# Congruences for Catalan and Motzkin numbers and related sequences

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**Abstract**

We prove various congruences for Catalan and Motzkin numbers as well as related sequences. The common thread is that all these sequences can be expressed in terms of binomial coefficients. Our techniques are combinatorial and algebraic: group actions, induction, and Lucas' congruence for binomial coefficients come into play. A number of our results settle conjectures of Benoit Cloitre and Reinhard Zumkeller. The Thue-Morse sequence appears in several contexts.

## 1 Introduction

Let $\mathbb{N}$ denote the nonnegative integers. The divisibility of the *Catalan numbers*

$$C_n = \frac{1}{n+1}\binom{2n}{n}, \qquad n \in \mathbb{N},$$

by primes and prime powers has been completely determined by Alter and Kubota [4] using arithmetic techniques. In particular, the fact that $C_n$ is odd precisely when

$n = 2^h - 1$ for some $h \in \mathbb{N}$ has attracted the attention of several authors including Deutsch [11], Eğecioğlu [15], and Simion and Ullman [32] who found combinatorial explanations of this result. In the next section we will derive the theorem which gives the largest power of 2 dividing any Catalan number by using group actions. In addition to its generality, this technique has the advantage that when $n = 2^h - 1$ there is exactly one fixed point with all the other orbits having size divisible by 2. For other congruences which can be proven using the action of a group, see Sagan's article [29].

By contrast, almost nothing is known about the residues of the *Motzkin numbers*

$$M_n = \sum_{k \geq 0} \binom{n}{2k} C_k, \qquad n \in \mathbb{N}.$$

In fact, the only two papers dealing with this matter of which we are aware are the recent articles of Luca [24] about prime factors of $M_n$ and of Klazar and Luca [22] about the periodicity of $M_n$ modulo a positive integer. In section 3 we will characterize the parity of the Motzkin numbers as well as three related sequences. Surprisingly, the characterizations involve a sequence which encodes the lengths of the blocks in the Thue-Morse sequence. The block-length sequence was first studied by Allouche et. al. [2]. For more information about the Thue-Morse sequence in general, the reader is referred to the survey article of Allouche and Shallit [3].

Section 4 is devoted to congruences for the central binomial and trinomial coefficients. We are able to use these results to describe the Motzkin numbers and their relatives modulo 3. They also prove various conjectures of Benoit Cloitre [10] and Reinhard Zumkeller [33]. The Thue-Morse sequence appears again. Our main tool in this section is Lucas' congruence for multinomial coefficients [25].

Our final section is a collection of miscellaneous results and conjectures about sequences related to binomial coefficients. These include the Apéry numbers, the central Delannoy and Eulerian numbers, Gould's sequence, and the sequence enumerating noncrossing graphs.

# 2 Catalan numbers

If $n, m \in \mathbb{N}$ with $m \geq 2$ then the *order of* $n$ *modulo* $m$ is

$$\omega_m(n) = \text{largest power of } m \text{ dividing } n.$$

If the base $m$ expansion of $n$ is

$$n = n_0 + n_1 m + n_2 m^2 + \cdots \tag{1}$$

then let

$$\Delta_m(n) = \{i \ : \ n_i = 1\}$$

and

$$\delta_m(n) = |\Delta_m(n)|$$

where the absolute value signs denotes cardinality. We will also use a pound sign for this purpose. If a subscript $m$ is not used then we are assuming $m = 2$ and in this case $\delta(n)$ is also the sum of the digits in the base 2 expansion of $n$.

We wish to prove the following theorem.

**Theorem 2.1** *For $n \in \mathbb{N}$ we have*

$$\omega(C_n) = \delta(n+1) - 1.$$

Note as an immediate corollary that $C_n$ is odd if and only if $n = 2^h - 1$ for some $h \in \mathbb{N}$. It is easy to prove this theorem from Kummer's result about the order of a binomial coefficient [23] (or see [12, pp. 270–271], also stated as Theorem 5.12 below). However, we wish to give a combinatorial proof.

We will use a standard interpretation of $C_n$ using binary trees. A *binary tree* $T$ is a tree with a root $r$ where every vertex has a left child, or a right child, or both, or neither. Note that this differs from the convention where a vertex in a binary tree must have no children or both children. It will also be convenient to consider $T = \emptyset$ as a binary tree. With this convention, any nonempty tree can be written as $T = (T', T'')$ where $T'$ and $T''$ are the subtrees generated by the left child and by the right child of $r$, respectively. (The subtree *generated* by a vertex $v$ of $T$ consists of $v$ and all its descendants.) Let $\mathcal{T}_n$ be the set of all binary trees on $n$ vertices. Then it is well-known that $|\mathcal{T}_n| = C_n$ for all $n \in \mathbb{N}$.

The *height* of a vertex $v$ is the length of the unique path from the root $r$ to $v$. A *complete binary tree* $T_h$ has all $2^i$ possible vertices at height $i$ for $0 \le i \le h$ and no other vertices. Let $G_h$ be the group of automorphisms of $T_h$ as a rooted tree. We will need some facts about $G_h$.

**Lemma 2.2** *We have the following*

*(1) If $h = 0$ then $G_0 = \{e\}$ where $e$ is the identity element, and if $h \ge 1$ then*

$$G_h = Z_2 \wr G_{h-1}$$

*where $Z_2$ is the cyclic group of order 2 and $\wr$ is wreath product.*

*(2) $\#G_h = 2^{2^h - 1}$.*

*(3) If $G_h$ acts on a set and $\mathcal{O}$ is an orbit of the action then $\#\mathcal{O}$ is a power of 2.*

**Proof** The proof of (1) follows by noting that $T_h = (T_{h-1}, T_{h-1})$ for $h \geq 1$. Then (2) is an easy induction on $h$ using (1). Finally, (3) is a consequence of (2) and the fact that for any group action the size of an orbit always divides the order of the group. ■

Now $G_n$ acts on $\mathcal{T}_n$ in the obvious way. It is this action which will permit us to calculate $\omega(C_n)$. Recall the *double factorial*

$$(2d)!! = (2d-1)(2d-3)\cdots 3\cdot 1.$$

**Lemma 2.3** *For $n \in \mathbb{N}$, let $d = \delta(n+1) - 1$. Then given any orbit $\mathcal{O}$ of $G_n$ acting on $\mathcal{T}_n$ we have*

$$\omega(\#\mathcal{O}) \geq d$$

*with equality for exactly $(2d)!!$ orbits.*

**Proof** We will induct on $n$ with the result being trivial for $n = 0$. For $n \geq 1$ let $T = (T', T'') \in \mathcal{T}_n$. We also let $n'$ and $n''$ be the number of vertices of $T'$ and $T''$ respectively, as well as setting $d' = \delta(n'+1) - 1$ and $d'' = \delta(n''+1) - 1$. Clearly $n+1 = (n'+1) + (n''+1)$. It follows that

$$d \leq d' + d'' + 1 \tag{2}$$

with equality if and only if we have a disjoint union $\Delta(n+1) = \Delta(n'+1) \uplus \Delta(n''+1)$.

Let $\mathcal{O}(T)$ denote the orbit of $T$. Then

$$|\mathcal{O}(T)| = \begin{cases} |\mathcal{O}(T')|^2 & \text{if } T' \cong T'', \\ 2|\mathcal{O}(T')||\mathcal{O}(T'')| & \text{otherwise.} \end{cases} \tag{3}$$

Also we have, by induction, $\omega(\#\mathcal{O}(T')) \geq d'$ and $\omega(\#\mathcal{O}(T'')) \geq d''$.

First consider the case when $T' \cong T''$. Then $n' = n''$ and so equation (2) gives $d < 2d' + 1$. Now from (3) we obtain

$$\omega(\#\mathcal{O}(T)) = 2\omega(\#\mathcal{O}(T')) \geq 2d' \geq d$$

as desired for the first half of the lemma. If we actually have $\omega(\#\mathcal{O}(T)) = d$ then this forces $2d' = d$. But since $n' = n''$ we also have $n+1 = 2(n'+1)$ and so $d = d'$. This can only happen if $d = d' = 0$ and consequently $n = 2^h - 1$ for some $h$. But by the third part of the previous lemma, $T_h$ is the unique tree with $2^h - 1$ vertices and $\omega(\#\mathcal{O}(T)) = 0$. Since in this case $(2d)!! = 0!! = 1$, we have proven the present lemma when $T' \cong T''$.

Now consider what happens when $T' \ncong T''$. Using equations (2) and (3) as before gives

$$\omega(\#\mathcal{O}(T)) = \omega(\#\mathcal{O}(T')) + \omega(\#\mathcal{O}(T')) + 1 \geq d' + d'' + 1 \geq d$$

and again the first half of the lemma follows. When $\omega(\#\mathcal{O}(T)) = d$ then we must have $\omega(\#\mathcal{O}(T')) = d'$, $\omega(\#\mathcal{O}(T'')) = d''$, and $\Delta(n+1) = \Delta(n'+1) \uplus \Delta(n''+1)$. Using (3) to count orbits and induction it follows that we will be done if we can show

$$(2d)!! = \frac{1}{2}\sum_{k=1}^{d} \binom{d+1}{k} (2k-2)!!(2d-2k)!! \tag{4}$$

for $d \geq 1$. Rewriting this equation in hypergeometric series form we obtain the equivalent identity

$${}_2F_1\left( \begin{matrix} -d-1, \ -1/2 \\ 1/2-d \end{matrix} ; 1 \right) = 0$$

which is true by Vandermonde's convolution. ■

We can now prove Theorem 2.1. Since the orbits of a group action partition the set acted on, we can use Lemma 2.2 (3) and Lemma 2.3 to write

$$C_n = \#\mathcal{T}_n = (2d)!!2^d + k2^{d+1}$$

for some $k \in \mathbb{N}$. Since $(2d)!!$ is odd we can conclude $\omega(C_n) = d = \delta(n+1) - 1$ as desired. ■

The reader may not be happy with the last step in the proof of Lemma 2.3 since its appeal to the theory of hypergeometric series is not combinatorial. So we wish to give a bijective proof of equation (4). For this, we will interpret the double factorial in terms of binary total partitions, an object introduced and enumerated by Schröder [30]. Given a set $S$ then a *binary total partition of* $S$ is an unordered rooted tree $B$ satisfying the following restrictions.

1. Every vertex of $B$ has 0 or 2 children.
2. Every vertex of $B$ is labeled with a subset of $S$ in such a way that
   (a) the root is labeled with $S$ and the leaves with the 1-element subsets of $S$,
   (b) if a vertex is labeled with $A$ and its children with $A', A''$ then $A = A' \uplus A''$.

For example, if $S = \{1, 2, 3, 4\}$ then one possible total binary partition is displayed in Figure 1. Let $b_d$ be the number of total binary partitions on set $S$ with $|S| = d$. Then

$$b_{d+1} = (2d)!!$$

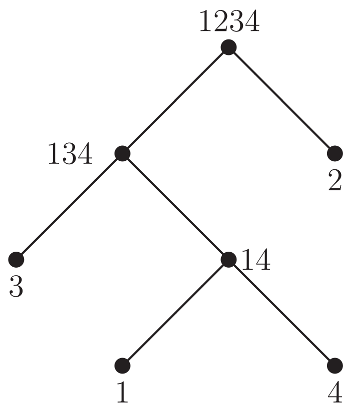


Figure 1: A total binary partition

For proofs of this fact, including a combinatorial one, see the text of Stanley [34, Example 5.2.6].

It is now easy to prove (4) combinatorially. The left side counts total binary partitions $B$ of a set $S$ with $|S| = d+1$. For the right side, note that each such $B$ can be formed uniquely by writing $S = S' \uplus S''$, letting $S'$ and $S''$ label the children of the root, and then forming total binary partitions on $S'$ and $S''$ to create the rest of $B$. If $\#S' = k$ then there are $\binom{d+1}{k}$ choices for $S'$ (after which, $S''$ is uniquely determined). The factors $(2k-2)!!$ and $(2d-2k)!!$ count the number of ways to put total binary partitions on $S'$ and $S''$, respectively. Finally, we must sum over all possible $k$ and divide by 2 since the tree is unordered. This completes the combinatorial proof of (4).

# 3 Motzkin numbers and related sequences

To find the parity of $M_n$ we must first introduce a related sequence. Define $\mathbf{c} = (c_0, c_1, c_2, \ldots) = (1, 3, 4, 5, 7, \ldots)$ inductively by $c_0 = 1$ and for $n \geq 0$

$$c_{n+1} = \begin{cases} c_n + 1 & \text{if } (c_n+1)/2 \notin \mathbf{c}, \\ c_n + 2 & \text{otherwise.} \end{cases} \tag{5}$$

Equivalently, $\mathbf{c}$ is the lexicographically least sequence of positive integers such that

$$m \in \mathbf{c} \text{ if and only if } m/2 \notin \mathbf{c}. \tag{6}$$

It follows that $\mathbf{c}$ contains all the positive odd integers $m$ since in this case $m/2$ is not integral.

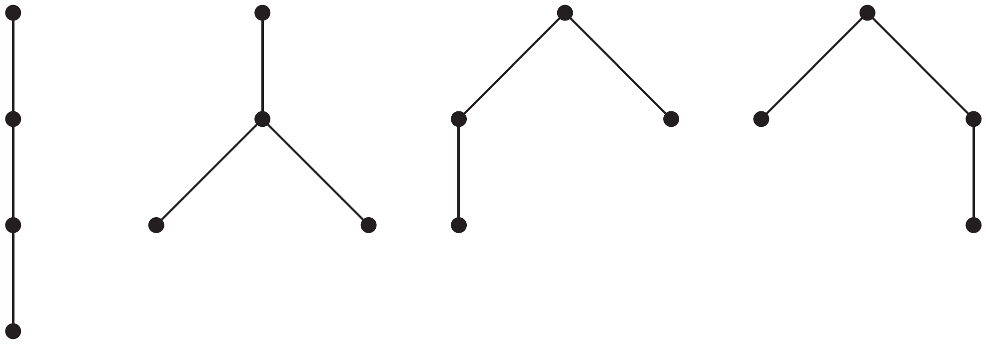

Figure 2: The 0-1-2 trees with three edges

The sequence $\mathbf{c}$ is intimately connected with the *Thue-Morse sequence* $\mathbf{t} = (t_0, t_1, t_2, \ldots) = (0, 1, 1, 0, 1, 0, 0, 1, \ldots)$ which is the 0-1 sequence defined inductively by $t_0 = 0$ and for $n \geq 1$

$$t_n = \begin{cases} t_{n/2} & \text{if } n \text{ even,} \\ 1 - t_{(n-1)/2} & \text{if } n \text{ odd.} \end{cases}$$

A *block* of a sequence is a maximal subsequence of consecutive, equal elements. One can show [2] that $c_n - c_{n-1}$ is the length of the $n$th block of $\mathbf{t}$ (where we start with the 0th block and set $c_{-1} = 0$).

Given a sequence $\mathbf{s} = (s_0, s_1, s_2, \ldots)$ and integers $k, l$ we let

$$k\mathbf{s} + l = (ks_0 + l, ks_1 + l, ks_2 + l, \ldots). \tag{7}$$

To simplify our notation, we will also write $k \equiv l \pmod{m}$ as $k \equiv_m l$ with the usual convention that if the modulus is omitted then $m = 2$. We can now characterize the parity of $M_n$.

**Theorem 3.1** *The Motzkin number $M_n$ is even if and only if either $n \in 4\mathbf{c} - 2$ or $n \in 4\mathbf{c} - 1$.*

**Proof** To prove this result we will need a combinatorial interpretation of $M_n$. A *0-1-2 tree* is an ordered tree where each vertex has at most two children (but a single child is not distinguished by being either left or right). It is known that $M_n$ is the number of 0-1-2 trees with $n$ edges. See the articles of Donaghey [13] and Donaghey and Shapiro [14] for details. The four 0-1-2 trees with three edges are shown in Figure 2

Now let $S_n$ be the number of *symmetric 0-1-2 trees* which are those with $n$ edges for which reflection in a vertical line containing the root is an automorphism of the tree. Only the first two trees in Figure 2 are symmetric. Clearly

$$M_n \equiv S_n \tag{8}$$

for all $n \in \mathbb{N}$. Furthermore,

$$S_{2n+1} = S_{2n} \tag{9}$$

since if a symmetric 0-1-2 tree has $2n+1$ edges then the root must have a single child and the subtree generated by that child must be a symmetric 0-1-2 tree with $2n$ edges. So to prove the theorem, it suffices to show that

$$S_{2n} \text{ is even if and only if } 2n \in 4\mathbf{c} - 2.$$

This can be restated that $S_{2n-2}$ is even iff $2n \in 4\mathbf{c}$ which is equivalent to $n \in 2\mathbf{c}$. So, by (6), it suffices to prove

$$S_{2n-2} \text{ is even if and only if } n \notin \mathbf{c}. \tag{10}$$

To prove (10), we will need a recursion involving $S_{2n-2}$. Let $T$ be a symmetric 0-1-2 tree with $2n-2$ edges. If the root of $T$ has one child then the subtree generated by that child is a symmetric 0-1-2 tree with $2n-3$ edges. If the root has two children then the subtree generated by one child can be any 0-1-2 tree with $n-2$ edges as long as the subtree generated by the other is its reflection. So using (8) and (9)

$$S_{2n-2} = S_{2n-3} + M_{n-2} \equiv S_{2n-4} + S_{n-2}. \tag{11}$$

We now prove (10) by induction, where the case $n = 1$ is trivial. Suppose first that $n \notin \mathbf{c}$. Then by (5) we have $n - 1 \in \mathbf{c}$ and by induction it follows that $S_{2n-4} = S_{2(n-1)-2}$ is odd. Also, since $n \notin \mathbf{c}$ we must have that $n$ is even. Furthermore, by (6) we have $n/2 \in \mathbf{c}$. By induction again, $S_{n-2} = S_{2(n/2)-2}$ is odd. So $S_{2n-4} + S_{n-2}$ is even and we are done with this direction.

When $n \in \mathbf{c}$, one can use similar reasoning to show that $S_{2n-4} + S_{n-2}$ is odd. One needs to consider the cases when $n$ is even and odd separately (and the latter case breaks into two subcases depending on whether $n-1$ is in $\mathbf{c}$ or not). But there are no really new ideas to the demonstration, so we omit the details. ■

We should note that Theorem 3.1 can also be derived from the results in [22], although it is not explicitly stated there. This theorem also permits us to determine the parity of various related sequences which we will now proceed to do.

A *Motzkin path of length* $n$ is a lattice path in the lattice $\mathbb{N} \times \mathbb{N}$ with steps $(1, 1)$, $(1, -1)$, and $(1, 0)$ starting at $(0, 0)$ and ending at $(n, 0)$. It is well known that $M_n$ is the number of Motzkin paths of length $n$. (Note that we do not need any condition about staying above the $x$-axis since we are working in $\mathbb{N} \times \mathbb{N}$.) Define a *Motzkin* *prefix of length* $n$ to be a lattice path which forms the first $n$ steps of a Motzkin path of length $m \geq n$. Equivalently, a Motzkin prefix is exactly like a Motzkin path except that the endpoint is not specified. Let $P_n$, $n \geq 0$, be the number of Motzkin prefixes of length $n$. This is sequence A005773 in Sloane's Encyclopedia [33]. The $P_n$ also count directed rooted animals with $n+1$ vertices as proved by Gouyou-Beauchamps and Viennot [18].

**Corollary 3.2** *The number $P_n$ is even if and only if $n \in 2\mathbf{c} - 1$.*

**Proof** Let $s_n$ be the number of Motzkin paths of length $n$ which are symmetric with respect to reflection in the vertical line $x = n/2$. Clearly $M_n \equiv s_n$ for all $n \geq 0$. There is also a bijection between Motzkin prefixes of length $n$ and symmetric Motzkin paths of length $2n$ gotten by concatenating the prefix with its reflection in the line $x = n$. So $P_n = s_{2n}$. Combining this with the previous congruence and Theorem 3.1 completes the proof. ■

Next we consider the *Riordan numbers* [33, A005043], $\gamma_n$, which count the number of ordered trees with $n$ edges where every nonleaf has at least two children. These are called *short bushes* by Bernhart [7]. If we relax the degree restriction so that the root can have any number of children then the resulting trees are called *bushes*. It is known [13, 14] that $M_n$ is the number of bushes with $n+1$ edges. It follows that

$$M_n = \gamma_{n+1} + \gamma_n \tag{12}$$

since every bush with $n+1$ edges is either a short bush or has a root with one child which generates a short bush with $n$ edges.

**Corollary 3.3** *The number $\gamma_n$ is even if and only if $n \in 2\mathbf{c} - 1$.*

**Proof** Given the previous corollary, it suffices to show that $\gamma_n$ and $P_n$ have the same parity. So it suffices to show that the two sequences satisfy the same recursion and boundary condition modulo 2. Now $\gamma_0 = 1 = P_0$ and we have just seen that

$$\gamma_{n+1} \equiv \gamma_n + M_n.$$

So consider the prefixes $p$ counted by $P_{n+1}$. If $p$ goes through $(n, 0)$ then there are two possible last steps for $p$ and so such paths need not be considered modulo 2. If $p$ goes through $(n, m)$ where $m > 0$ then those $p$ ending with a $(1, 1)$ step can be paired with those ending with a $(1, -1)$ step and ignored. So we are left with prefixes going through $(n, m)$ and $(n + 1, m)$ where $m > 0$. Such prefixes are equinumerous with those ending at $(n, m)$. And since $m > 0$, this is precisely the set of Motzkin prefixes which are not Motzkin paths. So

$$P_{n+1} \equiv P_n - M_n \equiv P_n + M_n$$

as desired. $\blacksquare$

Finally, consider the sequence counting *restricted hexagonal polyominos* [33, A002212]. The reader can find the precise definition of these objects in the paper of Harary and Read [19]. We will use an equivalent definition in terms of trees which can be obtained from the polyomino version by connecting the centers of adjacent hexagons. A *ternary tree* is a rooted tree where every vertex has some subset of three possible children: a left child, a middle child, or a right child. Just as with our definition of binary trees, this differs from the all or none convention for ternary trees. A *hex tree* is a ternary tree where no node can have two adjacent children. (A middle child would be adjacent to either a left or a right child but left and right children are not adjacent.) Let $H_n$, $n \geq 0$, be the number of hex trees having $n$ edges.

**Corollary 3.4** *The number $H_n$ is even if and only if $n \in 4\mathbf{c} - 2$ or $n \in 4\mathbf{c} - 1$.*

**Proof** In view of Theorem 3.1, it suffices to show that $H_n$ and $M_n$ have the same parity. Call a hex tree *symmetric* if the reflection in a line containing the root leaves it invariant, and let $h_n$ be the number of such trees with $n$ edges. There is an obvious bijection between symmetric hex trees and symmetric 0-1-2 trees. So

$$H_n \equiv h_n = S_n \equiv M_n$$

as desired. $\blacksquare$

# 4 Central binomial and trinomial coefficients

Our main tool in this section will be the following famous congruence of Lucas. If the base $p$ expansion of $n$ is

$$n = n_0 + n_1 p + n_2 p^2 + \cdots$$

then it will be convenient to denote the sequence of digits by

$$(n)_p = (n_0, n_1, n_2, \ldots) = (n_i).$$

**Theorem 4.1 (Lucas [25])** *Let $p$ be a prime and let $(n)_p = (n_i)$ and $(k)_p = (k_i)$. Then*

$$\binom{n}{k} \equiv_p \prod_i \binom{n_i}{k_i}. \qquad \blacksquare \tag{13}$$

The following corollary will be useful as well. It is also a special case of the theorem of Kummer cited in the discussion following the statement of Theorem 2.1. But this result will be sufficient for our purposes.

**Corollary 4.2** *Let $p$ be prime. If there is a carry when adding $k$ to $n-k$ in base $p$ then*

$$\binom{n}{k} \equiv_p 0.$$

**Proof** Using the notation of the previous theorem, if there is a carry out of the $i$th place where $i$ is minimum, then we have $n_i < k_i$. So $\binom{n_i}{k_i} = 0$ and thus the product side of (13) is zero. ■

Most of our results in this section will have to do with congruences modulo 3 so it will be useful to have the following notation. Given $i, j$ distinct integers in $\{0,1,2\}$ we let

$$T(ij) = \{n \in \mathbb{N} \ : \ (n)_3 \text{ contains only digits equal to } i \text{ or } j\}.$$

We begin with the central binomial coefficients. Recall that $\delta_3(n)$ is the number of ones in the base three expansion of $n$. The next result settles conjectures of Benoit Cloitre and Reinhard Zumkeller [33, A074938–40].

**Theorem 4.3** *The central binomial coefficients satisfy*

$$\binom{2n}{n} \equiv_3 \begin{cases} (-1)^{\delta_3(n)} & \textit{if } n \in T(01), \\ 0 & \textit{otherwise.} \end{cases}$$

**Proof** If $n$ has a 2 in its ternary expansion then there is a carry when adding $(n)_3$ to itself. So the second half of the theorem follows from the previous corollary. On the other hand, if $n \in T(01)$ then $2n \in T(02)$ and $(2n)_3$ has twos exactly where $(n)_3$ has ones. So by Lucas' Theorem

$$\binom{2n}{n} \equiv_3 \binom{2}{1}^{\delta_3(n)} \equiv_3 (-1)^{\delta_3(n)}$$

giving the first half. ■

It is easy to generalize the previous theorem to arbitrary prime modulus. To state the result, we need to define

$$\delta_{p,j}(n) = \text{number of elements of } (n)_p \text{ equal to } j \tag{14}$$

where $0 \le j < p$. Since the proof of the general case is the same as the one just given, we omit it.

**Theorem 4.4** *Let $p$ be prime and let $S$ be the set of all $n \in \mathbb{N}$ such that all elements of $(n)_p$ are less than or equal to $p/2$. Then*

$$\binom{2n}{n} \equiv_p \begin{cases} \prod_j \binom{2j}{j}^{\delta_{p,j}(n)} & \text{if } n \in S, \\ 0 & \text{otherwise.} \end{cases} \qquad \blacksquare$$

It turns out that there is a connection between the central binomial coefficients modulo 3 and the Thue-Morse sequence $\mathbf{t}$. This may seem surprising because $\mathbf{t}$ is essentially a modulo 2 object. However, Theorem 4.3 will allow us to reduce questions about $\binom{2n}{n}$ mod 3 to questions about bit strings. We will need another one of the many definitions of $\mathbf{t}$ for the proof, namely

$$t_n = \rho(\delta(n)) \tag{15}$$

where $\rho(k)$ is the remainder of $k$ on division by 2. We will also need the notation that $\mathbf{a} \equiv_m \mathbf{b}$ as sequences if and only if $a_n \equiv_m b_n$ for all $n \in \mathbb{N}$. The next result is again a conjecture of Cloitre [33, A074939].

**Theorem 4.5** *We have*

$$\left(n \ : \ \binom{2n}{n} \equiv_3 1\right) \equiv_3 \mathbf{t}.$$

**Proof** Let us call the sequence on the left of the previous congruence $\mathbf{a}$. Then from Theorem 4.3 we have that $n \in \mathbf{a}$ exactly when $n \in T(01)$ and $(n)_3$ has an even number of ones. From this it follows by an easy induction that $n = a_i$ if and only if $(n)_3 = (n_0, n_1, n_2, \ldots)$ where $(i)_2 = (n_1, n_2, \ldots)$ and $n_0$ is zero or one depending on whether $\delta(i)$ is even or odd, respectively. So by (15) we have

$$a_i = n \equiv_3 n_0 = \rho(\delta(i)) = t_i$$

for all $i \geq 0$. $\blacksquare$

There is an analogous conjecture of Cloitre for those central binomial coefficients with residue $-1$ modulo 3 [33, A074938]. Since the proof is much the same as the previous one, we omit it.

**Theorem 4.6** *We have*

$$\left(n \ : \ \binom{2n}{n} \equiv_3 -1\right) \equiv_3 1 - \mathbf{t}. \qquad \blacksquare$$

We next consider the central trinomial coefficients [33, A002426]. Let $T_n$ be the largest coefficient in the expansion of $(1 + x + x^2)^n$. It is easy [6] to express $T_n$ in terms of trinomial coefficients

$$T_n = \sum_{k \geq 0} \binom{n}{k,\ k,\ n-2k} \tag{16}$$

where we use the convention that if any multinomial coefficient has a negative number on the bottom then the coefficient is zero. Lucas' Theorem and its corollary generalize in the expected way to multinomial coefficients. So now we can find the residue of $T_n$ modulo a prime.

**Theorem 4.7** *Let $p$ be prime and let $(n)_p = (n_i)$. Then*

$$T_n \equiv_p \prod_i T_{n_i}$$

**Proof** This follows directly from expressing each trinomial coefficient in equation (16) as a product using Lucas' congruence, removing terms which are zero because the sum of the digits on the bottom is greater than the digit on the top, and then using the distributive law. ■

As an immediate corollary, we obtain a simple criterion for divisibility of $T_n$ by a prime which was conjectured independently by David Wilson [35] and the present authors.

**Corollary 4.8** *Let $p$ be a prime and let $(n)_p = (n_i)$. Then $T_n$ is divisible by $p$ if and only if $T_{n_i}$ is divisible by $p$ for some digit $n_i$ in the base $p$ expansion of $n$.* ■

As an example of how this result can be used to quickly compute $T_n$ modulo a prime, we completely determine its residues modulo 3.

**Corollary 4.9** *The central trinomial coefficients satisfy*

$$T_n \equiv_3 \begin{cases} 1 & \text{if } n \in T(01), \\ 0 & \text{otherwise.} \end{cases}$$

**Proof** Since $T_2 = 3 \equiv_3 0$, the previous corollary shows that $T_n \equiv_3 0$ for any $n$ with a 2 in its base 3 expansion. Similarly, if $n \in T(01)$ then since $T_0 = T_1 = 1$ we get $T_n \equiv_3 1$. ■

Since the $T_n$ are related to a number of the other sequences which we have been studying, we can use the previous result to determine their behavior modulo 3. We will apply linear operations to sets the same way we do to sequences (7).

**Corollary 4.10** *The Motzkin numbers satisfy*

$$M_n \equiv_3 \begin{cases} -1 & \text{if } n \in 3T(01) - 1, \\ 1 & \text{if } n \in 3T(01) \text{ or } n \in 3T(01) - 2, \\ 0 & \text{otherwise.} \end{cases}$$

**Proof** Barcucci, Pinzani, and Sprugnoli [6] have shown that

$$2M_n = 3T_n + 2T_{n+1} - T_{n+2}. \tag{17}$$

Reducing this equation modulo 3 and applying the previous theorem finishes the proof. ■

**Corollary 4.11** *The Motzkin prefix numbers satisfy*

$$P_n \equiv_3 \begin{cases} 1 & \text{if } n \in 3T(01), \\ -1 & \text{if } n \in 3T(01) + 1 \text{ or } 3T(01) - 1, \\ 0 & \text{otherwise.} \end{cases}$$

**Proof** If $p$ is a Motzkin prefix of length $n$ going through $(n-1, m)$ for some $m > 0$ then there are three ways to end the prefix and so they cancel out modulo 3. If $p$ goes through $(n-1, 0)$ then the first $n-1$ steps of $p$ form a Motzkin path and there are two possible last steps. So $P_n \equiv_3 2M_{n-1}$. Now apply the previous corollary to finish. ■

**Corollary 4.12** *The Riordan numbers satisfy*

$$\gamma_n \equiv_3 \begin{cases} 1 & \text{if } n \in T(01) - 1, \\ 0 & \text{otherwise.} \end{cases}$$

**Proof** Using the recursions (12) and (17) it is easy to prove inductively that $\gamma_n \equiv_3 T_{n+1}$. Theorem 4.9 now completes the proof. ■

# 5 Miscellaneous results and conjectures

We end with various results and conjectures related to what we have done in the previous sections.

## 5.1 Catalan numbers to other moduli

Theorem 2.1 implies that the $k$th block of zeros in the sequence of Catalan numbers modulo 2 has length $2^k - 1$ (where we start numbering with the first block). Alter and Kubota [4] have generalized this result to arbitrary primes and prime powers. One of their main theorems is as follows.

**Theorem 5.1** *(Alter and Kubota) Let $p \geq 3$ be a prime and let $q = (p+1)/2$. The length of the $k$th block of zeros of the Catalan sequence modulo $p$ is*

$$\left(p^{\omega_q(k)+\delta_{3,p}+1} - 3\right)/2$$

*where $\delta_{3,p}$ is the Kronecker delta.* ■

We can improve on this theorem in several regards. First of all, when $p = 3$ we can use our results to give a complete characterization of the residue of $C_n$ and not just say when it is zero. Suppose $(n)_3 = (n_i)$. Then we let

$$T^*(01) = \{n \ : \ n_i = 0 \text{ or } 1 \text{ for all } i \geq 1\}$$

and

$$\delta_3^*(n) = \text{number of } n_i = 1 \text{ for } i \geq 1.$$

**Theorem 5.2** *The Catalan numbers satisfy*

$$C_n \equiv_3 \begin{cases} (-1)^{\delta_3^*(n+1)} & \textit{if } n \in T^*(01) - 1, \\ 0 & \textit{otherwise.} \end{cases}$$

**Proof** The result is easy to verify for $n \leq 1$ so we assume $n \geq 2$. Directly from our definition of $C_n$ we have

$$C_n = \frac{4n-2}{n+1} C_{n-1}$$

If $n \equiv_3 0$ or 1 then $n+1$ is invertible modulo 3 and in fact $(4n-2)/(n+1) \equiv_3 1$. So for $k \geq 1$ we have $C_{3k-1} \equiv_3 C_{3k} \equiv_3 C_{3k+1}$. Thus it suffices to prove the theorem for $n \equiv_3 0$. Notice that in this case $C_n \equiv_3 \binom{2n}{n}$. Furthermore $n+1 \in T^*(01)$ if and only if $n \in T(01)$. And lastly $\delta_3^*(n+1) = \delta_3(n)$. Applying Theorem 2.1 finishes the proof. ■

We should verify that we can derive the $p = 3$ block lengths in Theorem 5.1 from Theorem 5.2. First from the latter result it follows that the $k$th block must start at an integer $3a-1$ and end at $3b-1$ for $a, b \in \mathbb{N}$. To simplify notation, let $\omega = \omega_2(k)$. Now $(a)_3$ must contain a 2 and $(a-1)_3$ does not. It follows that $(a)_3 = (a_0, a_1, a_2, \ldots)$ where $a_0 = 2$ and $(a_1, a_2, \ldots) = (k-1)_2$. Furthermore, since $b+1$ is the smallest integer larger than $a$ whose expansion contains no twos, the first $\omega+1$ elements of $(b)_3$ must all equal 2 and the rest must agree with the corresponding entries of $(a)_3$. By the same token, we must have $a_1 = a_2 = \ldots = a_\omega = 1$. Now one calculates the number of integers in the $k$th block by considering the first $\omega+1$ digits of $a$ and $b$ to get a count of

$$3(a-b+1) = 3[(3^{\omega+1}-1)-(3^\omega+3^{\omega-1}+\cdots+3+2)+1] = (3^{\omega+2}-3)/2$$

as desired. Note that not only have we been able to determine the length and starting and ending points of the block (which was also done by Alter and Kubota) but our demonstration is combinatorial as opposed to the original proof of Theorem 5.1 which is arithmetic. We had to use Lucas' Theorem to get to this result, but that theorem also has a combinatorial demonstration using group actions [29].

When $p \geq 5$, the residues of $C_n$ become more complicated, but one could use the same techniques in principle to compute them. In particular, if one is only interested in divisibility then one can derive Theorem 5.1 from Theorem 4.4 as we did for the $p=3$ case above.

It is also interesting another setting where a congruence involving the Catalan numbers has arisen. Albert, Atkinson, and Klazar [1] have studied *simple permutations* which are those permutations of $\{1,2,\ldots,n\}$ mapping no nontrivial subinterval of this set onto an interval. Then the number of such simple permutations is $2(-1)^{n+1} - \mathrm{Com}_n$ where $\mathrm{Com}_n$ is the coefficient of $x^n$ in the compositional inverse of the formal power series $\sum_{n\geq 1} n!x^n$ [33, A059372]. One of the results in [1] is that

$$\mathrm{Com}_n \equiv_3 C_{n-1}.$$

Their proof of this result uses generating functions, so it would be interesting to find a combinatorial one. Also, one would like to know the behavior of $\mathrm{Com}_n$ modulo other odd primes. (Albert et. al. have results for powers of two.)

The careful reader will note that we have not yet derived the residues of the hex tree numbers $H_n$ modulo three. It is time to fill that lacuna.

**Theorem 5.3** *The hex tree numbers satisfy*

$$H_n \equiv_3 \begin{cases} (-1)^{\delta_3^*(m+1)} & \text{if } n = 2m \text{ where } m \in T^*(01)-1, \\ 0 & \text{otherwise.} \end{cases}$$

**Proof** Suppose $T$ is a hex tree which has a vertex with a single child. Finding the first such vertex, say in depth-first order, one can associate with $T$ the two other hex trees which differ from $T$ only by moving the child into the two other possible positions. So modulo 3, $H_n$ is congruent to the number of hex tree with $n$ edges where every vertex has 0 or 2 children. So to be nonzero modulo 3, we must have

$n = 2m$ and the resulting trees are in bijection with binary trees on $m$ vertices (merely remove the $m$ leaves of the hex tree). Thus $H_n \equiv_3 C_m$ and we are now done by Theorem 5.2. ■

## 5.2 Motzkin numbers to other moduli

For the Motzkin numbers, one can prove results for larger primes by using Theorem 4.7 and equation (17). For example, we can prove the following conjecture of Tewodros Amdeberhan [5]. To put it in the context of Theorem 3.1, note that the Thue-Morse block sequence **c** can also be described [2] as the increasing sequence of all numbers of the form

$$(2i+1)4^j \text{ where } i, j \in \mathbb{N}.$$

**Theorem 5.4** *The Motzkin number $M_n$ is divisible by 5 if and only if $n$ is of the form*

$$(5i+1)5^{2j} - 2,\ (5i+2)5^{2j-1} - 1,\ (5i+3)5^{2j-1} - 2,\ (5i+4)5^{2j} - 1 \tag{18}$$

*where $i, j \in \mathbb{N}$ and $j \geq 1$.*

**Proof** From equation (17) we see that $M_n \equiv_5 0$ if and only if

$$3T_n + 2T_{n+1} - T_{n+2} \equiv_5 0. \tag{19}$$

The reader can easily compute all solutions to this equation modulo 5. Note that by Theorem 4.7 we have $T_n \not\equiv_5 0$ for all $n$ since this is true for $n \leq 4$. So only the twelve solutions to (19) which do not contain any zeros need be considered.

To finish the proof, we consider each congruence class for $n$ modulo 5 separately. The three cases where there is no carry in passing from $(n)_5$ to $(n+1)_5$ or to $(n+2)_5$ are similar as are the two cases where there is such a carry. So we will only do one of each in detail.

Suppose first that $n \equiv_5 1$. Then Theorem 4.7 yields (with multiplicative inverses taken modulo 5)

$$T_{n+1} \equiv_5 T_2 T_1^{-1} T_n \equiv_5 3T_n.$$

The only solutions of (19) satisfying this condition are

$$(T_n, T_{n+1}, T_{n+2}) \equiv_5 (1,3,4),\ (2,1,3),\ (3,4,2),\ (4,2,1).$$

By the same reasoning, we also have

$$T_{n+2} \equiv_5 T_3 T_1^{-1} T_n \equiv_5 2T_n.$$

But none of the previous four solutions also satisfy this condition. Thus $T_n$ is never divisible by 5 if $n \equiv_5 1$.

Now suppose $n \equiv_5 4$. The same type of argument as in the previous paragraph shows that $T_{n+2} \equiv_5 T_{n+1}$ which reduces us to looking at solutions

$$(T_n, T_{n+1}, T_{n+2}) \equiv_5 (1, 2, 2),\ (2, 4, 4),\ (3, 1, 1),\ (4, 3, 3)$$

of (19). Note that in all cases we have $T_{n+1} = 2T_n$.

To finish this case, let $\omega_5(n+1) = k > 0$ and let $d \neq 0$ be the units digit of $(n+1)/5^k$. It follows, as usual, that $T_{n+1} \equiv_5 T_d T$ where $T = \prod_{i>k} T_{n_i}$. Also, since $T_4 \equiv_5 -1$, we have $T_n \equiv_5 (-1)^k T_{d-1} T$. Combining the equations for $T_n$ and $T_{n+1}$ to eliminate $T$ gives

$$T_{n+1} \equiv_5 (-1)^k T_d T_{d-1}^{-1} T_n.$$

Comparing this with the last equation of the previous paragraph shows that we must have $T_d \equiv_5 (-1)^k 2T_{d-1}$. Checking the four possible values for $d$ and the two possible parities for $k$ gives a solution exactly when $k$ is odd and $d = 2$ or $k$ is even and $d = 4$. These are the 2nd and 4th solutions in (18). ■

As far as prime powers, the following conjecture is also due in part to Amdeberhan [5].

**Conjecture 5.5** *We have $M_n \equiv_4 0$ if and only if*

$$n = (4i+1)4^{j+1} - 1 \text{ or } n = (4i+3)4^{j+1} - 2 \text{ where } i, j \in \mathbb{N}.$$

*Furthermore we never have $M_n \equiv_8 0$.*

## 5.3 Gould's sequence

*Gould's sequence* [33, A001316] consists of the numbers $G_n$ which count the number of odd entries in the $n$th row of Pascal's triangle. More generally, we can calculate $G_n(p)$ which is the number of entries in the $n$th row of Pascal's triangle which are not zero modulo the prime $p$. Recall the definition of $\delta_{p,j}(n)$ in (14).

**Theorem 5.6** *Let $p$ be prime. Then*

$$G_n(p) = \prod_{1 \le j < p} (j+1)^{\delta_{p,j}(n)}.$$

*Furthermore, every entry of the $n$th row of Pascal's triangle is nonzero modulo $p$ if and only if*

$$n = qp^k - 1$$

*where $1 \le q < p$ and $k \in \mathbb{N}$. In particular*

$$G_n = 2^{\delta(n)}$$

*and every entry of the $n$th row of Pascal's triangle is odd if and only if $n = 2^k - 1$ where $k \in \mathbb{N}$.*

**Proof** Suppose $\binom{n}{k} \not\equiv_p 0$ where $(n)_p = (n_i)$ and $(k)_p = (k_i)$. If $n_i = j$ then we will not have a carry in the $i$th place if and only if $0 \le k_i \le j$. So there are $j+1$ choices for $k_i$ and taking the product of the number of choices for each $i$ gives the first statement of the theorem.

Now suppose that every entry of the $n$th row is nonzero modulo $p$. Since there are no carries for all $k$, all the elements of $(n)_p$ must equal $p-1$ except for possibly the last (leading) one $n_l$. Since there can never be a carry out of $n$'s last place, we have the desired characterization of those $n$ under consideration. ■

## 5.4 Sums of binomial and trinomial coefficients

The partial sums of central binomial coefficients [33, A006134] also have nice congruence properties. The proof of the next result is easily obtained by using Theorem 4.3 and induction on $n$, so we omit it. In conjunction with Theorem 4.5, it settles a conjecture of Cloitre [33, A083096].

**Theorem 5.7** *We have*

$$\sum_{k=0}^{n} \binom{2k}{k} \equiv_3 \begin{cases} (-1)^{\delta_3(n)} & \text{if } n \in 3T(01), \\ 0 & \text{otherwise.} \end{cases} \quad ■$$

With regards to Corollary 4.8 about divisibility of the central trinomial coefficients, both Wilson and the authors have independently made a conjecture which we have been unable to prove by our techniques.

**Conjecture 5.8** *Let $p$ be a prime and consider $n < p$. Then $T_n$ is divisible by $p$ if and only if $T_{p-n-1}$ is divisible by $p$.*

## 5.5 Apéry numbers and central Delannoy numbers

We can generalize our results about the central trinomial numbers as follows. Given positive integers $r, s$ we define the corresponding *generalized Apéry numbers* to be

$$a_n(r,s) = \sum_{k \ge 0} \binom{n}{k}^r \binom{n+k}{k}^s. \tag{20}$$

Note that since $r, s$ are positive, each term in this sum will have a factor of

$$\binom{n}{k}\binom{n+k}{k} = \binom{n+k}{k,\ k,\ n-k}.$$

Using this fact we can prove the following result. Since the demonstration is similar to that of Theorem 4.7, it is omitted.

**Theorem 5.9** *Let $p$ be a prime and let $(n)_p = (n_i)$. Then*

$$a_n(r,s) \equiv_p \prod_i a_{n_i}(r,s). \qquad \blacksquare$$

The *central Delannoy numbers* [33, A001850] are $D_n = a_n(1,1)$. Also, the *Apéry numbers* [33, A005258 and A005259] are $a_n = a_n(2,1)$ and $A_n = a_n(2,2)$. We will call them the *Apéry numbers of the first and second kinds*, respectively. They arose in Apéry's work on the irrationality of $\zeta(2)$ and $\zeta(3)$. For an exposition of Apéry's work, see the article of van der Poorten [27]. Gessel [17] proved Theorem 5.9 for the Apéry numbers of the second kind. His motivation was to prove generalizations of various conjectures of Chowla, Cowles, and Cowles [9]. These conjectures were also proved by Mimura [26] and (partly) by Radoux [28].

A number of Gessel's other results and their demonstrations generalize easily to $a_n(r,s)$ for arbitrary $r, s$. As an example, we prove the following.

**Theorem 5.10** *Let $p > 3$ be a prime and suppose $r \geq 2, s \geq 1$. Then*

$$a_{pn}(r,s) \equiv a_n(r,s) \pmod{p^3}.$$

**Proof** Partitioning the sum (20) depending on whether $p$ divides $k$ gives $a_{pn}(r,s) = S_1 + S_2$ where

$$S_1 = \sum_{j\geq 0} \binom{pn}{pj}^r \binom{pn+pj}{pj}^s$$

and

$$S_2 = \sum_{k=1}^{p-1}\sum_{j\geq 0} \binom{pn}{pj+k}^r \binom{pn+pj+k}{pj+k}^s.$$

Kazandzidis [21] has shown that for a prime $p > 3$ one has

$$\binom{pn}{pj} \equiv \binom{n}{j} \pmod{p^3}.$$

So we have $S_1 \equiv a_n(r,s) \pmod{p^3}$ and thus can finish the proof by showing that $S_2 \equiv 0 \pmod{p^3}$.

Note that we have the polynomial congruence

$$(1+x)^{p-1}(1+x) = (1+x)^p \equiv 1 + x^p \pmod{p},$$

from which it follows that $\binom{p-1}{i} \equiv (-1)^i \pmod{p}$ for $0 \leq i < p$. Using Lucas' Theorem and this congruence gives

$$\binom{pn}{pj+k} = \frac{pn}{pj+k}\binom{pn-1}{pj+k-1} \equiv \frac{pn}{k}\binom{n-1}{j}\binom{p-1}{k-1}$$

$$\equiv (-1)^{k-1}\frac{pn}{k}\binom{n-1}{j} \pmod{p^2}.$$

Also, by Lucas again,

$$\binom{pn+pj+k}{pj+k} \equiv \binom{n+j}{j} \pmod{p}.$$

Now if $a \equiv pb \pmod{p^2}$ and $c \equiv d \pmod{p}$ then $a^r c^s \equiv (pb)^r d^s \pmod{p^3}$ because $r \geq 2$. So

$$\begin{aligned} S_2 &\equiv \sum_{k=1}^{p-1}\sum_{j\geq 0}(-1)^{r(k-1)}\left(\frac{pn}{k}\right)^r\binom{n-1}{j}^r\binom{n+j}{j}^s \pmod{p^3} \\ &= (pn)^r\left[\sum_{k=1}^{p-1}\frac{(-1)^{r(k-1)}}{k^r}\right]\left[\sum_{j\geq 0}\binom{n-1}{j}^r\binom{n+j}{j}^s\right]. \end{aligned}$$

If $r \geq 3$ then this expression is clearly divisible by $p^3$. If $r = 2$ then it is known that $\sum_{k=1}^{p-1} 1/k^2$ is divisible by $p$ (for a proof see [20, p. 90]), so we still have divisibility of the entire expression by $p^3$. This completes the proof of the theorem. ■

Theorem 5.9 is a generalized Apéry number analogue of Lucas' Theorem. From it, we immediately get the following corollary.

**Corollary 5.11** *Let $p$ be a prime and let $(n)_p = (n_i)$. Then $p$ divides $a_n(r,s)$ if and only if $p$ divides $a_{n_i}(r,s)$ for some $n_i$.*

We have already seen how one can get a similar result about binomial coefficients, namely Corollary 4.2, from Lucas' Theorem. But as mentioned then, there is a stronger statement due to Kummer [23].

**Theorem 5.12 (Kummer)** *Let $p$ be prime. Then $\omega_p\binom{k+l}{k}$ is the number of carries in performing the base $p$ addition $(k)_p + (l)_p$.* ■

Beukers [8] conjectured an analogue of Kummer's Theorem for Apéry numbers of the second kind modulo 5 and 11. We believe that this generalizes.

**Conjecture 5.13** *Let $p$ be prime and let*

$$S = \{m < p \ : \ a_m(r, s) \equiv_p 0\}.$$

*Given $n$ with $(n)_p = (n_i)$, let $q$ be the number of indices $i$ with $n_i \in S$. Then $p^q$ divides $a_n(r, s)$.*

Finally, we note that Theorem 5.9 can be used to settle a conjecture of Cloitre [10] and also to show that the Apéry numbers of the first kind and central Delannoy numbers are the same modulo 3. Since the reader will have no trouble filling in the details at this point, the proofs are omitted.

**Theorem 5.14** *Let $r, s$ be positive integers. Then*

$$a_n(r, s) \equiv_3 \begin{cases} (-1)^{\delta_3(n)} & \text{if } s \text{ is even,} \\ 1 & \text{if } s \text{ is odd and } n \in T(02), \\ 0 & \text{otherwise.} \end{cases} \quad \blacksquare$$

**Theorem 5.15** *The Apéry numbers of the first kind and the central Delannoy numbers satisfy*

$$a_n \equiv_3 D_n \equiv_3 \begin{cases} 1 & \text{if } n \in T(02), \\ 0 & \text{otherwise.} \end{cases} \quad \blacksquare$$

## 5.6 Central Eulerian numbers

The *Eulerian numbers* [33, A008292] are denoted $A(n, k)$ and count the number of permutations in the symmetric group $\mathfrak{S}_n$ which have $k - 1$ descents. They can be written as

$$A(n, k) = \sum_{i=0}^{k} (-1)^i (k - i)^n \binom{n+1}{i}.$$

Since the odd numbered rows have an odd number of elements, we define the *central Eulerian numbers* to be

$$E_n = A(2n - 1, n) = \sum_{i=0}^{n} (-1)^i (n - i)^{2n-1} \binom{2n}{i}.$$

We have the following congruence for these numbers.

**Theorem 5.16** *The central Eulerian numbers satisfy*

$$E_n \equiv_3 \begin{cases} 1 & \text{if } n \in T(01)+1, \\ 0 & \text{otherwise.} \end{cases}$$

**Proof** Note that $k^{2n-1} = k$ for $k = 0, \pm 1$. So we have

$$E_n \equiv_3 \sum_{i=0}^{n} (-1)^i (n-i) \binom{2n}{i}.$$

Applying the binomial recursion to this sum twice yields, after massive cancellation,

$$E_n \equiv_3 (-1)^{n-1} \binom{2n-2}{n-1}.$$

Now Theorem 4.3 will finish the proof provided $n + \delta_3(n)$ is always even. But this is easy to show by induction on $n$, so we are done. ■

Rows in the Eulerian triangle are symmetric, so even numbered rows have two equal elements in the middle. We will call these elements *bicentral*. Cloitre conjectured the residues of these elements modulo 3. Since the proof of this result is similar to the one just given, we will omit it.

**Theorem 5.17** *The bicentral Eulerian numbers satisfy*

$$A(2n,n) \equiv_3 \begin{cases} 1 & \text{if } n \in 3T(01)+1, \\ -1 & \text{if } n \in 3T(01) \text{ or } 3T(01)+2, \\ p0 & \text{otherwise.} \end{cases} \quad ■$$

## 5.7 Noncrossing connected graphs

Noncrossing set partitions are an important object of study in combinatorics. An excellent survey of the area can be found in the article of Simion [31]. Noncrossing graphs are a generalization of noncrossing partitions which have been studied by Flajolet and Noy [16]. Consider vertices labeled $1, \ldots, n$ and arranged clockwise in this order around a circle. A graph on this vertex set is *noncrossing* if, when the edges are drawn with straight line segments between the vertices, none of the edges cross. Let $N_n$ be the number on noncrossing connected graphs on $n$ vertices [33, A007297]. Then it can be shown that

$$N_n = \frac{1}{n-1} \sum_{k \geq 0} \binom{3n-3}{n+k+1} \binom{k}{n-2}.$$

We have the following conjecture about the residue of $N_n$ modulo 3.

**Conjecture 5.18** *The number of noncrossing connected graphs satisfies*

$$N_n \equiv_3 \begin{cases} 1 & \textit{if } n = 3^i \textit{ or } n = 2 \cdot 3^i \textit{ for some } i \in \mathbb{N}, \\ -1 & \textit{if } n = 3^i + 3^j \textit{ for two distinct } i, j \in \mathbb{N}, \\ 0 & \textit{otherwise.} \end{cases}$$

In the first two cases, it is not hard to show that the congruence holds using Lucas' Theorem because of the very specific form of $(n)_3$. However, we have been unable to prove that for all remaining $n$ one always has $N_n$ divisible by 3. It would be even more interesting to give a combinatorial proof of this result based on symmetries of the graphs involved.

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