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Combinatorial proofs of hook generating functions for skew plane partitions

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Abstract

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We provide combinatorial proofs of two hook generating functions for skew plane partitions. One proof involves the Hillman–Grassl (1976) algorithm and the other uses a modification of Schützenberger's (1963, 1977) "jeu de taquin" due to Kadell (to appear). We also provide a bijection showing directly that these two generating functions are equal. Analogous results for skew shifted plane partitions are proved. Finally, some open questions are discussed.

1. Preliminaries

Stanley [9] was the first to derive the hook generating function for reverse plane partitions and a combinatorial proof of this result was given by Hillman and Grassl [2]. In an earlier paper [5] we showed how their algorithm could be generalized to give bijective proofs of other generating functions for partially ordered sets with hooklengths. It turns out that there are two hook-generating functions for skew plane partitions, also first derived algebraically by Stanley [10]. We will show that one can be proved using Hillman–Grassl and the other by a modified version of the Schützenberger "jeu de taquin" [7, 8] created by Kadell [3]. We also give a bijection which shows directly that these two product generating functions are equal. These proofs will be found in Section 2.

Similarly, shifted reverse plane partitions are enumerated by a hook generating function, as was first proved by Gansner [1]. We show that shifted plane partitions

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also have a pair of generating functions and use analogous techniques to derive the associated bijections; see Section 3. The shifted results as well as their proofs are new. The last section contains some open questions.

Many of these proofs have been discovered independently by Kevin Kadell (private communication). We appreciate his permission to include them here. First, however, we must give some definitions and notation.

Consider the plane

$$\Lambda = \{(i,j) \mid i,j \geqslant 1\}$$

viewed as an infinite array of boxes or cells arranged matrix-style in left-justified rows. Let $\lambda = (\lambda_1, \ldots, \lambda_l)$ be a fixed partition considered as a Ferrers diagram sitting in the upper-left corner of Λ. This gives rise to the skew shape

$$\Lambda/\lambda = \{(i,j) \mid (i,j) \in \Lambda,\ (i,j) \notin \lambda\}.$$

A *skew plane partition of shape* Λ/λ is a filling, P, of Λ/λ with nonnegative integers called *parts* such that rows and columns decrease weakly. For example, if $\lambda = (3,1)$ then one such skew plane partition (0 parts omitted) is

$$P = \begin{matrix} \blacksquare & \blacksquare & \blacksquare & 4 & 4 \\ \blacksquare & 3 & 3 & 2 & \\ 4 & 3 & 3 & 1 & \end{matrix}$$

If $P_{i,j}$ denotes the part of P in cell (i,j), then we say that P is a *skew plane partition of* n if $\sum_{(i,j)\in\Lambda/\lambda} P_{i,j} = n$. Our example is a skew plane partition of $4+4+3+3+2+4+3+3+1=27$. Let

$$pp_{\Lambda/\lambda}(n) = \text{number of plane partitions of } n \text{ having shape } \Lambda/\lambda.$$

We will be interested in product forms for the generating function of $pp_{\Lambda/\lambda}(n)$. For this, we need to define two types of hooks.

If $(i,j)\in\lambda$ then this cell has the usual hook of all cells directly to the right or directly below:

$$H_{i,j} = \{(i,j')\in\lambda \mid j' \geqslant j\} \cup \{(i',j)\in\lambda \mid i' \geqslant i\}.$$

If, instead, $(i,j)\in\Lambda/\lambda$ then we take the reflection of a normal hook in an anti-diagonal $i+j=\text{constant}$, i.e., using those cells to the left or above (i,j):

$$H_{i,j} = \{(i,j')\in\Lambda/\lambda \mid j' \leqslant j\} \cup \{(i',j)\in\Lambda/\lambda \mid i' \leqslant i\}.$$

In either case, the *hooklength* of cell (i, j) is $h_{i,j} = |H_{i,j}|$, where $|\cdot|$ denotes the cardinality. For example, if $\lambda = (4, 4, 3, 1)$ then the cells in the hooks of $(2, 2) \in \lambda$ are shown as circles in

■ ■ ■ ■

■ ● ● ●

■ ● ■

■

while those of $(4, 6) \notin \lambda$ are the circles in

■ ■ ■ ■ □ ○ □ ⋯

■ ■ ■ ■ □ ○ □ ⋯

■ ■ ■ □ □ ○ □ ⋯

■ ○ ○ ○ ○ ○ □ ⋯

□ □ □ □ □ □ □ ⋯

⋮ ⋮ ⋮ ⋮ ⋮ ⋮ ⋮ ⋮

Thus, $h_{2,2} = 4$ and $h_{4,6} = 8$.

2. Plane partitions

We will give combinatorial proofs of the two product formulae for the generating function for skew plane partitions. We will also show by a direct bijection that the two products are equal.

Theorem 2.1. *If λ is a fixed shape, then*

$$\sum_{n \geqslant 0} pp_{\Lambda/\lambda}(n) x^n = \prod_{(i,j) \in \Lambda/\lambda} \frac{1}{1 - x^{h_{i,j}}} \tag{1}$$

$$= \prod_{k \geqslant 1} \frac{1}{(1 - x^k)^k} \prod_{(i,j) \in \lambda} \frac{1}{1 - x^{h_{i,j}}}. \tag{2}$$

Proof. (1): We merely use a reflection of the normal Hillman–Grassl algorithm in an anti-diagonal. (This corresponds to the fact that the associated algebraic proof derives (1) as a limiting case of the ordinary hook generating function for reverse plane partitions.) Details of this approach have already appeared in [5] for the case $\lambda = \emptyset$, and the general case is virtually the same, so here we will only sketch the proof for completeness.

It suffices to find a bijection

$$P \leftrightarrow \kappa = (h_{i_1, j_1}, h_{i_2, j_2}, \ldots),$$

where P is a plane partition of shape Λ/λ and κ is a partition all of whose parts are hooklengths of Λ/λ such that

$$\sum_{(i,j)\in P} P_{i,j} = \sum_k h_{i_k, j_k}.$$

We will define a path p in P and then subtract one from every part on the path. The definition of p is as follows:

HG1. Start p at (a,b), the rightmost highest cell of P containing a nonzero entry.

HG2. Continue by iterating

$$(i,j)\in p \;\Rightarrow\; \begin{cases} (i+1,j)\in p & P_{i+1,j}=P_{i,j}, \\ (i,j-1)\in p & \text{otherwise.} \end{cases}$$

In other words, move left unless forced to move down in order not to violate the weakly decreasing condition along the rows (once the ones are subtracted).

HG3. Terminate p when the preceding induction rule fails. At this point we must be at the left end of some row, say row r.

It is easy to see that after subtracting one from the elements in p, the array remains a plane partition and the amount subtracted is $h_{r,b}$.

For example, the following diagram shows an array P with the cells of the path p enclosed in boxes, as well as the resulting plane partition P' after subtraction:

$$P = \begin{array}{cccccc} \blacksquare & \blacksquare & \blacksquare & \blacksquare & 3 & \boxed{3} \\ \blacksquare & \blacksquare & \blacksquare & \blacksquare & \boxed{3} & \boxed{3} \\ \blacksquare & \blacksquare & \blacksquare & \boxed{4} & \boxed{3} & 2 \\ \blacksquare & \boxed{5} & \boxed{4} & \boxed{4} & 2 & \\ 5 & 4 & 3 & 3 & 2 & \end{array} \;\rightarrow\; P' = \begin{array}{cccccc} \blacksquare & \blacksquare & \blacksquare & \blacksquare & 3 & 2 \\ \blacksquare & \blacksquare & \blacksquare & \blacksquare & 2 & 2 \\ \blacksquare & \blacksquare & \blacksquare & 3 & 2 & 2 \\ \blacksquare & 4 & 3 & 3 & 2 & \\ 5 & 4 & 3 & 3 & 2 & \end{array}$$

In this case $(a,b)=(1,6)$ and $r=4$; so, the number of ones subtracted is $h_{4,6}=8$. Make $h_{r,b}$ the first part of κ and continue the process by finding a path in P', subtracting ones to find the second part of κ, etc. The algorithm terminates when every entry of P has been zeroed out.

To reverse the process, given a partition of hooklengths, we must rebuild the plane partition. First, however, we must know in what order the hooklengths were removed. The following lemma, whose proof is omitted, answers that question.

Lemma 2.2. *In the decomposition of P into hooklengths, $h_{i,j}$ was removed before $h_{i',j'}$, if and only if*

$$j>j', \text{ or } j=j' \text{ and } i\leqslant i'.$$

Proof of Theorem 2.1 (*continued*). Now arrange the hooklengths in κ according to the total order given in the lemma and start adding them back, starting with the last hooklength and the plane partition of all zeros. In general, to add $h_{r,b}$ to P, we construct a reverse path q along which to add ones:

GH1. Start q at the leftmost cell in row r
GH2. Continue by

$$(i,j)\in q \;\Rightarrow\; \begin{cases} (i-1,j)\in q & \text{if } P_{i-1,j}=P_{i,j}, \\ (i,j+1)\in q & \text{otherwise.} \end{cases}$$

GH3. Terminate q when it passes through the highest cell of Λ/λ in column b.

This is a step-by-step inverse of the construction of the path p, as can be verified in the previous example. Thus, to finish the proof it suffices to show that q is well defined – i.e., that it must pass through the highest cell in column b. We leave this verification to the reader.

(2): First we must describe the modified version of "jeu de taquin" that we will need. Pick any cell $c=(i,j)\in\lambda$ which is at the end of its row and column. If P is a plane partition of shape Λ/λ, then we can perform a *backward jeu de taquin slide* into cell c using the following algorithm:

(B1) **While** $P_{i,j+1}\neq 0$ **or** $P_{i+1,j}\neq 0$ **do**
  (B2) **if** $P_{i,j+1}\geqslant P_{i+1,j}$ **then** slide $P_{i,j+1}$ into cell c
    else slide $P_{i+1,j}-1$ into cell c. **fi**
  (B3) Let $c:=$ the cell of the element that slid in step B2. **od**

Of course, the coordinates (i,j) of c also get changed by the assignment statement in step B3. Note also that 1 is subtracted from every element that moves up during the slide. If the result of a slide on P into c is P' and the total amount subtracted is d, then we will write $P'=j^c(P)$ and $d=d^c(P)$. It is easy to verify that P' is still a plane partition.

To illustrate, we have boxed the elements on the path of a slide into $c=(2,2)$ on the following partition and displayed the result after the slide is complete:

$$P=\begin{matrix} \blacksquare & \blacksquare & 4 \\ \blacksquare & \boxed{\blacksquare} & 3 \\ 4 & \boxed{4} & \boxed{3} \\ 3 & 3 & \boxed{2} \\ 2 & 2 & \boxed{1} \\ 1 & 1 & \end{matrix} \quad\rightarrow\quad P'=j^{(2,2)}(P)=\begin{matrix} \blacksquare & \blacksquare & 4 \\ \blacksquare & 3 & 3 \\ 4 & 3 & 1 \\ 3 & 3 & \\ 2 & 2 & \\ 1 & 1 & \end{matrix}$$

In this case $d^c(P)=3$.

Now to the proof of (2). By Theorems of MacMahon [4] and Stanley [9], the two products on the right-hand side of the equality count normal plane partitions (those where $\lambda=\emptyset$) and reverse plane partitions of shape λ (arrays obtained by replacing the boxes of λ by nonnegative integers such that rows and columns increase weakly), respectively. Thus, it suffices to find a bijection

$$P \leftrightarrow (Q, R),$$

where P is a plane partition of shape Λ/λ, Q is a normal plane partition and R is a reverse plane partition of shape λ, such that

$$\sum_{(i,j)\in\Lambda/\lambda} P_{i,j} = \sum_{(i,j)\in\Lambda} Q_{i,j} + \sum_{(i,j)\in\lambda} R_{i,j}.$$

First we discuss the map $P \to (Q, R)$. The basic idea is that we will use slides on P to obtain the normal array Q while R keeps track of the amount subtracted at each stage. Specifically, let $c_1, \ldots, c_n$ be the cells of $\lambda=(\lambda_1, \ldots, \lambda_l)$ listed in the order

$$(l, \lambda_l), (l, \lambda_l-1), \ldots, (l, 1), (l-1, \lambda_{l-1}), \ldots, (1, 1), \tag{3}$$

i.e., list each row from right to left, starting with the lowest row and working up. Define

$$Q = j^{c_n}(\cdots(j^{c_1}(P))).$$

Further, let p_k be the path corresponding to the slide into cell c_k. Finally, after performing j^c on some intermediate partition P', where $c=(i,j)$, we let

$$R_{i,\lambda_i-j+1} = d^c(P'),$$

i.e., we fill R by rows from *left to right* starting with the lowest row and working up. Using the previous example for our initial P, we make the following computation:

$$Q:\ \begin{matrix} \blacksquare & \blacksquare & 4 \\ \blacksquare & \blacksquare & 3 \\ 4 & 4 & 3 \\ 3 & 3 & 2 \\ 2 & 2 & 1 \\ 1 & 1 & \end{matrix}\ ,\quad \begin{matrix} \blacksquare & \blacksquare & 4 \\ \blacksquare & 3 & 3 \\ 4 & 3 & 1 \\ 3 & 3 & \\ 2 & 2 & \\ 1 & 1 & \end{matrix}\ ,\quad \begin{matrix} \blacksquare & \blacksquare & 4 \\ 3 & 3 & 3 \\ 3 & 2 & 1 \\ 3 & 1 & \\ 2 & & \\ 1 & & \end{matrix}\ ,\quad \begin{matrix} \blacksquare & 4 & 2 \\ 3 & 3 & \\ 3 & 2 & \\ 3 & 1 & \\ 2 & & \\ 1 & & \end{matrix}\ ,\quad \begin{matrix} 4 & 2 & 2 \\ 3 & 1 & \\ 3 & & \\ 3 & & \\ 2 & & \\ 1 & & \end{matrix}$$

$$R:\ \begin{matrix} \blacksquare & \blacksquare \\ \blacksquare & \blacksquare \end{matrix}\ ,\quad \begin{matrix} \blacksquare & \blacksquare \\ 3 & \blacksquare \end{matrix}\ ,\quad \begin{matrix} \blacksquare & \blacksquare \\ 3 & 4 \end{matrix}\ ,\quad \begin{matrix} 2 & \blacksquare \\ 3 & 4 \end{matrix}\ ,\quad \begin{matrix} 2 & 3 \\ 3 & 4 \end{matrix}$$

Thus,

$$\begin{array}{ccc} \blacksquare & \blacksquare & 4 \\ \blacksquare & \blacksquare & 3 \\ 4 & 4 & 3 \\ 3 & 3 & 2 \\ 2 & 2 & 1 \\ 1 & 1 & \end{array} \to \left(\begin{array}{ccc} 4 & 2 & 2 \\ 3 & 1 & \\ 3 & & \\ 3 & & \\ 2 & & \\ 1 & & \end{array} , \quad \begin{array}{cc} 2 & 3 \\ 3 & 4 \\ & \\ & \\ & \\ & \end{array} \right)$$

We must show that this map is well-defined. It is easy to see that Q is a normal plane partition and that R has the right shape. We need to verify that the rows and columns of R are increasing weakly. This will follow from Lemmas 2.3 and 2.4, respectively.

Lemma 2.3. *Let $p=p_k$ and $p'=p_{k+1}$ be the paths corresponding to backward slides into adjacent cells c_k and c_{k+1} in the same row. If (i,j) is the rightmost cell of p in row i then the rightmost cell of p' in row i lies in a column $<j$, i.e., p' lies to the left of p.*

Proof of Lemma 2.3. Since c_{k+1} lies directly to the left of c_k, it suffices to verify that if p' reaches $(i,j-1)$, then its next step will be down. Let x and y be the elements in cells $(i+1,j-1)$ and $(i+1,j)$ before the slide into c_k; see Fig. 1(a). So, $x \geqslant y$ since this array is a skew plane partition.

	$j-1$	j
i	$\square$	$\square$
$i+1$	x	y

(a) Before p_k

	$j-1$	j
i	$\square$	$y-1$
$i+1$	x	$\square$

(b) After p_k

Fig. 1. Slide situations.

Now, by the hypothesis on p, after the c_k slide we have x and $y-1$ in cells $(i+1,j-1)$ and (i,j), respectively; see Fig. 1(b). Thus, when p' reaches $(i,j-1)$, it must continue to $(i+1,j-1)$ since $x>y-1$. $\square$

Lemma 2.4. *Let $p=p_k$ and $p'=p_l$ be the paths corresponding to forward slides into cells c_k and c_l, respectively, where $c_k=(\lambda_r, \lambda_r-s)$ and $c_l=(\lambda_{r-1}, \lambda_{r-1}-s)$ for some r,s. If (i,j) is the lowest cell of p in column j then the lowest cell of p' in column j lies in a row $<i$, i.e., p' lies above p.*

Proof of Lemma 2.4. We will induct on k. Since c_l lies above and to the right of c_k, it suffices to verify that if p' reaches $(i-1,j)$, then its next step will be right. Let m be the largest integer such that cell $(i,j+t)$ is the lowest cell on path p_{k-t} in column $j+t$ for $0 \leqslant t \leqslant m$. Let x and y be the elements in cells $(i-1,j+m+1)$ and $(i,j+m+1)$, respectively, just before sliding along the path p_{k-m}. So, we have the situation in Fig. 2(a). Thus, $x \geqslant y$.

	j		$j+m+1$
$i-1$	□	$\cdots$	x
i	□	$\cdots$	y

(a) Before p_{k-m}

	j		$j+m+1$
$i-1$	□	$\cdots$	x
i	y	$\cdots$	□

(b) After p_k

	j	$j+1$
$i-1$	□	x'
i	y'	□

(c) Before p_l

Fig. 2. More slide situations.

By our assumption about the p_{k-t}'s, the situation after completion of p_k must look like the one in Fig. 2(b). Further slides from the same row as p_k can change the entry in cell (i,j) to some y' but, since the elements passing through a given cell decrease weakly, we must have $y' \leqslant y$. Also, because of the previous lemma, x does not change with such slides.

As for the slides from row λ_{r-1}, those before p_{l-m} cannot change x or y', by induction and Lemma 2.3 applied to p_{k-m+1}. For similar reasons, no slide before p_l can change y'. To see how the slides $p_{l-m}, \ldots, p_{l-1}$ affect the elements in row $i-1$, note that, by the previous lemma, no element that moves up a row during a given slide can be moved again by subsequent slides starting in the same row as the given one. Thus, the element x' that occupies the $(i-1, j+1)$ cell before p_l (see Fig. 3(c)) must either have come from cell $(i, j+1)$ or from row i. The first case cannot happen since p_{l-1} and previous slides from that row are above p_{k-1}. In the second case, since an element can be moved a maximum of t times in t slides, x' must have occupied a cell weakly to the left of x in Fig. 2(b). Thus, $x' \geqslant x$. Putting everything together, we have

$$x' \geqslant x \geqslant y \geqslant y'.$$

Hence x' will move left into cell (i,j) during the slide p_l and we are done with the proof of the lemma. □

Proof of Theorem 2.1 (*continued*). We now need to create the inverse map

$$(Q, R) \rightarrow P.$$

First we formulate the inverse of a backward slide, called (oddly enough) a *forward slide.* For such a slide, we are given a skew plane partition Q of shape Λ/λ and a cell $c=(i,j)$, which is the leftmost zero cell of Q in row i. Now perform the following steps:

(F1) **While** $(i, j-1) \in \Lambda/\lambda$ **or** $(i-1, j) \in \Lambda/\lambda$ **do**
 (F2) **if** $P_{i,j-1} \leqslant P_{i-1,j}$ **then** slide $P_{i,j-1}$ into cell c
  else slide $P_{i-1,j}+1$ into cell c. **fi**
 (F3) Let $c :=$ the cell of the element that slid in step F2. **od**

If only one of the two elements of the **if** clause above is defined, then that one slides automatically into c (with 1 added if necessary). The reader can check that a forward slide into cell (5, 3) of P' in the example after the definition of a backward slide restores P. It is easy to see that, in general, forward slides can be used to reverse backward slides and vice versa.

Now suppose that the pair (Q, R) is given. Order the cells of λ as in (3) and perform forward slides on Q associated with $c_n, c_{n-1}, \ldots, c_1$ in turn: if $c_k = (i, j)$ then the associated slide will be into the leftmost zero cell which lies in row $i + R_{i, \lambda_i - j + 1}$ in the current version of Q. The final version of Q will be the image of the pair, P.

It is clear that the composition of our previous map with this one is the identity. To make sure that the other composition is too, we need to verify that the forward slides made on Q vacate the cells $c_n, c_{n-1}, \ldots, c_1$ in that order. This is accomplished by analogs of Lemmas 2.3 and 2.4. Since their proofs are similar to what we have already seen, we will merely state the results.

Lemma 2.5. *Let $p = p_k$ and $p' = p_{k-1}$ be the paths of forward slides corresponding to adjacent cells c_k and c_{k-1} in the same row. If (i, j) is the leftmost cell of p in row i then the leftmost cell of p' in row i lies in a column $> j$, i.e., p' lies to the right of p.*

Lemma 2.6. *Let $p = p_k$ and $p' = p_l$ be the paths of backward slides corresponding to cells c_k and c_l, respectively, where $c_k = (\lambda_r', \lambda_r - s)$ and $c_l = (\lambda_{r+1}', \lambda_{r+1} - s)$ for some r, s. If (i, j) is the highest cell of p in column j then the highest cell of p' in column j lies in a row $> i$, i.e., p' lies below p.*

Proof of Theorem 2.1 (*continued*). (1) = (2): To show directly that the two products are equal, we merely need to demonstrate that the same exponents appear in both denominators. Clearly, it suffices to find an injection

$$f : \lambda \to \Lambda / \lambda$$

such that

(f1) for all cells $(i, j) \in \lambda$, we have $h_{i,j} = h_{f(i,j)}$, and

(f2) the multiset ("set" with repetitions) of hooklengths for the cells in Λ and $\Lambda / \lambda - f(\lambda)$ are the same.

To define this injection, it will be convenient to introduce the notion of a row strip.

The *rth row strip of* λ is the set of all cells of the shape λ that are r cells from the bottom of their respective columns. For example, we have marked the cells of the rth row strip in the following diagram with the integer r for $1 \leqslant r \leqslant 4$:

4	3	3	2
3	2	2	1
2	1	1	
1			

Similarly, the *rth row strip of* Λ/λ is the set of all cells of Λ/λ that are r cells from the top of their respective columns. Marking a skew shape with r's gives the following figure:

■	■	■	■	1	1	1	⋯
■	■	■	■	2	2	2	⋯
■	■	■	1	3	3	3	⋯
■	1	1	2	4	4	4	⋯
1	2	2	3	5	5	5	⋯
⋮	⋮	⋮	⋮	⋮	⋮	⋮	⋮

Let σ_r and τ_r denote the rth row strips of λ and Λ/λ, respectively. We will define the injection f by defining it on each row strip,

$$f:\sigma_r \to \tau_r.$$

Specifically, let the cells of σ_r be $(i_1, 1), (i_2, 2), (i_3, 3), \ldots$ and define inductively

$$f(i_j, j) = \text{the rightmost box of } \tau_r \text{ weakly to the left of } (i_j + r, \lambda_{i_j}) \text{ which is not already in the image of } f \tag{4}$$

as j successively takes on the values 1, 2, 3, etc. Thus, $f(i_1, 1) = (i_1 + r, \lambda_{i_1})$. If $i_2 = i_1$ then $f(i_2, 2)$ is the element of τ_r in column $\lambda_{i_1} - 1$. But if $i_2 < i_1$ then $f(i_2, 2) = (i_2 + r, \lambda_{i_2})$, etc. For example, if $\lambda = (9, 8, 6, 2)$ and $r = 2$ then we have marked $(i,j) \in \sigma_2$ and $f(i,j) \in \tau_2$ with the same letter in the following diagram:

■	■	■	■	■	■	*g*	*h*	■
■	■	*c*	*d*	*e*	*f*	■	■	□
a	*b*	■	■	■	■	□	□	*g*
■	■	□	□	□	□	*d*	*c*	□
□	□	*f*	*e*	*b*	*a*	□	□	□
□	*h*	□	□	□	□	□	□	□

Note that $f(i_1, 1) = (i_1 + r, \lambda_{i_1})$ is indeed in τ_r and, by construction, has the same hooklength as (i,j). We must show that the rest of f is well-defined, in that the cell $f(i,j)$ exists (in which case f is clearly injective), and that conditions (f1) and (f2) are satisfied. This will be taken care of by the following lemma and the fact that the rth row strip of Λ has hooklengths $\{r, r+1, r+2, \ldots\}$.

Lemma 2.7. *The function defined by equation* (4) *is well-defined and satisfies*

(1) *for all cells* $(i,j) \in \sigma_r$, *we have* $h_{i,j} = h_{f(i,j)}$, *and*

(2) *the hooklengths of the cells of* $\tau_r - f(\sigma_r)$, *read from left to right, are precisely* $\{r, r+1, r+2, \ldots\}$.

Proof of Lemma 2.7. We induct on the number of rows of λ. If $\lambda=(\lambda_1,\ldots,\lambda_l)$, then let $\bar{\lambda}=(\lambda_2,\ldots,\lambda_l)$. Now the row strips and their images in columns $j\leqslant\lambda_2$ of $\bar{\lambda}$ and λ (for $i\geqslant 2$) are exactly the same. So, by induction, f is well-defined and preserves hooklengths there. Also, $|f(\sigma_r\cap\bar{\lambda})|=\lambda_{r+1}$. So, there are $\lambda_2-\lambda_{r+1}$ elements of τ_r in columns $j\leqslant\lambda_2$ which are not in $f(\sigma_r\cap\bar{\lambda})$. Thus, by induction again, the hooklengths of these cells must be $r, r+1,\ldots,r+\lambda_2-\lambda_{r+1}-1$.

As far as the columns j, with $\lambda_2<j\leqslant\lambda_1$ are concerned, induction and the previous sentence combine to show that there τ_r has hooklengths from $r+\lambda_2-\lambda_{r+1}$ to

$$r+\lambda_2-\lambda_{r+1}+(\lambda_1-\lambda_2)-1=r+\lambda_1-\lambda_{r+1}-1.$$

Thus, the hooklengths available in columns $j\leqslant\lambda_1$ make $f(\sigma_r\cap\lambda_1)$ well-defined and hooklength-preserving if we use rule (4). Furthermore, there are $\lambda_r-\lambda_{r+1}$ elements of σ_r in row 1. So, the hooklengths unused by f in columns $j\leqslant\lambda_1$ form an interval from r to

$$r+\lambda_1-\lambda_{r+1}-1-(\lambda_r-\lambda_{r+1})=r+\lambda_1-\lambda_r-1.$$

Finally, the cells of τ_r in columns $j>\lambda_1$ clearly have hooklengths

$$r+\lambda_1-\lambda_r,\ r+\lambda_1-\lambda_r+1,\ldots,$$

so we are done.
This completes the proof of Theorem 2.1. □

3. Shifted plane partitions

Consider the shifted plane

$$\Lambda^*=\{(i,j)\in\Lambda\mid i\leqslant j\},$$

so that now each row is shifted over one box from the row above. Let $\lambda^*=(\lambda_1^*,\ldots,\lambda_l^*)$ be a strict partition, i.e., one where $\lambda_1^*>\cdots>\lambda_l^*$. Then λ^* can be viewed as a shifted shape in the upper-left corner of Λ^* via

$$\lambda^*=\{(i,j)\in\Lambda^*\mid i\leqslant j\leqslant i+\lambda_i-1\}.$$

This gives rise to the skew shifted shape

$$\Lambda^*/\lambda^*=\{(i,j)\mid(i,j)\in\Lambda^*,\ (i,j)\notin\lambda^*\}.$$

A *skew plane partition of n with shape* Λ^*/λ^*, P^*, is defined in the obvious way. For example, if $\lambda=(3,1)$ then one such skew shifted plane partition is

$$P^*=\begin{array}{ccccc}\blacksquare&\blacksquare&\blacksquare&4&4\\ &\blacksquare&3&2&\\ &&3&1&\end{array}$$

Let

$$pp_{\Lambda^*/\lambda^*}(n) = \text{number of shifted plane partitions of } n \text{ having shape } \Lambda^*/\lambda^*.$$

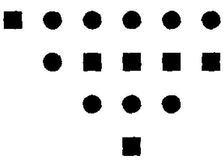
Shifted hooks are defined as follows. If $(i,j)\in\lambda^*$ then

$$H^*_{i,j}=\{(i,j')\mid j'\geqslant j\}\cup\{(i',j)\mid i'\geqslant i\}\cup\{(j+1,j')\mid j'\geqslant j+1\},$$

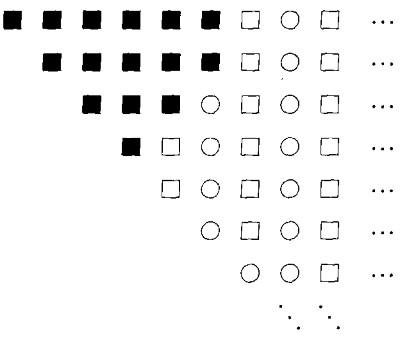
where all sets are contained in λ^*. This is just the normal hook if $j\geqslant l$, the number of parts of λ^* (i.e., (i,j) is not over the left staircase). If $j<l$ then the vertical portion of $H^*_{i,j}$ does a right turn and picks up all elements in row $j+1$. In the case $(i,j)\in\Lambda/\lambda$, we again take reflections to get

$$H^*_{i,j}=\{(i,j')\mid j'\leqslant j\}\cup\{(i',j)\mid i'\leqslant i\}\cup\{(i',i-1)\mid i'\leqslant i-1\},$$

where all sets are now in Λ^*/λ^*. Of course, the *shifted hooklength* of cell (i,j) is $h^*_{i,j}=|H^*_{i,j}|$. For example, if $\lambda^*=(6,5,3,1)$ then the cells in the hook of $(1,2)\in\lambda^*$ are shown as circles in



while those of $(7,8)\notin\lambda^*$ are the circles in

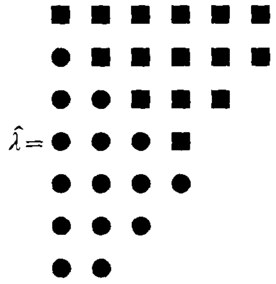


So, $h^*_{1,2}=9$ and $h^*_{7,8}=12$.

The way to motivate the definition of these hooks is as follows. Given the shifted shape λ^*, let $\hat{\lambda}$ denote the left-justified shape obtained by gluing together λ^* and its transpose λ^{*t}, i.e.,

$$\hat{\lambda}=\{(i,j)\mid i\leqslant j\leqslant i+\lambda^*_i-1 \text{ or } j+1\leqslant i\leqslant j+\lambda^*_j\}.$$

To illustrate, if $\lambda^* = (6, 5, 3, 1)$ as before, then



where the circles now indicate the cells of $\lambda^{*\prime}$. It is easy to see that if $(i,j)\in\Lambda^*/\lambda^*$ then $h^*_{i,j} = h_{i,j}$, where the normal hooklength is calculated in $\Lambda/\hat{\lambda}$. Similarly, if $(i,j)\in\lambda^{*\prime}$ then $h^*_{i,j} = h_{j+1,i}$, where the normal hooklength is in $\hat{\lambda}$. This is because, in both cases, the shifted hook is just the normal hook with one of its appendages bent. In what follows, we will also need a third type of hooklength. If $(i,j)\in\lambda^*$ then let $\hat{h}_{i,j} = h_{i,j}$, where the normal hooklength is computed in $\hat{\lambda}$.

We can now state the analog of Theorem 2.1.

Theorem 3.1. *If λ^* is a fixed shifted shape, then*

$$\sum_{n\geq 0} pp_{\Lambda^*/\lambda^*}(n)x^n = \prod_{(i,j)\in\Lambda^*/\lambda^*} \frac{1}{1-x^{h^*_{i,j}}} \tag{5}$$

$$= \prod_{k\geq 1} \frac{1}{(1-x^k)^{\lceil k/2 \rceil}} \prod_{(i,j)\in\lambda^*} \frac{1}{1-x^{\hat{h}_{i,j}}} \tag{6}$$

Proof. (5): Again, we are just reflecting the shifted Hillman–Grassl algorithm (see [5]) in an anti-diagonal. Because of the similarity with the proof of (1), we content ourselves with defining the path p^* along which to subtract ones in a given shifted skew plane partition P^*. The reader who has made it this far will find no difficulty in filling in the details of the rest of the algorithm:

SHG1. Start p^* at (a, b), the rightmost highest cell of P^* containing a nonzero entry.

SHG2. Continue by iterating

$$(i,j)\in p^* \Rightarrow \begin{cases} (i+1,j)\in p^* & \text{if } P^*_{i+1,j} = P^*_{i,j}, \\ (i,j-1)\in p^* & \text{otherwise.} \end{cases}$$

SHG3. The induction rule in SHG2 will fail at some cell (r, s) at the left end of a row; so, subtract ones along this portion of p^*.

SHG4. **If** $r<s$ **then stop**
else (now $r=s$) continue p^* by $(r-1, r-1)\in p^*$ and iterate

$$(i,j)\in p^* \Rightarrow \begin{cases} (i,j+1)\in p^* & \text{if } P^*_{i,j+1}=P^*_{i,j}, \\ (i-1,j)\in p^* & \text{otherwise } \textbf{fi}. \end{cases}$$

SHG5. Now the induction rule in SHG4 will fail at some cell (t, u) at the top of a column.

It is easy to see that after subtracting one from the elements in p^*, the array remains a shifted plane partition and the amount subtracted is $h^*_{r,b}$ or $h^*_{u+1,b}$ depending upon whether the path terminates in step SHG4 or SHG5, respectively. (The crucial observation is that the second half of p^*, if it exists, cannot intersect the first half because of the subtraction in SHG3.)

(5)=(6): We obtain the analog of the map f of the proof that (1)=(2) as follows. Using column strips (rather than row strips), define an injection

$$f:\hat{\lambda} \to \Lambda/\hat{\lambda}.$$

A simple argument shows that if we restrict the domain of f to Λ^* then the range also becomes included in the shifted plane. Furthermore, those cells in $\Lambda^*-f(\lambda^*)$ have hooklengths given by the first product in (6). □

4. Open questions

First of all, the reader will have noticed that we gave no direct proof that the product (6) counts shifted skew plane partitions. There is a shifted version of the "jeu de taquin" [6, 11], but it is not clear how to apply it in this case. Krattenthaler (private communication) has pointed out that the second product in (6) counts shifted reverse plane partitions R^* of shape λ^* such that $R_{i,i}$ is even for all i. However, this does not seem to help.

In [5] we also consider a third family of partitions with hooklengths: rooted trees. A rooted tree τ is a finite partially ordered set with a unique minimal element (called the root) whose Hasse diagram is a tree in the graph-theoretic sense of the term. A reverse τ-partition is an assignment T of nonnegative integers to the vertices of τ such that if $v\leqslant w$ in the partial order in τ then $T(v)\leqslant T(w)$ as integers. This is the tree analog of a reverse plane partition or a reverse shifted plane partition. The hooks in this case are just

$$H_v=\{w\in\tau \mid w\geqslant v\}.$$

In all three cases, the generating function for those reverse partitions summing to n is a finite product in terms of hooklengths. However, we have been unable to define a notion of skewness for trees that will yield a nice generating function for the corresponding (nonreverse) partitions. Perhaps one of our readers will have better luck.

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