On divisibility of Narayana numbers by primes

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Abstract

Using Kummer's Theorem, we give a necessary and sufficient condition for a Narayana number to be divisible by a given prime. We use this to derive certain properties of the Narayana triangle.

1 The main theorem

Let $\mathbb{N}$ denote the nonnegative integers and let $k, n \in \mathbb{N}$. The *Narayana numbers* [10, A001263] can be defined as

$$N(n,k) = \frac{1}{n}\binom{n}{k}\binom{n}{k+1}$$

where $0 \le k < n$. The Narayana numbers (in fact, a q-analogue of them) were first studied by MacMahon [6, Article 495] and were later rediscovered by Narayana [7]. They are closely related to the *Catalan numbers* [10, A000108]

$$C_n = \frac{1}{n+1}\binom{2n}{n}$$

and in fact $\sum_k N(n,k) = C_n$. The Narayana numbers can be arranged in a triangular array with $N(n,k)$ in row n and column k so that the row sums are the Catalan numbers. Like the numbers C_n, the numbers $N(n,k)$ have many combinatorial interpretations; see, for example, the article of Sulanke [11].

The main result of this note is a characterization of when $N(n,k)$ is divisible by a given prime p. To state it, we need some notation. Let $\Delta_p(n) = (n_i)$ denote the sequence of digits of n in base p so that $n = \sum_i n_i p^i$. Similarly we define $\Delta_p(k) = (k_i)$. If we are considering $k \leq n$ then it will be convenient to extend the range of definition of (k_i) so that both sequences have the same length by setting $k_i = 0$ if $p^i > k$. The *order* of n modulo p is the largest power of p dividing n and will be denoted $\omega_p(n)$. As usual, $k|n$ means that k divides n.

Kummer's Theorem [5] gives a useful way of finding the order of binomial coefficients. For example, Knuth and Wilf [4] used it to find the highest power of a prime which divides a generalized binomial coefficient.

Theorem 1.1 (Kummer) *Let p be prime and let $\Delta_p(n) = (n_i)$, $\Delta_p(k) = (k_i)$. Then $\omega_p\binom{n}{k}$ is the number of carries in performing the addition $\Delta_p(k) + \Delta_p(n-k)$. Equivalently, it is the number of indices i such that either $k_i > n_i$ or there exists an index $j < i$ with $k_j > n_j$ and $k_{j+1} = n_{j+1}, \ldots, k_i = n_i$.* ■

Now everything is in place to state and prove our principal theorem.

Theorem 1.2 *Let p be prime. Also let $\Delta_p(n) = (n_i)$, $\Delta_p(k) = (k_i)$ and $\omega = \omega_p(n)$. Then $p \nmid N(n,k)$ if and only if one of the two following conditions hold:*

1. *When $p \nmid n$ we have*
   - *(a) $k_i \leq n_i$ for all i, and*
   - *(b) $k_j < n_j$ where j is the first index with $k_j \neq p-1$ (if such an index exists).*
2. *When $p \mid n$ we have*
   - *(a) $k_i \leq n_i$ for all $i > \omega$, and*
   - *(b) $k_\omega < n_\omega$, and*
   - *(c)* $k_0 = k_1 = \ldots = k_{\omega-1} = \begin{cases} 0 & \text{if } p \mid k; \\ p-1 & \text{if } p \nmid k. \end{cases}$

Proof First suppose that p is not a divisor of n. Then p does not divide $N(n,k)$ if and only if p divides neither $\binom{n}{k}$ nor $\binom{n}{k+1}$. By Kummer's Theorem this is equivalent to $k_i \leq n_i$ and $(k+1)_i \leq n_i$ for all i. However, if j is the first index with $k_j \neq p-1$, then we have

$$(k+1)_i = \begin{cases} 0 & \text{if } i < j; \\ (k)_i + 1 & \text{if } i = j; \\ (k)_i & \text{if } i > j. \end{cases}$$

So these conditions can be distilled down to insisting that $k_j < n_j$ in addition to $k_i \leq n_i$ for all other i.

Now consider what happens when p divides n. Suppose first that p also divides k. So $(n)_i = 0$ for $i < \omega$, which is a nonempty set of indices, and $(k+1)_0 = 1$. It follows there are at least ω carries in computing $\Delta_p(k+1) + \Delta_p(n-k-1)$. By Kummer's Theorem again, $\omega_p\binom{n}{k+1} \geq \omega$. So p does not divide $N(n,k)$ if and only if it does not divide $\binom{n}{k}$ and $\omega_p\binom{n}{k+1} = \omega$. Applying Kummer's theorem once more shows that this will happen exactly when $k_i \leq n_i$ for all i with $k_\omega < n_\omega$. So in particular $k_i = 0$ for $i < \omega$ since then $n_i = 0$. This completes the case when p divides both n and k.

Finally, suppose $p \mid n$ but $p \nmid k$. Arguing as in the previous paragraph, we see that p is not a divisor of $N(n,k)$ if and only if $\omega_p\binom{n}{k} = \omega$ and p does not divide $\binom{n}{k+1}$. But if p is not a divisor of $\binom{n}{k+1}$ then, using Kummer's theorem, we must have $(k+1)_i = 0$ for $i < \omega$. So $(k)_i = p-1$ for $i < \omega$. Conditions 2(b) and (c) also follow as before. This completes the demonstration of the theorem. ■

2 Applications

It is well known that C_n is odd if and only if $n = 2^m - 1$ for some m. For a combinatorial proof of this which in fact establishes $\omega_2(C_n)$, see the article of Deutsch and Sagan [2]. Analogously, all the entries of the nth row of the Narayana triangle are odd. This is a special case of the following result.

Corollary 2.1 *Let p be prime and let $n = p^m - 1$ for some $m \in \mathbb{N}$. Then for all k, $0 \leq k \leq n-1$, we have $p \nmid N(n,k)$.*

Proof By Theorem 1.2 we just need to verify that 1(a) and (b) hold for all k. However, they must be true because $n_i = p-1$ for all i. ■

We clearly can not have a row of the Narayana triangle where every element is divisible by p since $N(n,0) = N(n,n-1) = 1$ for all n. But we can ensure that every entry except the first and last is a multiple of p.

Corollary 2.2 *Let p be prime and let $n = p^m$ for some $m \in \mathbb{N}$. Then $p \mid N(n,k)$ for $1 \leq k \leq n-2$.*

Proof Suppose that $n = p^m$ and that p does not divide $N(n,k)$. If p divides k, then condition 2(c) forces $k = 0$. If p does not divide k, then the same condition forces $k = n-1$. So these are the only two numbers not divisible by p in the nth row of Narayana's triangle. ■

3 Comments and Questions

I. Clearly one could use the same techniques presented here to determine $\omega_p(N(n,k))$. However, the cases become complicated enough that it is unclear whether this would be an interesting thing to do.

II. The characterization in Theorem 1.2 is involved enough that it may be hopeless to ask for a combinatorial proof. However, there should be a combinatorial way to derive the simpler statements in Corollaries 2.1 and 2.2, although we have not been able to do so.

As has already been mentioned, the order $\omega_2(C_n)$ can be established by combinatorial means, specifically through the use of group actions. Unfortunately, the action used by Deutsch and Sagan [2] is not sufficiently refined to preserve the objects counted by $N(n,k)$. For more information about how such methods can be used to prove congruences, the reader can consult Sagan's article [8] which also contains a survey of the literature.

Deutsch [1], Eğecioğlu [3], and Simion and Ullman [9] have all found combinatorial ways to explain the fact that C_n is odd if and only if $n = 2^m - 1$ for some m. Perhaps one or more of the viewpoints in these papers could be adapted to the Narayana numbers.

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