Corrigendum to “Log-concavity and log-convexity via distributive lattices”

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Abstract

A squence of real numbers $(a_n)_{n\geq 0}$ is log-concave if $a_n^2 \geq a_{n-1}a_{n+1}$ for all $n \geq 1$, or log-convex if the inequality is reversed. In “Log-concavity and log-convexity via distributive lattices” we developed a tool for combinatorially proving such inequalities. To do so, one constructs a distributive lattice L and two ideals I, J in L having cardinatlities $|L| = a_{n+1}$, $|I| = |J| = a_n$, and $|I \cap J| = a_{n-1}$. It was claimed that the sequence of values of a descent polynomial always forms a log-concave sequence. Unfortunately, the cardinality of the intersection of the ideals given in the proof was wrong. We give correct ideals to prove this assertion.

1 Background

Consider a sequence $(a_n)_{n\geq 0}$ of real numbers. The sequence is *log-concave* if, for all $n \geq 1$,

$$a_n^2 \geq a_{n-1}a_{n+1},$$

or *log-concave* if

$$a_n^2 \leq a_{n-1}a_{n+1},$$

whenever $n \geq 1$. Log-concave and log-convex sequences abound in combinatorics, algebra, and geometry.

In [LS26] we developed a new method for proving log-concavity and log-convexity. Let $(L, \trianglelefteq)$ be a finite distributive lattice. We will use $|\cdot|$ to denote the cardinality of a finite set. Say that $I \subseteq L$ is a *lower order ideal* if

$$x \in I \text{ and } y \trianglelefteq x \implies y \in I,$$

or that J is an *upper order ideal* provided

$$x \in J \text{ and } y \trianglerighteq x \implies y \in J.$$

The following is a simple but powerful consequence of the FKG inequality [FKG71].

oil **Lemma 1.1** (The Order Ideal Lemma, [LS26, Lemma 1.1])**.** *Let L be a distributive lattice and suppose that $I, J \subseteq L$ are ideals.*

(a) If I, J are both lower ideals or both upper ideals then

$$|I| \cdot |J| \leq |I \cap J| \cdot |L|.$$

(b) If one of I, J is a lower order ideal and the other is upper then

$$|I| \cdot |J| \geq |I \cap J| \cdot |L|. \qquad \square$$

Part (b) of the Order Ideal Lemma can be used to demonstrate log-concavity by constructing a lattice L, a lower order ideal I, and an upper order ideal J so that

$$|L| = a_{n+1}, \ |I| = |J| = a_n, \quad \text{and } |I \cap J| = a_{n-1}.$$

Then applying the lemma gives

$$a_n^2 = |I| \cdot |J| \geq |I \cap J| \cdot |L| = a_{n-1}a_{n+1}.$$

In [LS26] we used this lemma, among many applications, to show that the values of a descent polynomial (defined in the next section) form a log-concave sequence. But the ideals given in the proof did not have an intersection of the correct cardinality. The purpose of this corrigendum is to provide a correct proof by changing the ideals used.

2 Descent polynomials

We will just give the minimum amount of information to understand the corrigendum. For more details and examples, see [LS26, Section 6.1].

Let $\mathbb{N}$ be the nonnegative integers and, for $m, n \in \mathbb{N}$ define

$$[m, n] = \{m, m+1, \ldots, n\}.$$

In particular, we abbreviate

$$[n] = [1, n].$$

Let $\mathfrak{S}_n$ be the *symmetric group* of all permutations of $[n]$ written in 1-line form $\pi = \pi_1 \ldots \pi_n$. The *descent set* of π is

$$\operatorname{Des} \pi = \{i \mid \pi_i > \pi_{i+1}\} \subseteq [n-1].$$

Given a set S of positive integers, we let

$$D_n(S) = \{\pi \in \mathfrak{S}_n \mid \operatorname{Des} \pi = S\}$$

and

$$d_n(S) = |D_n(S)|.$$

It is a famous result of MacMahon [Mac04] that, for any S, we have $d_n(S)$ is a polynomial in n called the *descent polynomial* of S. We will prove that the sequence $(d_n(S))_{n\geq 0}$ is log-concave for all S.

To construct the necessary distributive lattice, we consider the *positional inversion table* of $\pi \in \mathfrak{S}_n$ which is

$$\kappa(\pi) = (\kappa_1, \kappa_2, \ldots, \kappa_n)$$

where

$$\kappa_i = \#\{j > i \mid \pi_j < \pi_i\}.$$

Note that

$$\kappa(\pi) \in [0, n-1] \times [0, n-2] \times \cdots \times [0, 0],$$

and we let

$$\mathcal{K}_n = \{\kappa \mid \kappa \in [0, n-1] \times [0, n-2] \times \cdots \times [0, 0]\}.$$

Say that $\kappa \in \mathcal{K}_n$ has *descent set*

$$\operatorname{Des} \kappa = \{i \mid \kappa_i > \kappa_{i+1}\}.$$

Note that $\operatorname{Des} \kappa$ does not include the indices i such that $\kappa_i = \kappa_{i+1}$.

The map $\mathfrak{S}_n \to \mathcal{K}_n$ by $\pi \mapsto \kappa(\pi)$ is a well-known bijection and so we will often use the two sets interchangeably. In particular, we have $\operatorname{Des} \pi = \operatorname{Des} \kappa(\pi)$. Now define

$$\mathcal{K}_n(S) = \{\kappa \in \mathcal{K}_n \mid \operatorname{Des} \kappa = S\}$$

and partially order this set component-wise. Note that $|\mathcal{K}_n(S)| = d_n(S)$. This order is related to the middle order of Bouvel, Ferrari, and Tenner [BFT25].

Lemma 2.1 ([LS26, Lemma 6.5])**.** *For any S the poset $\mathcal{K}_n(S)$ is a distributive lattice* □

We will need the following observations about $\mathcal{K}_n(S)$.

Lemma 2.2. *Suppose $S \neq \emptyset$ and let $m = \max S$. Suppose $\kappa = (\kappa_1, \ldots, \kappa_n) \in \mathcal{K}_n(S)$.*

(a) We have $\kappa_m \geq 1$.

(b) We have $\kappa_i = 0$ for $i > m$.

Proof. (a) Since $m \in S = \operatorname{Des}\kappa$ we must have $\kappa_m > \kappa_{m+1} \geq 0$. But this implies $\kappa_m \geq 1$.

(b) Suppose, towards a contradiction, that $\kappa_i > 0$ for some $i > m$. By definition of $\mathcal{K}_n$ we have $\kappa_n = 0$. So there must be an index $j \geq i > m$ such that $\kappa_j > 0$ and $\kappa_{j+1} = 0$. But then $j \in \operatorname{Des}\kappa$, contradicting the maximality of m. □

We now come to the result from [LS26] whose proof needs correction.

Theorem 2.3 ([LS26, Theorem 6.6]). *For any S, the sequence $(d_n(S))_{n\geq 0}$ is log-concave.*

Proof. The result is easy if $S = \emptyset$, so we assume S is nonempty. Keeping the notation of the previous lemma, we see that if $\kappa \in \mathcal{K}_n(S)$ then in fact

$$\kappa \in [0, n-1] \times [0, n-2] \times \cdots \times [0, n-m+1] \times [1, n-m] \times [0, 0]^{n-m}$$

where the superscript denotes the $(n-m)$-fold product.

Now consider the lattice $\mathcal{K}_{n+1}(S)$ so that $|\mathcal{K}_{n+1}(S)| = d_{n+1}(S)$. Inside this lattice, define the set

$$I = \mathcal{K}_n(S) \times [0, 0].$$

Clearly I is a lower order ideal with $I \cong D_n(S)$ so that $|I| = d_n(S)$. Also construct

$$J = \mathcal{K}_{n+1}(S) \cap ([1, n] \times [1, n-1] \times \cdots \times [1, n-m+2] \times [2, n-m+1] \times [0, 0]^{n-m+1}).$$

From its definition, J is an upper order ideal. To show $|J| = d_n(S)$ we will construct a poset isomorphism $J \cong D_n(S)$. If $\kappa \in J$ then factor it as a concatenation $\kappa = \lambda\mu$ where $|\lambda| = m$ and $|\mu| = n+1-m$. Map κ to $\kappa' = \lambda'\mu'$ where λ' is the sequence obtained from λ by subtracting 1 from each entry, and $\mu' = [0, 0]^{n-m}$. It easy easy to see that this is an isomorphism by constructing the inverse map.

Finally, we have

$$I \cap J = K_{n+1}(S) \cap ([1, n-1] \times [1, n-2] \times \cdots \times [1, n-m+1] \times [2, n-m] \times [0, 0]^{n-m+1}).$$

Using arguments similar to those in the previous paragraph, one shows that $I \cap J \cong D_{n-1}(S)$ which implies $|I \cap J| = d_{n-1}(S)$. Now we are done by part (b) of the Order Ideal Lemma. □

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