν/μ where r_i is the row number of the entry i in T. A semistandard tableau T *fits* ν/μ if $\mathrm{cw}(T)$ is the Yamanouchi symbol for some standard Young tableau of shape ν/μ. For example,

$$T = \begin{array}{ccc} 1 & 1 & 2 \\ 2 & 3 & \end{array}$$

fits $(4,3,1)/(2,1)$ since $\mathrm{cw}(T) = 21312 = r_1 \ldots r_5$ corresponds to the standard tableau

$$\begin{array}{cccc} & & 2 & 4 \\ & 1 & 5 & \\ 3 & & & \end{array}$$

or equivalently to the shape sequence

$$R : \mu = (2,1) \to (2,2) \to (3,2) \to (3,2,1) \to (4,2,1) \to (4,3,1) = \nu.$$

The coefficient $c_{\lambda\mu}^{\nu}$ is then equal to the number of semistandard tableaux T of shape λ that fit ν/μ.

The factorial Schur functions were introduced by Biedenharn and Louck [BL]. In this paper we study a generalization of them, $s_\lambda(x|a)$, due to Macdonald [M2] and Goulden-Greene [GG] (see definition in Section 2 below). These are polynomials in x and the doubly-infinite sequence of variables $a = (a_i)$. Specializing $a_i = 0$ for all i they turn into $s_\lambda(x)$. The functions $s_\lambda(x|a)$ form a basis in the symmetric polynomials in x over $\mathbb{C}[a]$ so one can define the corresponding Littlewood-Richardson coefficients $c_{\lambda\mu}^{\nu}(a)$, see (5). Our main result is Theorem 3.1 which gives a combinatorial rule for calculating a two-variable generalization $c_{\theta\mu}^{\nu}(a,b)$ of these coefficients (8), where θ is a skew diagram. In the case $|\nu| = |\theta| + |\mu|$ the rule turns into a rule for computing the intertwining number of the skew representations of the symmetric group corresponding to the diagrams θ and ν/μ [LS, JP, Z]. Specializing further to $\mu = \emptyset$ (respectively $\theta = \lambda$) we get the classical Littlewood–Richardson rule in the first (respectively second) formulation above. A completely different rule for calculating the $c_{\lambda\mu}^{\nu}(a)$ follows from the work of Billey and Shimozono [BS] on multiplying Schubert polynomials.

In Section 4 we consider the specialization $a_i = i - 1$. The corresponding coefficients $c_{\lambda\mu}^{\nu}(a)$ turn out to be the structure constants for the center of the universal enveloping algebra for the Lie algebra $\mathfrak{gl}(n)$ and for an algebra of invariant differential operators in certain distinguished bases. We also obtain a formula which relates these coefficients to the dimensions of skew diagrams. This implies a symmetry property of these coefficients.

2 Preliminaries

Let $x = (x_1, \dots, x_n)$ be a finite sequence of variables and let $a = (a_i)$, $i \in \mathbb{Z}$ be a doubly-infinite variable sequence. The *generalized factorial Schur function* for a partition λ of length at most n can be defined as follows [M2]. Let

$$(y|a)^k = (y - a_1) \cdots (y - a_k)$$

for each $k \geq 0$. Then

$$s_\lambda(x|a) = \frac{\det[(x_j|a)^{\lambda_i+n-i}]_{1\leq i,j\leq n}}{\prod_{i<j}(x_i - x_j)}. \tag{3}$$

There is an explicit combinatorial formula for $s_\lambda(x|a)$ analogous to (1):

$$s_\lambda(x|a) = \sum_T \prod_{\alpha\in\lambda} (x_{T(\alpha)} - a_{T(\alpha)+c(\alpha)}), \tag{4}$$

where T runs over all semistandard tableaux of shape λ with entries in $\{1, \dots, n\}$, $T(\alpha)$ is the entry of T in the cell $\alpha \in \lambda$ and $c(\alpha) = j - i$ is the content of $\alpha = (i, j)$.

The highest homogeneous component of $s_\lambda(x|a)$ in x obviously coincides with $s_\lambda(x)$. Therefore the polynomials $s_\lambda(x|a)$ form a basis for $R[x]^{S_n}$ where $R = \mathbb{C}[a]$, and one can define Littlewood-Richardson type coefficients $c_{\lambda\mu}^{\nu}(a)$ by

$$s_\lambda(x|a)s_\mu(x|a) = \sum_\nu c_{\lambda\mu}^{\nu}(a)s_\nu(x|a). \tag{5}$$

Comparing the highest homogeneous components in x on the both sides and using the Littlewood-Richardson Rule for the $s_\lambda(x)$ we see that

$$c_{\lambda\mu}^{\nu}(a) = \begin{cases} c_{\lambda\mu}^{\nu} & \text{if } |\nu| = |\lambda| + |\mu|, \\ 0 & \text{if } |\nu| > |\lambda| + |\mu|. \end{cases} \tag{6}$$

Contrary to the classical case, the coefficients $c_{\lambda\mu}^{\nu}(a)$ turn out to be nonzero if $|\nu| < |\lambda| + |\mu|$ and $\lambda, \mu \subseteq \nu$. This makes it possible to compute them using induction on $|\nu/\mu|$ while keeping λ fixed.

The starting point of our calculation is the fact that the polynomials $s_\lambda(x|a)$ possess some (characteristic) vanishing properties; see [S2, O1]. We use the following result from [O1]. For a partition ρ with $l(\rho) \le n$ define an n-tuple $a_\rho = (a_{\rho_1+n}, \dots, a_{\rho_n+1})$.

Theorem 2.1 (Vanishing Theorem) *Given partitions λ, ρ with $l(\lambda), l(\rho) \le n$*

$$s_\lambda(a_\rho|a) = \begin{cases} 0 & \text{if } \lambda \not\subseteq \rho, \\ \prod_{(i,j)\in\lambda}(a_{\lambda_i+n-i+1} - a_{n-\lambda^t_j+j}) & \text{if } \lambda = \rho. \end{cases}$$

where λ^t is the diagram conjugate to λ.

In particular, $s_\lambda(a_\lambda|a) \neq 0$ for any specialization of the sequence a such that $a_i \neq a_j$ if $i \neq j$. We reproduce the proof of the Vanishing Theorem from [O1] for completeness.

Proof. The ij-th entry of the determinant in the numerator of the right hand side of (3) for $x = a_\rho$ is

$$(a_{\rho_j+n-j+1} - a_1)\cdots(a_{\rho_j+n-j+1} - a_{\lambda_i+n-i}). \tag{7}$$

The condition $\lambda \not\subseteq \rho$ implies that there exists an index k such that $\rho_k < \lambda_k$. Then for $i \le k \le j$ we have

$$1 \le \rho_j + n - j + 1 \le \rho_k + n - k + 1 \le \lambda_k + n - k \le \lambda_i + n - i,$$

and so all the entries (7) with $i \le k \le j$ are zero. Hence, the determinant is zero which proves the first part of the theorem.

Let us set now $x = a_\lambda$ in (3). Then the ij-th entry of the determinant is

$$(a_{\lambda_j+n-j+1} - a_1)\cdots(a_{\lambda_j+n-j+1} - a_{\lambda_i+n-i}),$$

which equals zero for $i < j$ and is nonzero for $i = j$. This means that the matrix is lower triangular with nonzero diagonal elements. Taking their product and dividing by

$$\prod_{i<j}(a_{\lambda_i+n-i+1} - a_{\lambda_j+n-j+1})$$

we get the desired equation. ∎

3 Calculating the coefficients

We will be able to prove more general results by introducing a second infinite sequence of variables denoted $b = (b_i)$, $i \in \mathbb{Z}$. Let θ and μ be skew and normal (i.e., skewed by $\emptyset$) diagrams, respectively. Define Littlewood-Richardson type coefficients $c^{\nu}_{\theta\mu}(a, b)$ by the formula

$$s_\theta(x|b)s_\mu(x|a) = \sum_\nu c^{\nu}_{\theta\mu}(a, b)s_\nu(x|a), \tag{8}$$

where $s_\theta(x|b)$ is defined as in (4) with λ replaced by θ and a replaced by b.

As in Section 1, consider a sequence of diagrams

$$R : \mu = \rho^{(0)} \to \rho^{(1)} \to \ldots \to \rho^{(l-1)} \to \rho^{(l)} = \nu, \tag{9}$$

and let r_i be the row number of $\rho^{(i)}/\rho^{(i-1)}$. Construct the set $\mathcal{T}(\theta, R)$ of semistandard θ-tableaux T with entries from $\{1, \ldots, n = |x|\}$ such that T contains cells $\alpha_1, \ldots, \alpha_l$ with

$$\alpha_1 < \ldots < \alpha_l \quad \text{and} \quad T(\alpha_i) = r_i,\ 1 \le i \le l,$$

where $<$ is column order. Let us distinguish the entries in $\alpha_1, \ldots, \alpha_l$ by barring each of them. For example, if $n = 2$ and

$$R : (2, 1) \to (2, 2) \to (3, 2)$$

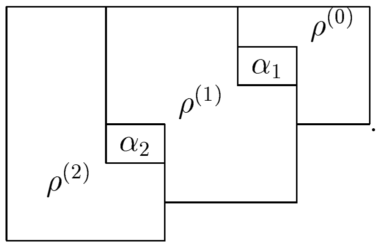
so that $r_1 r_2 = 21$ then for $\theta = (3, 2)/(1)$ we have

$$\mathcal{T}(\theta, R) = \left\{ \begin{array}{ccc} & 1 & 1 \\ \overline{1} & \overline{2} & \end{array},\ \begin{array}{ccc} & \overline{1} & \overline{2} \\ 1 & 2 & \end{array},\ \begin{array}{ccc} & 1 & \overline{2} \\ \overline{1} & 2 & \end{array},\ \begin{array}{ccc} & 1 & 2 \\ \overline{1} & \overline{2} & \end{array},\ \begin{array}{ccc} & \overline{1} & \overline{2} \\ 2 & 2 & \end{array} \right\}.$$

We also let

$$\mathcal{T}(\theta, \nu/\mu) = \biguplus_R \mathcal{T}(\theta, R),$$

where the union is over all sequences R of the form (9). Finally, for each cell α with $\alpha_i < \alpha < \alpha_{i+1}$, $0 \le i \le l$, set $\rho(\alpha) = \rho^{(i)}$. (Inequalities involving cells with out-of-range subscripts are ignored.) For instance, if $l = |\nu/\mu| = 2$ then the following schematic diagram gives the layout of the $\rho(\alpha)$

$\theta =$ 

We are now in a position to state the Littlewood-Richardson rule for the $c^\nu_{\theta\mu}(a,b)$. The reader should compare the following formula with the combinatorial one for the $s_\lambda(x|a)$ in (4).

Theorem 3.1 *The coefficient $c^\nu_{\theta\mu}(a,b)$ is zero unless $\mu\subseteq\nu$. If $\mu\subseteq\nu$ then*

$$c^\nu_{\theta\mu}(a,b)=\sum_{T\in\mathcal{T}(\theta,\nu/\mu)}\prod_{\substack{\alpha\in\theta\\ T(\alpha)\,unbarred}}\big((a_{\rho(\alpha)})_{T(\alpha)}-b_{T(\alpha)+c(\alpha)}\big)\,.$$

As immediate specializations of this result, note the following.

1. If $a=b$ and θ is normal then this is a Littlewood-Richardson rule for the $s_\lambda(x|a)$.

2. If $a=b$ and μ is empty then this is a rule for the expansion of skew factorial Schur polynomial.

3. If $|\nu|=|\theta|+|\mu|$ then $c^\nu_{\theta\mu}(a,b)$ is independent of a and b and equals the number of semistandard tableaux of shape θ that fit ν/μ. This coincides with the number of pictures between θ and ν/μ [JP, Z]. In particular, $c^\nu_{\theta\mu}(a,b)=c^\nu_{\theta\mu}$, an ordinary Littlewood-Richardson coefficient.

4. If $\mu=\emptyset$ and $\theta=\lambda$ is normal then this is a rule for the re-expansion of a factorial Schur polynomial in terms of those for a different sequence of second variables. In particular,

   $$s_\lambda(x|a)=\sum_{\nu\subseteq\lambda}g_{\lambda\nu}(a)s_\nu(x)$$

   where

   $$g_{\lambda\nu}(a)=(-1)^{|\lambda/\nu|}\sum_{T\in\mathcal{T}(\lambda,\nu)}\prod_{\substack{\alpha\in\lambda\\ T(\alpha)\text{unbarred}}}a_{T(\alpha)+c(\alpha)}.$$

   The particular cases of this formula with λ being a row or a column are contained in [GG].

We present the proof of Theorem 3.1 as a chain of propositions.

Note that the first claim of the theorem follows immediately from the Vanishing Theorem. Indeed, let ν be minimal (with respect to containment) among all partitions in (8) such that $c^\nu_{\theta\mu}(a,b)\neq 0$. Suppose $\nu\not\supseteq\mu$. Then setting $x=a_\nu$ in (8) and using the first part of the Vanishing Theorem gives

$$0=c^\nu_{\theta\mu}(a,b)s_\nu(a_\nu|a).$$

But by the Vanishing Theorem's second part we have $s_\nu(a_\nu|a) \neq 0$ and so a contradiction to $c^\nu_{\theta\mu}(a,b) \neq 0$.

We shall assume hereafter that $\mu \subseteq \nu$ and also write

$$|a_\rho| = a_{\rho_1+n} + \cdots + a_{\rho_n+1}.$$

Proposition 3.2 *If $\mu \subseteq \nu$ with $|\nu/\mu| = l$ then*

$$\frac{s_\mu(a_\nu|a)}{s_\nu(a_\nu|a)} = \sum_{\mu\to\rho^{(1)}\to\cdots\to\rho^{(l-1)}\to\nu} \frac{1}{(|a_\nu| - |a_{\rho^{(0)}}|)\cdots(|a_\nu| - |a_{\rho^{(l-1)}}|)},$$

where $\rho^{(0)} = \mu$.

Proof. Setting $x = a_\mu$ in (8) and using the Vanishing Theorem gives

$$c^\mu_{\theta\mu}(a,b) = s_\theta(a_\mu|b). \tag{10}$$

Further, for $\theta = (1)$ and $a = b$ relation (8) takes the form (cf. [OO, Theorem 9.1])

$$s_{(1)}(x|a)s_\mu(x|a) = s_{(1)}(a_\mu|a)s_\mu(x|a) + \sum_{\mu\to\rho} s_\rho(x|a)$$

which follows from (10), (6), and the Branching Theorem for the ordinary Schur functions.

Setting $x = a_\nu$ in the previous equation and using the Vanishing Theorem we get

$$s_{(1)}(a_\nu|a)s_\mu(a_\nu|a) = s_{(1)}(a_\mu|a)s_\mu(a_\nu|a) + \sum_{\mu\to\rho\subseteq\nu} s_\rho(a_\nu|a). \tag{11}$$

We have

$$s_{(1)}(a_\nu|a) - s_{(1)}(a_\mu|a) = |a_\nu| - |a_\mu|$$

and so (11) gives

$$\frac{s_\mu(a_\nu|a)}{s_\nu(a_\nu|a)} = \frac{1}{|a_\nu| - |a_\mu|} \sum_{\mu\to\rho\subseteq\nu} \frac{s_\rho(a_\nu|a)}{s_\nu(a_\nu|a)}.$$

Induction on $|\nu/\mu|$ completes the proof. ∎

It will be convienient to have a notation for sums like those occurring in the previous proposition. So let

$$H(\mu,\rho) = \sum_{\mu\to\rho^{(1)}\to\cdots\to\rho^{(r-1)}\to\rho} \frac{1}{(|a_\rho| - |a_{\rho^{(0)}}|)\cdots(|a_\rho| - |a_{\rho^{(r-1)}}|)}, \tag{12}$$

and

$$H'(\rho,\nu) = \sum_{\rho\to\rho^{(r+1)}\to\cdots\to\rho^{(l-1)}\to\nu} \frac{1}{(|a_\rho| - |a_{\rho^{(r+1)}}|)\cdots(|a_\rho| - |a_{\rho^{(l)}}|)}, \qquad (13)$$

where $\rho^{(0)} = \mu$ and $\rho^{(l)} = \nu$.

Proposition 3.3 *We have the formula*

$$c^\nu_{\theta\mu}(a,b) = \sum_{\mu\subseteq\rho\subseteq\nu} s_\theta(a_\rho|b) H(\mu,\rho) H'(\rho,\nu).$$

Proof. We use induction on $|\nu/\mu|$, noting that (10) is the base case $|\nu/\mu| = 0$. Set $x = a_\nu$ in (8) and divide both sides by $s_\nu(a_\nu|a)$. By Proposition 3.2 we get

$$c^\nu_{\theta\mu}(a,b) = s_\theta(a_\nu|b)H(\mu,\nu) - \sum_{\sigma\subset\nu} c^\sigma_{\theta\mu}(a,b)H(\sigma,\nu).$$

By the induction hypotheses we can write this as

$$\begin{aligned} c^\nu_{\theta\mu}(a,b) &= s_\theta(a_\nu|b)H(\mu,\nu) - \sum_{\sigma\subset\nu}\sum_{\mu\subseteq\rho\subseteq\sigma} s_\theta(a_\rho|b)H(\mu,\rho)H'(\rho,\sigma)H(\sigma,\nu) \\ &= s_\theta(a_\nu|b)H(\mu,\nu) - \sum_{\mu\subseteq\rho\subset\nu} s_\theta(a_\rho|b)H(\mu,\rho) \sum_{\rho\subseteq\sigma\subset\nu} H'(\rho,\sigma)H(\sigma,\nu). \end{aligned}$$

To complete the proof we note that

$$\sum_{\rho\subseteq\sigma\subseteq\nu} H'(\rho,\sigma)H(\sigma,\nu) = 0,$$

which follows from the identity

$$\sum_{i=1}^{k} \frac{1}{(u_1-u_2)\cdots(u_1-u_i)(u_k-u_i)\cdots(u_k-u_{k-1})} = 0,$$

which holds for any variables $u_1,\dots,u_k$ by induction on $k>1$. (In the denominator an empty product is, as usual, equal to 1.) ∎

Proposition 3.4 *We have the recurrence relation*

$$c^\nu_{\theta\mu}(a,b) = \frac{1}{|a_\nu| - |a_\mu|}\left(\sum_{\mu\to\mu'} c^\nu_{\theta\mu'}(a,b) - \sum_{\nu'\to\nu} c^{\nu'}_{\theta\mu}(a,b)\right). \qquad (14)$$

Proof. By Proposition 3.3 it suffices to check that

$$H(\mu,\rho)H'(\rho,\nu) = \frac{1}{|a_\nu| - |a_\mu|}\left(\sum_{\mu\to\mu'} H(\mu',\rho)H'(\rho,\nu) - \sum_{\nu'\to\nu} H(\mu,\rho)H'(\rho,\nu')\right).$$

This follows from the relations

$$\sum_{\mu\to\mu'} H(\mu',\rho) = (|a_\rho| - |a_\mu|)H(\mu,\rho)$$

and

$$\sum_{\nu'\to\nu} H'(\rho,\nu') = (|a_\rho| - |a_\nu|)H'(\rho,\nu). \qquad \blacksquare$$

Given a sequence

$$R: \mu = \rho^{(0)} \to \rho^{(1)} \to \ldots \to \rho^{(l-1)} \to \rho^{(l)} = \nu,$$

and an index $k \in \{1,\ldots,l\}$ introduce a set of θ-tableaux $\mathcal{T}_k(\theta,R)$ having entries from the set $\{1,\ldots,n\}$ as follows. Each tableau $T \in \mathcal{T}_k(\theta,R)$ contains cells $\alpha_1,\ldots,\alpha_{k-1},\alpha_{k+1},\ldots,\alpha_l$ such that

$$\alpha_1 < \ldots < \alpha_{k-1} < \alpha_{k+1} < \ldots < \alpha_l \quad \text{and} \quad T(\alpha_i) = r_i,\ 1 \le i \le l, i \ne k.$$

As usual, we distinguish the entries in the $\alpha_i, i \ne k$, by barring them. Now modify the $\rho(\alpha)$ for R by defining, for cells with unbarred entries,

$$\rho^+(\alpha) = \begin{cases} \rho^{(k)} & \text{if } \alpha_{k-1} < \alpha < \alpha_{k+1}, \\ \rho(\alpha) & \text{otherwise,} \end{cases}$$

and

$$\rho^-(\alpha) = \begin{cases} \rho^{(k-1)} & \text{if } \alpha_{k-1} < \alpha < \alpha_{k+1}, \\ \rho(\alpha) & \text{otherwise.} \end{cases}$$

Also define corresponding products

$$\begin{aligned}
\mathcal{S}(R) &= \sum_{T\in\mathcal{T}(\theta,R)} \prod_{\substack{\alpha\in\theta \\ T(\alpha)\text{unbarred}}} \left((a_{\rho(\alpha)})_{T(\alpha)} - b_{T(\alpha)+c(\alpha)}\right), \\
\mathcal{S}_k^+(R) &= \sum_{T\in\mathcal{T}_k(\theta,R)} \prod_{\substack{\alpha\in\theta \\ T(\alpha)\text{unbarred}}} \left((a_{\rho^+(\alpha)})_{T(\alpha)} - b_{T(\alpha)+c(\alpha)}\right),
\end{aligned}$$

and similarly for $\mathcal{S}_k^-(R)$. So Theorem 3.1 is equivalent to

$$c^\nu_{\theta\mu}(a,b) = \sum_R \mathcal{S}(R). \tag{15}$$

Proposition 3.5 *Given a sequence R we have*

$$\mathcal{S}(R)=\frac{1}{|a_\nu|-|a_\mu|}\sum_{k=1}^{l}\left(\mathcal{S}_k^+(R)-\mathcal{S}_k^-(R)\right). \tag{16}$$

Proof. It suffices to show that for each k we have

$$\mathcal{S}_k^+(R)-\mathcal{S}_k^-(R)=(|a_{\rho^{(k)}}|-|a_{\rho^{(k-1)}}|)\mathcal{S}(R).$$

Formula (16) will then follow from the relation

$$\sum_{k=1}^{l}(|a_{\rho^{(k)}}|-|a_{\rho^{(k-1)}}|)=|a_\nu|-|a_\mu|.$$

For a given $T\in\mathcal{T}_k(\theta,R)$ the factors in the formulas for $\mathcal{S}_k^+(R)$ and $\mathcal{S}_k^-(R)$ are identical except for the case where $\alpha_{k-1}<\alpha<\alpha_{k+1}$ and $T(\alpha)=r_k$. To see what happens when we divide $\mathcal{S}_k^+(R)-\mathcal{S}_k^-(R)$ by

$$|a_{\rho^{(k)}}|-|a_{\rho^{(k-1)}}|=(a_{\rho^{(k)}})_{r_k}-(a_{\rho^{(k-1)}})_{r_k},$$

fix T and consider its contribution to the quotient. We need the following easily proved formula, where we are thinking of $u=(a_{\rho^{(k)}})_{r_k}$, $v=(a_{\rho^{(k-1)}})_{r_k}$ and $m_i=b_{T(\alpha)+c(\alpha)}$ as α runs over all cells of T with $\alpha_{k-1}<\alpha<\alpha_{k+1}$ and $T(\alpha)=r_k$:

$$\frac{\prod_{i=1}^{s}(u-m_i)-\prod_{i=1}^{s}(v-m_i)}{u-v}=\sum_{j=1}^{s}(u-m_1)\cdots\widehat{(u-m_j)}(v-m_{j+1})\cdots(v-m_s)$$

(a hat indicates the factor is to be omitted). The right-hand side of this expression can now be interpreted as the contribution to $\mathcal{S}(R)$ of all tableaux gotten from T by barring one of the r_k between α_{k-1} and α_{k+1} in column order. ■

We now prove (15) by induction on $|\nu/\mu|$. Equation (10) takes care of the case $|\nu/\mu|=0$. By the induction hypothesis,

$$\sum_{\mu\to\mu'}c^{\nu}_{\theta\mu'}(a,b)=\sum_R\mathcal{S}_1^+(R)\quad\text{and}\quad\sum_{\nu'\to\nu}c^{\nu'}_{\theta\mu}(a,b)=\sum_R\mathcal{S}_l^-(R).$$

So formulas (14), (16) and the following proposition complete the proof of (15) and hence Theorem 3.1.

Proposition 3.6 *We have*

$$\sum_R\sum_{k=1}^{l-1}\mathcal{S}_k^-(R)=\sum_R\sum_{k=2}^{l}\mathcal{S}_k^+(R).$$

Proof. We can rewrite this formula as follows:

$$\sum_{R,k,T} \mathrm{wt}^-(R,k,T) = \sum_{R',k',T'} \mathrm{wt}^+(R',k',T'), \tag{17}$$

where $T \in \mathcal{T}_k(\theta, R)$, $k = 1, \ldots, l-1$ and $T' \in \mathcal{T}_{k'}(\theta, R')$, $k' = 2, \ldots, l$ with weights defined by

$$\mathrm{wt}^-(R,k,T) = \prod_{\substack{\alpha\in\theta \\ T(\alpha)\text{unbarred}}} \left((a_{\rho^-(\alpha)})_{T(\alpha)} - b_{T(\alpha)+c(\alpha)}\right)$$

and similarly define $\mathrm{wt}^+(R',k',T')$. To prove (17) we will construct a bijection $(R,k,T) \longleftrightarrow (R',k',T')$ preserving the weights in the sense that $\mathrm{wt}^-(R,k,T) = \mathrm{wt}^+(R',k',T')$. There are three cases.

Case 1. Suppose that the skew diagram $\rho^{(k+1)}/\rho^{(k-1)}$ consists of two cells in different rows and columns. Then R' is the sequence obtained from R by replacing $\rho^{(k)}$ by the other diagram $\rho'^{(k)}$ such that $\rho^{(k-1)} \to \rho'^{(k)} \to \rho^{(k+1)}$ while $k' = k+1$ and $T' = T$.

Case 2. Let $\rho^{(k+1)}/\rho^{(k-1)}$ have two cells in the same row. Then $R' = R$, $k' = k+1$ and $T' = T$.

Case 3. Let $\rho^{(k+1)}/\rho^{(k-1)}$ have two cells in the same column, say in rows r and $r+1$. Let $(i+1,j) = (i_1,j_1)$ be the cell of T containing the correponding $\overline{r+1}$. If there is an r in cell (i,j) then it must be unbarred. In this case let T' be T with the bar moved from the $r+1$ to the r, $R' = R$ and $k' = k+1$. Weights are preserved since $(a_{\rho^{(k-1)}})_r = (a_{\rho^{(k+1)}})_{r+1}$ and $T(\alpha) + c(\alpha)$ is invariant under the change.

Now suppose cell (i,j) of T contains an entry less than r or $(i,j) \notin \theta$ and let $j' = j_1'$ be the column of the left-most $r+1$ in row $i+1$. Since this subcase is more complicated than the others, the reader may wish to follow along with the example given after the end of this proof. Let s be the maximum integer such that for $1 \le t \le s$ we have

1. there is an $\overline{r+t}$ in cell $(i+t, j_t)$ for some j_t corresponding to a cell in the same column as those of $\rho^{(k+1)}/\rho^{(k-1)}$,
2. if $(i+t, j_t')$ contains the left-most $r+t$ in row $i+t$ then $(i+t, j_t)$ is between $(i+t-1, j_{t-1})$ and $(i+t-1, j_{t-1}')$ in column order. (Assume this is true vacuously when $t = 1$.)

Note that the condition on cell (i,j) implies that none of the $r+t$'s to the left of the one in $(i+t, j_t)$ can be barred.

We now form T' by moving the bar in cell $(i+t, j_t)$ to cell $(i+t, j'_t)$ and replacing the $r+t$'s in cells $(i+t, j'_t), (i+t, j'_t+1), \ldots, (i+t, j_t)$ by $r+t-1$'s. Note that the result will still be a semistandard tableau because of the assumption about (i, j) and the choice of elements to decrease. Since the elements from the given column of $\rho^{(k+1)}/\rho^{(k-1)}$ are still added in the correct order in T' it determines a valid R', complete except for the step where a cell is added in row $r+s$ of that column which should be done immediately following the addition of $r+s-1$. Then k' is the position of this $r+s$. Invariance of weights follows from considerations like those in the first subcase, noting that the contribution to wt^- of each entry decreased in T is the same as that of the element on its right to wt^+ in T'.

The inverse of this construction is similar and left to the reader. This completes the proof of the Theorem 3.1. ∎

As an example of the last subcase, suppose we have the R sequence

$$(3,2,2,2) \to (3,3,2,2) \to (3,3,3,2) \to (4,3,3,2) \to (4,3,3,2,1) \to (4,3,3,3,1)$$

with Yamanouchi symbol $r_1 \ldots r_5 = 23154$. Let $k = 1$ so $r = 2$ and consider

$$T = \begin{array}{cccccc} 1 & 1 & 1 & 1 & \overline{1} & 1 \\ 2 & 3 & 3 & 3 & 3 & \overline{3} \\ 4 & 4 & \overline{4} & 4 & \overline{5} & \end{array} .$$

Then $(i+1, j) = (2, 6)$ and $s = 2$ with $r+1, r+2 = 3, 4$ so

$$T' = \begin{array}{cccccc} 1 & 1 & 1 & 1 & \overline{1} & 1 \\ 2 & \overline{2} & 2 & 2 & 2 & 2 \\ \overline{3} & 3 & 3 & 4 & \overline{5} & \end{array} .$$

Column reading the barred elements of T' and inserting $r+2 = 4$ after $r+1 = 3$ gives the Yamanouchi symbol 15234 corresponding to the R' sequence

$$(3,2,2,2) \to (4,2,2,2) \to (4,2,2,2,1) \to (4,3,2,2,1) \to (4,3,3,2,1) \to (4,3,3,3,1)$$

and $k' = 5$.

4 Multiplication rules for Capelli operators and quantum immanants

Let E_{ij}, $i, j = 1, \ldots, n$ denote the standard basis of the general linear Lie algebra $\mathfrak{gl}(n)$. Denote by $Z(\mathfrak{gl}(n))$ the center of the universal enveloping algebra $U(\mathfrak{gl}(n))$.

Given $\kappa = (\kappa_1, \ldots, \kappa_n) \in \mathbb{C}^n$ consider a $\mathfrak{gl}(n)$-module $L(\kappa)$ of highest weight κ. That is, $L(\kappa)$ is generated by a nonzero vector v such that

$$\begin{aligned} E_{ii} \cdot v &= \kappa_i \, v, \quad \text{for} \quad i = 1, \ldots, n; \\ E_{ij} \cdot v &= 0 \quad \text{for} \quad 1 \leq i < j \leq n. \end{aligned}$$

Any element $z \in Z(\mathfrak{gl}(n))$ acts as a scalar $\omega(z) = \omega_\kappa(z)$ in $L(\kappa)$ and this scalar is independent of the choice of the highest weight module $L(\kappa)$. Moreover, $\omega(z)$ is a symmetric polynomial in the shifted variables $x_1, \ldots, x_n$ where $x_i = \kappa_i + n - i$. The mapping $z \mapsto \omega(z)$ defines an algebra isomorphism

$$\omega : \, Z(\mathfrak{gl}(n)) \to \mathbb{C}[x]^{S_n}$$

called the *Harish-Chandra isomorphism*; see e.g. Dixmier [D, Section 7.4].

For any positive integer m consider the natural action of the complex Lie group $GL(n)$ in the algebra $\mathcal{P}$ of polynomials on the vector space $\mathbb{C}^n \otimes \mathbb{C}^m$. The corresponding Lie algebra $\mathfrak{gl}(n)$ then acts by differential operators

$$\pi(E_{ij}) = \sum_{a=1}^{m} x_{ia} \partial_{ja},$$

where the x_{ia} are the coordinates on $\mathbb{C}^n \otimes \mathbb{C}^m$ and the ∂_{ia} are the corresponding partial derivatives. This representation is uniquely extended to an algebra homomorphism

$$\pi : \, U(\mathfrak{gl}(n)) \to \mathcal{PD}$$

where $\mathcal{PD}$ is the algebra of polynomial coefficient differential operators in the x_{ia}. The image of $Z(\mathfrak{gl}(n))$ under π is contained in the subalgebra $\mathcal{PD}^G$ of differential operators invariant with respect to the action of the group $G = GL(n) \times GL(m)$. Moreover, if $m \geq n$ then this restriction is an algebra isomorphism which can be called the *Capelli isomorphism*; see [H, HU] for further details. So if $m \geq n$ we have the triple isomorphism

$$\mathbb{C}[x]^{S_n} \xleftarrow{\omega} Z(\mathfrak{gl}(n)) \xrightarrow{\pi} \mathcal{PD}^G.$$

Distinguished bases in the three algebras which correspond to each other under these isomorphisms were constructed in [O1] (see also [N, O2, M3]).

In the algebra $\mathbb{C}[x]^{S_n}$ the basis is formed by the polynomials $s_\lambda(x|a)$ with $l(\lambda) \leq n$ and the sequence a specialized to $a_i = i - 1$ for all $i \in \mathbb{Z}$. We shall denote these polynomials by $s^*_\lambda(x)$. They were first introduced in [BL]. Explicitly,

$$s^*_\lambda(x) = \sum_T \prod_{\alpha \in \lambda} (x_{T(\alpha)} - T(\alpha) - c(\alpha) + 1),$$

where T runs over all semistandard tableaux of shape λ with entries in $\{1, \ldots, n\}$. We shall denote by $f^{\nu}_{\lambda\mu}$ the coefficient $c^{\nu}_{\lambda\mu}(a)$ in this specialization of a. In other words, the $f^{\nu}_{\lambda\mu}$ can be defined by the formula

$$s^*_\lambda(x) s^*_\mu(x) = \sum_\nu f^{\nu}_{\lambda\mu}\, s^*_\nu(x).$$

To describe the basis in $Z(\mathfrak{gl}(n))$ we introduce some more notation. Given k matrices A, B, ..., C of size $p \times q$ with entries from an algebra $\mathcal{A}$ we regard their tensor product $A \otimes B \otimes \cdots \otimes C$ as an element

$$\sum A_{a_1 i_1} B_{a_2 i_2} \cdots C_{a_k i_k} \otimes e_{a_1 i_1} \otimes e_{a_2 i_2} \otimes \cdots \otimes e_{a_k i_k} \in \mathcal{A} \otimes (\mathrm{Mat}_{pq})^{\otimes k},$$

where Mat_{pq} denotes the space of complex $p \times q$-matrices and the e_{ai} are the standard matrix units. The symmetric group S_k acts in a natural way in the tensor space $(\mathbb{C}^n)^{\otimes k}$, so that we can identify permutations from S_k with elements of the algebra $(\mathrm{Mat}_{nn})^{\otimes k}$.

For a diagram λ with $l(\lambda) \leq n$ denote by T_0 the λ-tableau obtained by filling in the cells by the numbers $1, \ldots, k = |\lambda|$ from left to right in successive rows starting from the first row. We let R_λ and C_λ denote the row symmetrizer and column antisymmetrizer of T_0 respectively. By $c_\lambda(r)$ we denote the content of the cell occupied by r. Introduce the matrix $E = (E_{ij})$ whose ij-th entry is the generator E_{ij} and set

$$\mathbb{S}_\lambda = \frac{1}{h(\lambda)} \operatorname{tr}(E - c_\lambda(1)) \otimes \cdots \otimes (E - c_\lambda(k)) \cdot R_\lambda C_\lambda,$$

where the trace is taken over all the tensor factors Mat_{nn}, and $h(\lambda)$ is the product of the hook-lengths of the cells of λ:

$$h(\lambda) = \prod_{\alpha \in \lambda} h_\alpha.$$

The elements $\mathbb{S}_\lambda$ with $l(\lambda) \leq n$ form a basis in the algebra $Z(\mathfrak{gl}(n))$. In [O1] they were called the *quantum immanants*.

Let us now describe the basis in the algebra $\mathcal{PD}^G$. The representation π can be written in a matrix form as follows

$$\pi : \, E \mapsto X D^t,$$

where X and D are the $n \times m$ matrices formed by the coordinates x_{ia} and the derivatives ∂_{ia}, respectively, while D^t is the matrix transposed to D. Introduce the following differential operators

$$\Delta_\lambda = \frac{1}{k!} \operatorname{tr} X^{\otimes k} \cdot (D^t)^{\otimes k} \cdot \chi^\lambda,$$

where χ^λ is the irreducible character of S_k corresponding to λ. Explicitly,

$$\Delta_\lambda = \frac{1}{k!} \sum_{i_1,\dots,i_k} \sum_{a_1,\dots,a_k} \sum_{s\in S_k} \chi^\lambda(s) x_{i_1a_1} \cdots x_{i_ka_k} \partial_{i_{s(1)}a_1} \cdots \partial_{i_{s(k)}a_k}.$$

The operators Δ_λ with $l(\lambda) \le n$ form a basis in $\mathcal{PD}^G$. They are called the *higher Capelli operators.*

The following identities were proved in [O1] (for other proofs see [N, O2, M3]):

$$\omega(\mathbb{S}_\lambda) = s^*_\lambda(x) \quad \text{and} \quad \pi(\mathbb{S}_\lambda) = \Delta_\lambda.$$

Using Theorem 3.1 we obtain the following multiplication rules for the elements $\mathbb{S}_\lambda$ and the operators Δ_λ.

Theorem 4.1 *We have*

$$\mathbb{S}_\lambda \mathbb{S}_\mu = \sum_\nu f^\nu_{\lambda\mu}\, \mathbb{S}_\nu$$

and

$$\Delta_\lambda \Delta_\mu = \sum_\nu f^\nu_{\lambda\mu}\, \Delta_\nu$$

where the coefficients $f^\nu_{\lambda\mu}$ are given by

$$f^\nu_{\lambda\mu} = \sum_{T\in\mathcal{T}(\lambda,\nu/\mu)} \prod_{\substack{\alpha\in\theta\\ T(\alpha) unbarred}} \big(\rho(\alpha)_{T(\alpha)} + n - 2\,T(\alpha) - c(\alpha) + 1\big) \tag{18}$$

with R, $\mathcal{T}(\lambda,\nu/\mu)$, and $\rho(\alpha)$ defined in Theorem 3.1. ∎

Proposition 3.3 enables us to obtain another formula for $f^\nu_{\lambda\mu}$. For a skew diagram ν/μ let

$$h(\nu/\mu) = \frac{|\nu/\mu|!}{\dim \nu/\mu},$$

where $\dim \nu/\mu$ is the number of standard ν/μ-tableaux. In particular, if μ is empty $h(\nu)$ is the product of the hook-lengths of the cells of ν.

In the specialization of the sequence a under consideration we obtain from (12) and (13) that

$$H(\mu,\rho) = \frac{1}{h(\rho/\mu)} \quad \text{and} \quad H'(\rho,\nu) = \frac{(-1)^{|\nu/\rho|}}{h(\nu/\rho)}.$$

Moreover, by Proposition 3.2,

$$\frac{s_\lambda(a_\rho|a)}{s_\rho(a_\rho|a)} = \frac{1}{h(\rho/\lambda)}$$

and by the Vanishing Theorem

$$s_\rho(a_\rho|a) = h(\rho).$$

Thus Proposition 3.3 becomes the following.

Proposition 4.2 *One has the formula*

$$f^\nu_{\lambda\mu} = \sum_{\lambda,\mu\subseteq\rho\subseteq\nu} (-1)^{|\nu/\rho|} \frac{h(\rho)}{h(\nu/\rho)h(\rho/\lambda)h(\rho/\mu)}. \qquad \blacksquare \tag{19}$$

Formula (19) implies the following symmetry property of the coefficients $f^\nu_{\lambda\mu}$.

Corollary 4.3 *If* $l(\lambda^t),\ l(\mu^t),\ l(\nu^t) \leq n$ *then*

$$f^{\nu^t}_{\lambda^t\mu^t} = f^\nu_{\lambda\mu}.$$

Proof. This follows immediately from the relation $h(\nu^t/\mu^t) = h(\nu/\mu)$. $\blacksquare$

Remarks. 1. It follows from (18) that the coefficients $f^\nu_{\lambda\mu}$ are integers while the summands on right hand side of (19) need not be. In fact the numbers $h(\nu/\mu)$ need not be integers either, e.g., $h((3,2)/(1)) = 24/5$.

2. Note that since in the case of $|\nu| = |\lambda|+|\mu|$ the $f^\nu_{\lambda\mu}$ coincide with the classical Littlewood-Richardson coefficients $c^\nu_{\lambda\mu}$, the latter can be computed using (19) as well, but this does not appear to be very useful for practical purposes. For example, consider $\lambda = \mu = (1^n)$ and $\nu = (2^r 1^{n-r})$, then (19) gives

$$f^\nu_{\lambda\mu} = \sum_{k=0}^{r} (-1)^{r-k} \frac{(n+1)!}{k!(r-k)!(n-k+1)}$$

while directly from (18) we get

$$f^\nu_{\lambda\mu} = (n-r)! \tag{20}$$

As a final example, take $m = n$ in the definition of Δ_λ. Then for $\lambda = (1^n)$ we get the classical Capelli operator [C]:

$$\Delta_{(1^n)} = \det X \det D.$$

We find from (18) that the coefficients $f^\nu_{(1^n)(1^n)}$ are zero except for $\nu = (2^r 1^{n-r})$, $r = 0, 1, \ldots, n$. So by (20) the square of the Capelli operator is given by

$$(\det X \det D)^2 = \sum_{r=0}^{n} (n-r)!\, \Delta_{(2^r 1^{n-r})}.$$

Acknowledgment. We would like to thank G. D. James, A. N. Kirillov, A. Lascoux and S. Sahi for valuable discussions.

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