A Generalization of Rota's NBC Theorem

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Abstract

We generalize Rota's theorem characterizing the Möbius function of a geometric lattice in terms of subsets of atoms containing no broken circuit and give applications to the weak Bruhat order of a finite Coxeter group and the Tamari lattices. We also give a direct proof of the fact that in the geometric case any total order of the atoms can be used. Simple involutions are used in both proofs. Finally we show how involutions can be used in similar situations, specifically in a special case of Rota's Crosscut Theorem as well as in related proofs of Walker on Hall's Theorem and Reiner on characteristic and Poincaré polynomials.

1 Rota's theorem and its generalization

One of the most beautiful and useful theorems in algebraic combinatorics is Rota's theorem [14] characterizing the Möbius function of a geometric lattice in terms of subsets of atoms which are NBC, i.e., contain no broken circuit. In this note we will generalize Rota's theorem to any lattice satisfying a simple condition and give applications to the weak Bruhat order of a Coxeter group and the Tamari lattices. The proof of Rota's theorem is an easy application of the simplest version of the Involution Principle of Garsia and Milne [6]. We also use an involution to show directly that in the geometric case the number of NBC sets is the same for any total ordering of the atoms. Finally we discuss a related proof for a special case of Rota's Crosscut Theorem as well as proofs of Walker concerning the Möbius function as a reduced Euler characteristic and of Reiner connecting characteristic and Poincaré polynomials.

We first review Rota's original theorem. Let L be a finite poset with minimal element $\hat{0}$. The *Möbius function* of L is the function $\mu : L \to \mathbf{Z}$ ($\mathbf{Z}$ being the integers) which is uniquely defined by

$$\sum_{y \leq x} \mu(y) = \delta_{\hat{0}x} \tag{1}$$

where the right side is the Kronecker delta. In particular, if L is the lattice of divisors of an integer then μ is the number-theoretic Möbius function.

Suppose that L is a lattice and let $\wedge$ and $\vee$ denote the meet (greatest lower bound) and join (least upper bound) operations, respectively. Let $\mathcal{A}(L)$ be the set of *atoms* of L, i.e, all $a \neq \hat{0}$ such that there is no $x \in L$ with $\hat{0} < x < a$. We say that L is *atomic* if every $x \in L$ is a join of atoms.

Assume further that L is *ranked* with rank function ρ, which means that for all $x \in L$ the quantity

$$\rho(x) = \text{length of a maximal } \hat{0} \text{ to } x \text{ chain}$$

is well defined (independent of the chain). Such a lattice is *semimodular* if

$$\rho(x \wedge y) + \rho(x \vee y) \leq \rho(x) + \rho(y)$$

for all $x, y \in L$. It is easy to prove, using this inequality and induction, that if $B \subseteq \mathcal{A}(L)$ then $\rho(\vee B) \leq |B|$ where the vertical bars denote cardinality. So define B to be *independent* if $\rho(\vee B) = |B|$ and *dependent* otherwise. If B is independent then we say it is a *base* for $x = \vee B$. If C is a minimal (with respect to inclusion) dependent set then we say that C is a *circuit*. Now put a total order on $\mathcal{A}(L)$ which we will denote $\trianglelefteq$ to distinguish it from the partial order $\leq$ in L. A circuit

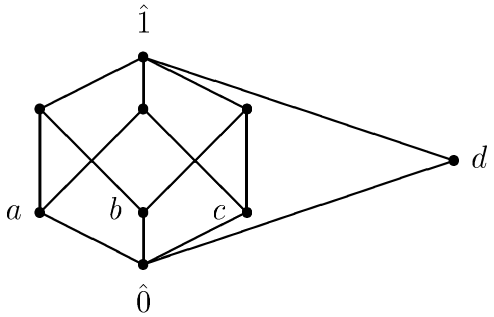


Figure 1: An example lattice L

C has corresponding *broken circuit* $\bar{C} = C \setminus c$ where c is the smallest atom in C. Finally, $B \subseteq \mathcal{A}(L)$ is *NBC* if it contains no broken circuit. Note that such a set must be independent. Rota's theorem can now be stated.

Theorem 1.1 (Rota) *Let L be a geometric (i.e., atomic and semimodular) lattice. Then for any total ordering of $\mathcal{A}(L)$ we have*

$$\mu(x) = (-1)^{\rho(x)} (\textit{number of NBC bases of } x). \blacksquare \tag{2}$$

To generalize this result to lattices, we first need to redefine some terms since L may no longer be ranked. Call $B \subseteq \mathcal{A}(L)$ *independent* if $\vee\bar{B} < \vee B$ for any proper subset $\bar{B}$ of B. Thus if C is dependent then $\vee\bar{C} = \vee C$ for some $\bar{C} \subset C$. Note that it follows directly from the definitions that a superset of a dependent set is dependent, or equivalently that a subset of an independent set is independent. The definitions of base, circuit and broken circuit can now be kept as before. If C is a circuit, it will be convenient to adopt the notation $\bar{C} = C \setminus c$ for the corresponding broken circuit. This done, our generalization is as follows.

Theorem 1.2 *Let L be a finite lattice. Let $\trianglelefteq$ be any total ordering of $\mathcal{A}(L)$ such that for all broken circuits $\bar{C} = C \setminus c$ we have*

$$\vee\bar{C} = \vee C.$$

Then for all $x \in L$ we have

$$\mu(x) = \sum_{B} (-1)^{|B|} \tag{3}$$

where the sum is over all NBC bases B of x.

Before presenting the proof, let us do an example. Consider the lattice L in Figure 1 with the atoms ordered $a \lhd b \lhd c \lhd d$. The circuits of L are $\{a, b, d\}$,

$\{a,c,d\}$ and $\{b,c,d\}$ with corresponding broken circuits $\{b,d\}$ and $\{c,d\}$. It is easy to verify that these circuits satisfy the hypothesis of Theorem 1.2. Also, the element $x = \hat{1}$ has two NBC bases, namely $\{a,d\}$ and $\{a,b,c\}$. It follows that

$$\mu(\hat{1}) = (-1)^2 + (-1)^3 = 0$$

which is readily checked from the definition of the Möbius function.

Proof (of Theorem 1.2). Let

$$\tilde{\mu}(x) = \sum_B (-1)^{|B|}.$$

Then since (1) uniquely defines μ, it suffices to show that $\sum_{y \le x} \tilde{\mu}(y) = \delta_{\hat{0}x}$. If $x = \hat{0}$ then both sides of this equation are clearly equal to 1. So we assume that $x > \hat{0}$ and show that

$$\sum_{y \le x} \tilde{\mu}(y) = 0. \tag{4}$$

Consider the set

$$\mathcal{S} = \{B \ : \ B \text{ is a base for some } y \le x\}$$

with sign function

$$\epsilon(B) = (-1)^{|B|}.$$

Clearly $\sum_{B \in \mathcal{S}} \epsilon(B)$ is the left side of (4), so to prove this identity it suffice to find a sign-reversing involution on $\mathcal{S}$.

Let a_0 be the smallest atom under x. Define a map $\iota : \mathcal{S} \to \mathcal{S}$ by

$$\iota(B) = B \bigtriangleup a_0$$

where $\bigtriangleup$ is the symmetric difference operator. This is clearly a sign-reversing involution as long as it is well-defined, i.e., as long as B NBC implies $\iota(B)$ NBC.

There are now two cases. If $\iota(B) = B \setminus a_0$ then $\iota(B)$ is still NBC because it is a subset of B. Otherwise let $\bar{B} := \iota(B) = B \cup a_0$ and suppose $\bar{B}$ contains broken circuit $\bar{C} = C \setminus c$. If $a_0 \notin \bar{C}$ then $\bar{C} \subseteq B$ contradicting B being NBC. If $a_0 \in \bar{C}$ then we must have

$$c \lhd a_0 \tag{5}$$

because of the way circuits are broken. But now, using the theorem's hypothesis,

$$c \le \vee C = \vee \bar{C} \le \vee \bar{B} \le x.$$

Thus $c \unrhd a_0$ since a_0 is the least atom under x, contradicting (5). ■

Note that when L is geometric, then all NBC bases of a given $x \in L$ have the same number of elements, namely $\rho(x)$. Thus the right sides of (2) and (3) do really coincide in this case. Furthermore, the hypothesis of Theorem 1.2 explains why any ordering of $\mathcal{A}(L)$ works in Rota's Theorem: $\vee\bar{C} = \vee C$ for any $\bar{C}$ obtained by removing a single atom from the circuit C. On the other hand, it would be nice to have a direct proof of this fact using involutions, which we present next.

Proposition 1.3 *Let L be a geometric lattice and let $\mathcal{O}_1$ and $\mathcal{O}_2$ be two total orderings of $\mathcal{A}(L)$. Then for all $x \in L$ we have*

$$\textit{number of NBC bases of } x \textit{ in } \mathcal{O}_1 = \textit{number of NBC bases of } x \textit{ in } \mathcal{O}_2. \quad (6)$$

Proof. It is enough to prove the proposition in the case where $\mathcal{O}_2$ is obtained from $\mathcal{O}_1$ by transposing the order of two atoms $c \lhd_1 d$ adjacent in $\mathcal{O}_1$. For $i = 1, 2$ let $\mathcal{BC}_i$ (respectively, $\mathcal{NBC}_i$) be the set of broken circuits (respectively, NBC bases of x) in the two orders. If C is any circuit containing both c, d and no a smaller than both then

$$\bar{C}^1 = C \setminus c \in \mathcal{BC}_1 \quad \text{and} \quad \bar{C}^2 = C \setminus d \in \mathcal{BC}_2. \quad (7)$$

Any other circuit gives rise to

$$\bar{C} = C \setminus a \in \mathcal{BC}_1 \cap \mathcal{BC}_2. \quad (8)$$

Now define a map $f : \mathcal{NBC}_1 \setminus \mathcal{NBC}_2 \to \mathcal{NBC}_2 \setminus \mathcal{NBC}_1$ by

$$f(B) = B \bigtriangleup \{c, d\}.$$

We must show that f is well-defined.

First note that B cannot contain any broken circuit of the form (8) since $B \in \mathcal{NBC}_1$, but it must contain one of the form (7) since $B \notin \mathcal{NBC}_2$. So for some circuit C we have $B \supseteq \bar{C}^2$ but $B \not\supseteq \bar{C}^1$, since $B \in \mathcal{NBC}_1$. Thus we must have $c \in B$ and $d \notin B$ and so

$$f(B) = (B \setminus c) \cup d = \bar{B} \cup d \quad (9)$$

where $\bar{B} = B \setminus c$.

We claim that $f(B) \notin \mathcal{NBC}_1$. Indeed $\bar{C}^1 = (\bar{C}^2 \setminus c) \cup d$ and $\bar{C}^2 \subseteq B$. Thus equation (9) yields $\bar{C}^1 \subseteq f(B)$.

Now we claim that $f(B) \in \mathcal{NBC}_2$. Suppose not. But $f(B)$ can't contain a broken circuit of the form (7) since $c \notin f(B)$ and $c \in \bar{C}^2$ for all C of this type. Thus $f(B)$ must contain a broken circuit of type (8), say $f(B) \supseteq \bar{D}$. However, $B \not\supseteq \bar{D}$ and so by (9) we can write

$$\bar{D} = D' \cup d$$

where $D' \subseteq B$. Also $\bar{D} = D \setminus a$ where $a \lhd d$ (in both orders) since $d \in \bar{D}$. In fact a is smaller than every atom in C since no element of C is smaller than c, d. Write

$$D = D' \cup a \cup d \quad \text{and} \quad C = \bar{C}^2 \cup d$$

where the unions are disjoint. The fact that L is geometric implies that there exists a circuit $E \subset C \cup D$ which does not contain $d \in C \cap D$. Now if $a \notin E$ then

$$E \subseteq D' \cup \bar{C}^2 \subseteq B$$

which contradicts $B \in \mathcal{NBC}_1$. On the other hand, if $a \in E$ then $\bar{E} = E \setminus a$ since a is minimal in $C \cup D$. But then $\bar{E} \subseteq B$ contradicting $B \in \mathcal{NBC}_1$ again. This final contradiction finishes the proof that $f(B) \in \mathcal{NBC}_2$ and that f is well-defined.

Using precisely the same reasoning, one shows that f has a well-defined inverse $f^{-1} : \mathcal{NBC}_2 \setminus \mathcal{NBC}_1 \to \mathcal{NBC}_1 \setminus \mathcal{NBC}_2$ given by

$$f^{-1}(B') = B' \bigtriangleup \{c, d\} = (B' \setminus d) \cup c.$$

Thus f is a bijection and this proves the theorem. Of course, this could also be expressed in terms of an involution $\iota : \mathcal{NBC}_1 \bigtriangleup \mathcal{NBC}_2 \to \mathcal{NBC}_1 \bigtriangleup \mathcal{NBC}_2$ where $\iota(B) = B \bigtriangleup \{c, d\}$. ■

A related result is the fact that the Tutte polynomial [15] as defined by external and internal activities is order-independent. Tutte's original proof of this fact in the paper just cited is quite involved. It would be interesting to find an easier proof using involutions.

2 Applications

We now give two examples of lattices which are not geometric, but whose Möbius functions can be computed using Theorem 1.2. We first note a general result that follows from our main theorem.

Corollary 2.1 *Let L be a finite lattice such that $\mathcal{A}(L)$ is independent. Then the Möbius values of L are all 0 or ± 1. Specifically, if $x \in L$ then*

$$\mu(x) = \begin{cases} (-1)^{|B|} & \text{if } x = \vee B \text{ for some } B \subseteq \mathcal{A}(L), \\ 0 & \text{else.} \end{cases}$$

Proof. If $\mathcal{A}(L)$ is independent then so is any $B \subseteq \mathcal{A}(L)$. Furthermore, there are no circuits so any such B is NBC. Finally, independence of $\mathcal{A}(L)$ implies that $\vee B \neq \vee B'$ for any $B \neq B'$. The corollary now follows from Theorem 1.2. ■

We note that Corollary 2.1 also follows easily from a special case of Rota's Crosscut Theorem [14], proved by involutions in Section 3.

We now derive the Möbius function of the weak Bruhat order of a Coxeter group which is a result of Björner [2]. (We do not consider the strong ordering because it is not a lattice in general.) Any terminology from the theory of Coxeter groups not defined here can be found in Humphreys' book [10]. Let (W, S) be a finite Coxeter system so that W is a finite Coxeter group and S is a set of simple generators of W. The *length of* $w \in W$, $l(w)$, is the smallest l such that

$$w = s_1 s_2 \cdots s_l \tag{10}$$

where $s_i \in S$. If $v, w \in W$ then we write $v \geq w$ if there is an $s \in S$ with $v = ws$ and $l(v) = l(w)+1$. (It is easy to see that $l(v) = l(ws) = l(w) \pm 1$, cf. Lemma 3.3.) Extending this relation by transitive closure, we obtain the weak Bruhat poset P_W on W. Equivalently, this is the partial order obtained from the Cayley graph of W with respect to S by directing edges away from the identity element.

The atoms of P_W are just the elements of S. The $\hat{1}$ of P_W is the element of maximum length, $w_0 = \vee S$. If $J \subset S$ is any proper subset, then these elements generate a corresponding parabolic subgroup W_J which is a proper subgroup of W. So none of the elements $w_0(J) = \vee J$ is equal to w_0 and so $S = \mathcal{A}(W_P)$ is independent. Thus Corollary 2.1 applies and we have proved the following result.

Proposition 2.2 (Björner) *Let (W, S) be a Coxeter system and let P_W be the corresponding weak Bruhat order. Then for $w \in W$ we have*

$$\mu(w) = \begin{cases} (-1)^{|J|} & \textit{if } w = w_0(J) \textit{ for some } J \subseteq S, \\ 0 & \textit{else.} \end{cases} \qquad \blacksquare$$

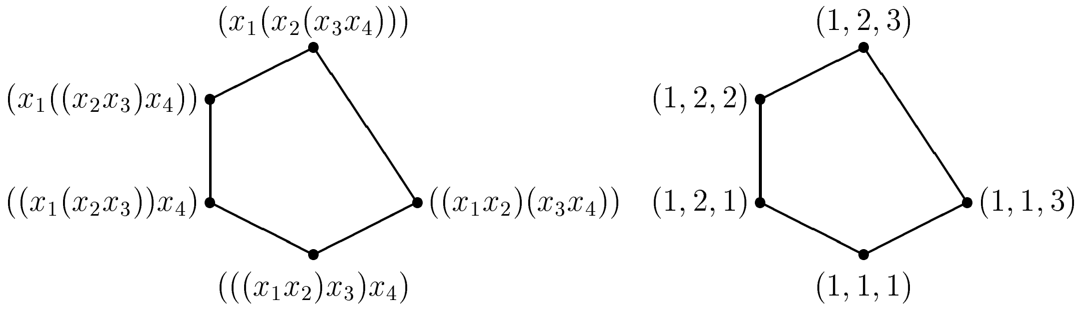
Björner actually derives the Möbius function from any interval $[v, w]$ in P_W. But this follows easily from the preceding proposition since there is a poset isomorphism $[v, w] \equiv [\hat{0}, v^{-1}w]$.

Next we consider the Tamari lattices [5, 7, 9]. Consider the set of all proper parenthesizations of the word $x_1 x_2 \ldots x_{n+1}$. It is well known that the number of such is the Catalan number $C_n = \frac{1}{n+1}\binom{2n}{n}$. Partially order this set by saying that π is covered by σ if

$$\pi = \ldots((AB)C)\ldots \qquad \text{and} \qquad \sigma = \ldots(A(BC))\ldots$$

for some subwords A, B, C. This poset is the *Tamari lattice* T_n and T_3 is illustrated in Figure 2 (a).

A *left bracket vector*, $(v_1, \ldots, v_n)$, is an integer vector satisfying the conditions



(a) Parenthesized version (b) Left bracket version

Figure 2: The Tamari lattice T_3

1. $1 \leq v_i \leq i$ for all i and
2. if $S_i = \{v_i, v_i + 1, \ldots, i\}$ then for any pair S_i, S_j either one set contains the other or $S_i \cap S_j = \emptyset$.

The number of left bracket vectors with n components is also C_n. In fact given any parenthesized word π we have an associated left bracket vector $v(\pi) = (v_1, \ldots, v_n)$ defined as follows. To calculate v_i, start at x_i in π and move to the left, counting the number of x's and the number of left parentheses you meet until these two numbers are equal. Then $v_i = j$ where x_j is the last x which is passed before the numbers balance. It is not hard to show that this gives a bijection between parenthesizations and left bracket vectors, thus inducing a partial order on the latter. This version of T_3 is shown in Figure 2 (b).

We will need the following theorem which is proved (in a dual version) in [9].

Theorem 2.3 (Huang and Tamari) *The poset T_n is a lattice. In fact, if $v(\pi) = (v_1, \ldots, v_n)$ and $v(\sigma) = (w_1, \ldots, w_n)$ then*

$$v(\pi \vee \sigma) = (\max\{v_1, w_1\}, \ldots, \max\{v_n, w_n\}). \blacksquare$$

We can now calculate the Möbius function of the Tamari lattice. This calculation has been done before by a number of different people. J. M. Pallo [13] derived the result by a method equivalent to ours. Paul Edelman [private communication] demonstrated that the Möbius function is always ± 1 by showing that the associated order complex has the homotopy type of a wedge of spheres. Finally Björner and Wachs [3] used their theory of nonpure shellings to get Edelman's result.

Proposition 2.4 *Let $\pi \in T_n$ have vector $v(\pi) = (v_1, \ldots, v_n)$. Then*

$$\mu(\pi) = \begin{cases} (-1)^t & \text{if } v_i \in \{1, i\} \text{ for all } i \\ 0 & \text{else} \end{cases}$$

where t is the number of $v_i = i \neq 1$. In particular

$$\mu(T_n) = (-1)^{n-1}.$$

Proof. Note that T_n has $n-1$ atoms $a_2, \ldots, a_n$ where $v(a_i)$ has $v_i = i$ and all other $v_j = 1$. From Theorem 2.3 we see that the atom set is independent. Thus Corollary 2.1 applies and the given formulae follow easily. ■

3 Crosscuts, Euler characteristics, and characteristic polynomials

We now present some proofs of related results using involutions. The following is a special case of Rota's Crosscut Theorem [14].

Theorem 3.1 (Rota) *If L is a finite lattice and $x \in L$ then define*

$$a_i(x) = \text{ number of sets of } i \text{ atoms whose join is } x.$$

We have

$$\mu(x) = a_0 - a_1 + a_2 - \cdots.$$

Proof. The proof follows the same lines as that of Theorem 1.2. In this case the set is

$$\mathcal{S} = \{A \ : \ A \subseteq \mathcal{A}(L) \ \text{ and } \ \vee A \leq x\}$$

with sign function

$$\epsilon(A) = (-1)^{|A|}.$$

Given any fixed atom $a \leq x$ we define the involution

$$\iota(A) = A \bigtriangleup a.$$

This is clearly well-defined and the proof follows. ■

It would be interesting to find of a proof of the Crosscut Theorem in its full generality using involutions. The stumbling block is that to apply this method one would need to have a crosscut C such that for every $x \in L$ not covering $\hat{0}$ we have

$C \cap [\hat{0}, x]$ is a crosscut of that interval. But this condition forces C to be the set of atoms.

The Möbius function of any partially ordered set L can be viewed as a reduced Euler characteristic. If $x \in L$ then a *chain of length i in the open interval* $(\hat{0}, x)$ is

$$c: \ x_0 < x_1 < \ldots < x_i$$

where $\hat{0} < x_j < x$ for all j. Let

$$c_i(x) = \text{number of chains of length } i \text{ in } (\hat{0}, x).$$

Note that if $x > \hat{0}$ then $c_{-1}(x) = 1$ because of the empty chain. Walker [16, Theorem 1.6] notes that the following result, usually known as Philip Hall's Theorem [8, 14], can be proved using involutions.

Theorem 3.2 (Hall) *If L is any partially ordered set with a $\hat{0}$ and $x \in L$, then*

$$\mu(x) = \begin{cases} 1 & \text{if } x = \hat{0}, \\ -c_{-1}(x) + c_0(x) - c_1(x) + \cdots & \text{else.} \end{cases} \tag{11}$$

Proof. Again, the proof follows the lines of Theorem 1.2. Let

$$\mathcal{S} = \{\hat{0}\} \cup \{(c, y) : \ c \text{ is a chain in } (\hat{0}, y),\ \hat{0} < y \leq x\}$$

with sign function

$$\epsilon(\hat{0}) = 1 \ \text{ and } \ \epsilon(c, y) = (-1)^{l(c)}$$

where $l(c)$ is the length of the chain. The involution ι is defined by

$$\hat{0} \overset{\iota}{\leftrightarrow} (\emptyset, x)$$

and for $(c, y) \in \mathcal{S} \setminus \{\hat{0}, (\emptyset, x)\}$ let

$$\iota(c, y) = \begin{cases} (c \setminus c', c') & \text{if } y = x, \\ (c < y, x) & \text{else.} \end{cases}$$

where c' is the largest element of c, and $c < y$ is the chain formed by adjoining y to c. The fact that this is a sign-reversing involution can now be used to show that the right side of (11) satisfies the same recursion as $\mu(x)$. ∎

If a geometric lattice comes from a hyperplane arrangement, even more can be said about its Möbius function. Any terms in the following discussion which are not defined can be found in the book of Orlik and Terao [11]. Let W be a finite Euclidean reflection group acting on a vector space V. Let $\mathcal{A}_W$ be the corresponding hyperplane arrangement with intersection lattice L_W, i.e, L_W is the set of all subspaces of V that can be obtained as intersections of hyperplanes in $\mathcal{A}_W$ ordered by *reverse* inclusion.

Define the *absolute length of* $w \in W$, $\hat{l}(w)$, to be the smallest l such that w can be written as

$$w = t_1 t_2 \cdots t_l \tag{12}$$

with the t_i coming from the set of all reflections $T \subseteq W$. This differs from the definition of ordinary length given in (10) in that one is not restricted to a set S of simple reflections. An expression of the form (12) will be called *absolutely reduced.* We will need the following result about absolute length.

Lemma 3.3 *Let W be a finite reflection group and consider $w \in W$. If $t \in W$ is any reflection then*

$$\hat{l}(wt) = \hat{l}(w) \pm 1.$$

Proof. If $w = t_1 t_2 \cdots t_k$ is an absolutely reduced expression then $wt = t_1 \cdots t_k t$ so that $\hat{l}(wt) \le \hat{l}(w) + 1$. Now replacing w by wt in the last inequality yields $\hat{l}(wt) \ge \hat{l}(w) - 1$. Finally, we cannot have $\hat{l}(wt) = \hat{l}(w)$ since $\det(wt) = -\det(w)$ and $\det(u) = (-1)^{\hat{l}(u)}$ for any $u \in W$. ■

For any element $w \in W$ let

$$V^w = \{v \in V \ : \ w(v) = v\}.$$

It follows from an easy-to-prove result of Carter [4] that if $V^w = X$ for some subspace $X \in L_W$ then $\hat{l}(w) = \operatorname{codim} X$. This makes the statement of the following theorem unambiguous.

Theorem 3.4 *Let W be a finite reflection group with corresponding intersection lattice L_W. Then for any $X \in L_W$ we have*

$$\mu(X) = (-1)^{\hat{l}}(\textit{number of } w \in W \textit{ with } V^w = X).$$

where $\hat{l} = \hat{l}(w)$ of some (any) w with $V^w = X$.

Proof. This proof was discovered by Victor Reiner [personal communication] using the ideas in our proof of Theorem 1.2. I thank him for letting me reproduce it here.

Let $\tilde{\mu}(X) = (-1)^{\hat{l}}(\text{number of } w \in W \text{ with } V^w = X)$. Then we must show that

$$\sum_{Y \le X} \tilde{\mu}(Y) = \delta_{\hat{0},X} \tag{13}$$

If $X = \hat{0} = V$ then both sides of (13) are clearly 1. If $X > \hat{0}$ then consider the set

$$W' = \{w \in W \ : \ V^w \supseteq X\}$$

with sign function

$$\epsilon(w) = (-1)^{\hat{l}(w)}.$$

Clearly the right side of (13) is given by $\sum_{w \in W'} \epsilon(w)$. But W' is just the stabilizer of X, and so is a non-trivial reflection group in its own right. Let t be any fixed reflection in W' and define an involution $\iota : W' \to W'$ by

$$\iota(w) = wt.$$

By Lemma 3.3 this is sign-reversing and so we are done. ∎

We should note that there is a direct connection between absolutely reduced expressions and NBC bases. Specifically, in [1] Barcelo, Goupil and Garsia show that if $H_1, \ldots, H_m$ is an NBC base of $\mathcal{A}_W$ then the corresponding product of reflections $r_{H_1} \cdots r_{H_m}$ is totally reduced and this gives a bijection between NBC bases and W.

We end by showing how Theorem 3.4 relates the characteristic polynomial of L_W to the Poincaré polynomial of W. The *characteristic polynomial* of L_W is the generating function for its Möbius function:

$$\chi(L_W, t) = \sum_{X \in L_W} \mu(X) t^{\dim X}.$$

The *Poincaré polynomial* of W is the generating function for its elements by absolute length:

$$\pi(W, t) = \sum_{w \in W} t^{\hat{l}(w)}.$$

Theorem 3.5 *Let W be a finite reflection group in V, $\dim V = n$, with corresponding intersection lattice L_W. Then*

$$\pi(W, t) = (-t)^n \chi(L_W, -1/t).$$

Proof. Using Theorem 3.4 and the lemma of Carter cited previously, we have the following series of equalities

$$\begin{aligned} (-t)^n \chi(L_W, -1/t) &= \sum_{X \in L_W} \mu(X)(-t)^{\operatorname{codim} X} \\ &= \sum_{w \in W} t^{\hat{l}(w)} \\ &= \pi(W, t). \ \blacksquare \end{aligned}$$

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