2.3 PARTITIONS

INTRODUCTION

[This should be placed after Andrews' part of the introduction] Young tableaux were introduced by the Reverend Alfred Young [1873–1940]. They are of interest in combinatorics, and the theories of symmetric functions, group representations, and invariants. Here we concentrate on the first two areas. The reader can find information about the others in the references.

GLOSSARY

[These definitions should be alphabetically merged with Andrews']

hook: the set of cells in a Ferrers' diagram directly to the right or directly below a given cell.

hooklength: the number of cells in a Ferrers' diagram directly to the right or directly below a given cell.

symmetrtic function: a polynomial in variables $x_1, \ldots, x_n$ which is invariant under the action of the symmetric group S_n. (See 5.2.2.)

Schur function: the symmetric fucntion which is the generating function for all semistandard tableaux of a given shape.

Semistandard Young tableau: a Young tableau with the rows weakly increasing and the columns increasing.

Standard Young tableau: a semistandard Young tableau with the cells in bijection with the integers $1, \ldots, n$ (= number of cells).

Young tableau: an arrary obtained by replacing each cell of a Ferrers diagram by a positive integer.

[Andrews' sections skipped here]

2.3.1 Stirling coefficients

2.3.2 Stirling coefficient identities

2.3.3 Partitions of integers, Ferrers diagrams

2.3.4 YOUNG TABLEAUX

Definitions:

A **Young tableau (YT) of shape** λ is an arrary, T, obtained by replacing each cell of the Ferrers diagram of λ by a positive integer (see 2.3.3.).

A Young tableau is **semistandard** (an SSYT) if the rows weakly increase and the columns strictly increase.

A semistandard Young tableau of shape $\lambda \vdash n$ is **standard** (an SYT) if its entries are exactly $1, \ldots, n$. The number of SYT of shape λ is denoted f_λ.

Let $\mathbb{R}[\mathbf{x}]$ be the real polynomial ring in the variables $\mathbf{x} = \{x_1, \ldots, x_n\}$ (see 5.4.6). Then $f \in \mathbb{R}[\mathbf{x}]$ is **symmetric** if $\pi f = f$ for all $\pi \in S_n$, the symmetric group (see 5.2.2) where $\pi f(x_1, \ldots, x_n) = f(x_{\pi(1)}, \ldots, x_{\pi(n)})$.

The **elementary symmetric function** of degree k in n variables is

$$e_k(\mathbf{x}) = e_k(x_1, \ldots, x_n) = \sum_{1 \le i_1 < \ldots < i_k \le n} x_{i_1} \cdots x_{i_n}.$$

And for a partition $\lambda = (\lambda_1, \ldots, \lambda_l)$ we let $e_\lambda(\mathbf{x}) = e_{\lambda_1}(\mathbf{x}) \cdots e_{\lambda_l}(\mathbf{x})$.

The **complete homogeneous symmetric function** of degree k in n variables is

$$h_k(\mathbf{x}) = e_k(x_1, \ldots, x_n) = \sum_{1 \le i_1 \le \ldots \le i_k \le n} x_{i_1} \cdots x_{i_n}.$$

And for a partition $\lambda = (\lambda_1, \ldots, \lambda_l)$ we let $h_\lambda(\mathbf{x}) = h_{\lambda_1}(\mathbf{x}) \cdots h_{\lambda_l}(\mathbf{x})$.

A Young tableau T has corresponding **monomial** $\mathbf{x}^T = \prod_{(i,j) \in \lambda} x_{T_{i,j}}$.

Given partition $\lambda = (\lambda_1, \ldots, \lambda_l)$ and $n \ge l$, the associated **Schur function** is

$$s_\lambda(x_1, \ldots, x_n) = \sum_T \mathbf{x}^T$$

where the sum is over all SSYT T of shape λ with entries at most n.

Facts:

1. The coefficient of $x_1 x_2 \cdots x_n$ in $s_\lambda(x_1, \ldots, x_n)$ is f_λ.
2. If $\lambda = (k)$ (one row) then $s_\lambda(\mathbf{x}) = h_k(\mathbf{x})$. If $\lambda = (1^k)$ (one column) then $s_\lambda(\mathbf{x}) = e_k(\mathbf{x})$.
3. The generating functions (see 3.1) for the $e_k(\mathbf{x})$ and $h_k(\mathbf{x})$ are

$$\begin{aligned} \sum_{k \ge 0} e_k(\mathbf{x}) t^k &= \prod_{i \ge 1} (1 + x_i t) \\ \sum_{k \ge 0} h_k(\mathbf{x}) t^k &= \prod_{i \ge 1} \frac{1}{1 - x_i t}. \end{aligned}$$

4. The recurrence relations (see 3.2) for the $e_k(\mathbf{x})$ and $h_k(\mathbf{x})$ are

$$\begin{aligned}
e_k(x_1,\ldots,x_n) &= e_k(x_1,\ldots,x_{n-1}) + x_n e_{k-1}(x_1,\ldots,x_{n-1}) \\
h_k(x_1,\ldots,x_n) &= h_k(x_1,\ldots,x_{n-1}) + x_n h_{k-1}(x_1,\ldots,x_n).
\end{aligned}$$

5. Specializations of the the $e_k(\mathbf{x})$ and $h_k(\mathbf{x})$ give binomial coefficients (see 2.2.2) and Stirling numbers (see 2.3.1)

$$\begin{aligned}
\binom{n}{k} &= e_k(\overbrace{1,\ldots,1}^{n}) \\
&= h_k(\overbrace{1,\ldots,1}^{n-k+1}) \\
|s(n,k)| &= e_{n-k}(1,2,\ldots,n-1) \\
S(n,k) &= h_{n-k}(1,2,\ldots,k).
\end{aligned}$$

6. If $\lambda \vdash n$ then the Schur function $s_\lambda(\mathbf{x})$ is the cycle index (see 2.4.1) for the characters of the irreducible representation of the symmetric group S_n corresponding to λ. Also f_λ is the degree of this representation.

7. The symmetric functions in $\mathbf{x} = \{x_1,\ldots,x_n\}$ form a graded algebra Λ_n (see 5.4.5) whose homogeneous piece of degree k is denoted Λ_n^k.

8. The polynomials $e_\lambda(\mathbf{x}), h_\lambda(\mathbf{x}), s_\lambda(\mathbf{x})$ are all symmetric. As λ runs over all partitions of k each of the three families runs over a basis for Λ_n^k which thus has dimension $p(k)$ = number of partitions of k (see 2.3.3).

Examples:

1. If $\lambda = (3,2)$ then a complete list of SYT is

$$\begin{matrix}1&2&3\\4&5&\end{matrix}\;,\quad \begin{matrix}1&2&4\\3&5&\end{matrix}\;,\quad \begin{matrix}1&2&5\\3&4&\end{matrix}\;,\quad \begin{matrix}1&3&4\\2&5&\end{matrix}\;,\quad \begin{matrix}1&3&5\\2&4&\end{matrix}$$

2. If $\lambda = (2,2)$ then a complete list of SSYT with entries at most 3 is

$$\begin{matrix}1&1\\2&2\end{matrix}\,,\quad \begin{matrix}1&1\\3&3\end{matrix}\,,\quad \begin{matrix}2&2\\3&3\end{matrix}\,,\quad \begin{matrix}1&1\\2&3\end{matrix}\,,\quad \begin{matrix}1&2\\2&3\end{matrix}\,,\quad \begin{matrix}1&2\\3&3\end{matrix}$$

The last tableau has monomial $\mathbf{x}^T = x_1x_2x_3x_3 = x_1x_2x_3^2$. The Schur function for this list is

$$s_{(2,2)}(x_1,x_2,x_3) = x_1^2x_2^2 + x_1^2x_3^2 + x_2^2x_3^3 + +x_1^2x_2x_3 + x_1x_2^2x_3 + x_1x_2x_3^2.$$

3. As examples of elementary and complete homogeneous symmetric functions:

$$\begin{aligned}
e_2(x_1,x_2,x_3) &= x_1x_2 + x_1x_3 + x_2x_3 \\
e_{(2,1)}(x_1,x_2,x_3) &= (x_1x_2 + x_1x_3 + x_2x_3)(x_1 + x_2 + x_3) \\
h_2(x_1,x_2,x_3) &= x_1x_2 + x_1x_3 + x_2x_3 + x_1^2 + x_2^2 + x_3^2 \\
h_{(2,1)}(x_1,x_2,x_3) &= (x_1x_2 + x_1x_3 + x_2x_3 + x_1^2 + x_2^2 + x_3^2)(x_1 + x_2 + x_3).
\end{aligned}$$

2.3.5 TABLEAUX IDENTITIES

Definitions:

The **hook** of cell (i,j) in the Ferrers diagram for a partition λ is

$$H_{i,j} = \{(k,j) \in \lambda :\ k \geq i\} \cup \{(i,k) \in \lambda :\ k \geq j\}.$$

The **hooklength** of cell (i,j) is the number of cells in its hook, i.e.,

$$h_{i,j} = |H_{i,j}|.$$

The **content** of cell (i,j) is

$$c_{i,j} = j - i.$$

The **minimum weight** of a partition λ is

$$m(\lambda) = \sum_{i \geq 1} i\lambda_i.$$

This is the smallest possible sum of the entries of an SSYT of shape λ.

The if q is a variable then the **principal specialization** of a symmetric function is obtained by letting $x_i = q^i$ for all i.

If G is a group (see 5.2) then an **involution** is $g \in G$ such that g^2 is the identity. Let $\mathrm{inv}(n)$ be the number of involutions in the symmetric group S_n.

Facts:

1. *Frame-Robinson-Thrall Hook Formula* [1954] The number of SYT of fixed shape λ is

$$f_\lambda = \frac{n!}{\prod_{(i,j)\in\lambda} h_{i,j}}.$$

2. *Frobenius Determinantal Formula* [1900] The number of SYT of fixed shape $\lambda = (\lambda_1, \ldots, \lambda_l)$ is the determinant

$$f_\lambda = n! \left| \frac{1}{(\lambda_i + j - i)!} \right|_{1 \leq i,j \leq l}.$$

3. The principal specialization of $s_\lambda(\mathbf{x})$ is

$$s_\lambda(q, q^2, \ldots, q^n) = q^{m(\lambda)} \prod_{(i,j)\in\lambda} \frac{1 - q^{c_{i,j}+n}}{1 - q^{h_{i,j}}}.$$

This can be use to derive the hook formula.

4. *Jacobi-Trudi Determinants* [1841,1864] The Schur function of $\lambda = (\lambda_1, \ldots, \lambda_l)$ is the determinant

$$s_\lambda(\mathbf{x}) = |h_{\lambda_i + j - i}|_{1 \le i,j \le l}\,.$$

This can be used to prove the determinantal formula for f_λ. There is also a dual form, letting the conjugate of λ (see 2.3.3) be $\lambda' = (\lambda'_1, \ldots, \lambda'_m)$

$$s_\lambda(\mathbf{x}) = \left|e_{\lambda'_i + j - i}\right|_{1 \le i,j \le m}\,.$$

5. *Jacobi Alternant Quotient* [1841] The Schur function in $\mathbf{x} = \{x_1, \ldots, x_n\}$ is the determinant quotient

$$s_\lambda(\mathbf{x}) = \frac{\left|x_i^{\lambda_j + n - j}\right|_{1 \le i,j \le n}}{\left|x_i^{n-j}\right|_{1 \le i,j \le n}}.$$

6. We have the following summations involving the number of SYT:

$$\begin{aligned} \sum_{\lambda \vdash n} f_\lambda &= \mathrm{inv}(n) \\ \sum_{\lambda \vdash n} f_\lambda^2 &= n! \end{aligned}$$

7. If $\mathbf{x} = \{x_1, \ldots, x_n\}$ and $\mathbf{y} = \{y_1, \ldots, y_n\}$ then

$$\begin{aligned} \sum_{\lambda} s_\lambda(\mathbf{x}) &= \prod_{1 \le i \le n} \frac{1}{1 - x_i} \prod_{1 \le i < j \le n} \frac{1}{1 - x_i x_j} \\ \sum_{\lambda} s_\lambda(\mathbf{x}) s_\lambda(\mathbf{y}) &= \prod_{1 \le i,j \le n} \frac{1}{1 - x_i y_j} \qquad \text{[Littlewood, 1939]} \\ \sum_{\lambda} s_\lambda(\mathbf{x}) s_{\lambda'}(\mathbf{y}) &= \prod_{1 \le i,j \le n} (1 + x_i y_j) \qquad \text{[Littlewood, 1939]} \end{aligned}$$

These identities can be used to prove those in the previous example.

Examples:

1. For the partition $(3,2)$ we have $H_{1,1} = \{(1,1),\ (2,1),\ (1,2),\ (1,3)\}$. In the following digram each cell of $(3,2)$ is replaced with its hooklength.

$$\begin{matrix} 4 & 3 & 1 \\ 2 & 1 & \end{matrix}$$

The hook formula gives the number of SYT of shape $(3,2)$ (cf. 2.3.4, Example 1) to be

$$f_{(3,2)} = \frac{5!}{4 \cdot 3 \cdot 2 \cdot 1^2} = 5.$$

The determinantal formula gives the same result

$$f_{(3,2)} = 5! \begin{vmatrix} 1/3! & 1/4! \\ 1/1! & 1/2! \end{vmatrix} = 5.$$

2. In the following diagram each cell of $(2,2)$ is replaced by its content

$$\begin{matrix} 0 & 1 \\ -1 & 0 \end{matrix}$$

The product formula for the principle specialization with shape $(2,2)$ (cf. 2.3.4, Example 2) gives

$$s_{(2,2)}(q, q^2, q^3) = q^6 \frac{(1-q^3)(1-q^4)(1-q^2)(1-q^3)}{(1-q^3)(1-q^2)(1-q^2)(1-q^1)} = q^6 + q^7 + 2q^8 + q^9 + q^{10}.$$

The Jacobi-Trudi determinant yields

$$s_{(2,2)}(\mathbf{x}) = \begin{vmatrix} h_2(\mathbf{x}) & h_3(\mathbf{x}) \\ h_1(\mathbf{x}) & h_2(\mathbf{x}) \end{vmatrix} = h_2^2(\mathbf{x}) - h_3(\mathbf{x})h_1(\mathbf{x})$$

Since $(2,2)' = (2,2)$ (self-dual) we also have

$$s_{(2,2)}(\mathbf{x}) = \begin{vmatrix} e_2(\mathbf{x}) & e_3(\mathbf{x}) \\ e_1(\mathbf{x}) & e_2(\mathbf{x}) \end{vmatrix} = e_2^2(\mathbf{x}) - e_3(\mathbf{x})e_1(\mathbf{x}).$$

If $\mathbf{x} = \{x_1, x_2, x_3\}$ then as a quotient of alternants we have

$$s_{(2,2)}(\mathbf{x}) = \frac{\begin{vmatrix} x_1^4 & x_1^3 & 1 \\ x_2^4 & x_2^3 & 1 \\ x_3^4 & x_3^3 & 1 \end{vmatrix}}{\begin{vmatrix} x_1^2 & x_1 & 1 \\ x_2^2 & x_2 & 1 \\ x_3^2 & x_3 & 1 \end{vmatrix}}.$$

3. For the partitions of $n = 3$ we have

$$f_{(3)} = 1, \quad f_{(2,1)} = 2, \quad f_{(1,1,1)} = 1$$

so our summation formulas become

$$\begin{array}{lcccl} \sum_{\lambda \vdash 3} f_\lambda & = & 4 & = & \mathrm{inv}(3) \\ \sum_{\lambda \vdash 3} f_\lambda^2 & = & 6 & = & 3! \end{array}$$

If $\mathbf{x} = \{x_1, x_2\}$ and $\mathbf{y} = \{y_1, y_2\}$ then

$$\begin{aligned} \sum_\lambda s_\lambda(\mathbf{x}) &= \frac{1}{(1-x_1)(1-x_2)(1-x_1x_2)} \\ \sum_\lambda s_\lambda(\mathbf{x})s_\lambda(\mathbf{y}) &= \frac{1}{(1-x_1y_1)(1-x_1y_2)(1-x_2y_1)(1-x_2y_2)} \\ \sum_\lambda s_\lambda(\mathbf{x})s_{\lambda'}(\mathbf{y}) &= (1+x_1y_1)(1+x_1y_2)(1+x_2y_1)(1+x_2y_2) \end{aligned}$$

2.3.6 TABLEAUX ALGORITHMS

Definitions:

An **inner corner** of the partition λ is $(i,j) \in \lambda$ such that $(i+1,j),(i,j+1) \notin \lambda$.

An **outer corner** of the partition λ is $(i,j) \notin \lambda$ such that $(i-1,j),(i,j-1) \in \lambda$.

The Greene-Nijenhuis-Wilf [1979] Algorithm: This algorithm produces a SYT of given shape $\lambda \vdash n$ uniformly at random. It can also be used to prove the hook formula (see 2.3.5, Fact 1). One takes a random walk along hooks of λ until one gets to an inner corner (i,j). This cell is labeled n and then the process is repeated on $\lambda \setminus (i,j)$ to find a cell to label $n-1$, etc., until all cells are labeled.

ALGORITHM: **The Greene-Nijenhuis-Wilf Algorithm**

{*Initialize*} Pick cell $(i,j) \in \lambda$ with probability $1/n$.
{*Find a corner*} While (i,j) is not an inner corner: pick $(i',j') \in H_{i,j} \setminus (i,j)$ with probability $1/(h_{i,j}-1)$ and let $(i,j) := (i',j')$.
{*Now (i,j) is an inner corner*} Label (i,j) with n.
{*Update and iterate*} Let $\lambda = \lambda \setminus (i,j), n := n-1$ and return to Initialize if $n > 0$ or end if $n = 0$.

The Robinson-Schensted [1938,1961] Algorithm: This algorithm proves the second summation formula in 2.3.5, Fact 6, by giving a map from permutations $\pi = p_1 \dots p_n \in S_n$ to pairs (P,Q) of SYT of the same shape which is a bijection. The p_k are inserted sequentially into P using a bumping process where p_k displaces an entry of the first row, which then displaces an entry of the second, etc. until some entry comes to rest by adding a cell at the end of a row. The entry k is then put in Q in the same place as the new cell in P.

ALGORITHM: **The Robinson-Schensted Algorithm**

{*Initialize*} Let $P = Q = \emptyset$, $k := 1$, $p := p_1$, $i := 1$.
{*Bumping in P*} While there is an entry of row i of P greater than p let $P_{i,j}$ be the smallest such entry and exchange p and $P_{i,j}$. Let $i := i+1$ and iterate.
{*Now $p >$ all of row i*} Row i will have an outer corner (i,j) and let $P_{i,j} := p$.
{*Modify Q*} Let $Q_{i,j} := k$.
{*Update and iterate*} If $k < n$ then let $k := k+1$, $p := p_k$, $i := 1$ and return to Bumping in P, else end if $k = n$.

Facts:

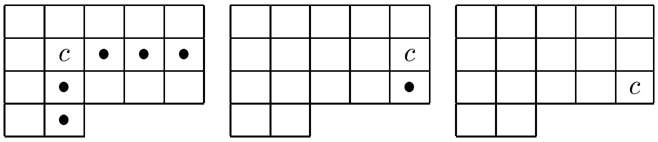
1. The Robinson-Schensted algorithm can also be used to prove the first summation formula in 2.3.5, Fact 6, by showing that if π maps to (P,Q) then π^{-1} maps to (Q,P) [Schützenberger 1963].

2. Knuth [1970] has generalized this algorithm to prove the Littlewood's identities (see 2.3.5, Fact 7).

Examples:

1. Here is an example of the *Find a corner* loop of the Greene-Nijenhuis-Wilf algorithm with $\lambda = (5, 5, 5, 2)$, $n = 17$. At each stage the current choice of cell $c = (i, j)$ is displayed along with dots in its hook where the next cell must be chosen.

$c = (i, j) \in \lambda$:



probability of c : 1/17 1/5 1/1

2. Here is an example of the full Robinson-Schensted algorithm for the permutation $\pi = 6, 2, 3, 1, 7, 5, 4$.

$$\begin{array}{lllllllll}
p_k: & & 6, & 2, & 3, & 1, & 7, & 5, & 4, \\[1em]
P: & \emptyset, & 6, & \begin{array}{l}2,\\6\end{array} & \begin{array}{ll}2&3,\\6&\end{array} & \begin{array}{ll}1&3,\\2&\\6&\end{array} & \begin{array}{lll}1&3&7,\\2&&\\6&&\end{array} & \begin{array}{lll}1&3&5,\\2&7&\\6&&\end{array} & \begin{array}{lll}1&3&4.\\2&5&\\6&7&\end{array} \\[2em]
k: & & 1, & 2, & 3, & 4, & 5, & 6, & 7, \\[1em]
Q: & \emptyset, & 1, & \begin{array}{l}1,\\2\end{array} & \begin{array}{ll}1&3,\\2&\end{array} & \begin{array}{ll}1&3,\\2&\\4&\end{array} & \begin{array}{lll}1&3&5,\\2&&\\4&&\end{array} & \begin{array}{lll}1&3&5,\\2&6&\\4&&\end{array} & \begin{array}{lll}1&3&5.\\2&6&\\4&7&\end{array}
\end{array}$$

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