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Basic Derivations for Subarrangements of Coxeter Arrangements

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Abstract. We prove that various subarrangements of Coxeter hyperplane arrangements are free. We do this by exhibiting a basis for the corresponding module of derivations. Our method uses a theorem of Saito [24] and Terao [30] which checks for a basis using certain divisibility and determinantal criteria. As a corollary, we find the roots of the characteristic polynomials for these arrangements, since they are just one more than the degrees in any basis of the module. We will also see some interesting applications of symmetric and supersymmetric functions along the way.

Keywords: hyperplane arrangement, derivation, basis, Coxeter

1. Introduction

Our aim is to show that certain subarrangements of the Coxeter arrangements are free by explicitly calculating bases for the corresponding modules of derivations. As immediate corollaries, we will be able to read off the roots of their characteristic polynomials. First, however, we need to set up some definitions and notation. We will follow the book of Orlik and Terao [19] as much as possible.

Let K be a field, and let

$$\mathcal{A} = \{H_1, \ldots, H_k\} \tag{1}$$

be an arrangement (finite set) of hyperplane subspaces in K^n. Thus all our hyperplanes will be *central*, i.e., going through the origin. Let $L = L(\mathcal{A})$ be the poset of intersections of these hyperplanes ordered by reverse inclusion. Thus L has a unique minimal element $\hat{0}$ corresponding to K^n, an atom corresponding to each H_i, and a unique maximal element $\hat{1}$ corresponding to $\cap_{1\leq i\leq k} H_i$. It is well known that L is a geometric lattice with rank function

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$$\text{rk } X = n - \dim X$$

for any $X \in L$. Let $\mu(X) = \mu(\hat{0}, X)$ denote the Möbius function of the lattice. Then the *characteristic polynomial of* L is

$$\chi(\mathcal{A}, t) = \sum_{X \in L(\mathcal{A})} \mu(X) t^{\dim X}. \tag{2}$$

If $\chi(\mathcal{A}, r) = 0$ then r is called an *exponent* of $\mathcal{A}$. (When $\mathcal{A}$ is an arrangement coming from a root system, then r is an exponent of the corresponding Weyl group.) The multiset (= set with repetitions) of exponents of $\mathcal{A}$ will be denoted $\exp(\mathcal{A})$.

Now consider the polynomial algebra $A = K[x_1, \ldots, x_n] = K[x]$ with the usual grading by total degree $A = \oplus_{i \geq 0} A_i$. A *derivation* is a K-linear map

$$\theta : A \to A$$

satisfying

$$\theta(fg) = f\theta(g) + g\theta(f)$$

for any $f, g \in A$. The set of all derivations is an A-module. It is graded by saying that θ has degree d if $\theta(A_i) \subseteq A_{i+d}$. This module is free with a basis given by the operators $\partial/\partial x_1, \ldots, \partial/\partial x_n$. It will often be convenient to display a derivation as a column vector whose entries are its components with respect to this basis. Thus if

$$\theta = p_1(x)\partial/\partial x_1 + \cdots + p_n(x)\partial/\partial x_n$$

where $p_i(x) \in K[x]$ for all i, then we write

$$\theta = \begin{bmatrix} p_1(x) \\ \vdots \\ p_n(x) \end{bmatrix} = \begin{bmatrix} \theta(x_1) \\ \vdots \\ \theta(x_n) \end{bmatrix}$$

Let $\mathbf{e}_1, \ldots, \mathbf{e}_n$ denote the coordinate vectors in K^n with the variables $x_1, \ldots, x_n$ being considered as elements of the corresponding dual basis. So any hyperplane $H \subseteq K^n$ is defined by an equation

$$\alpha_H(x_1, \ldots, x_n) = 0$$

where α_H is a linear polynomial. Thus, the arrangement $\mathcal{A}$ in (1) is determined by the form

$$Q = Q(\mathcal{A}) = \prod_i \alpha_{H_i}(x).$$

Consider the associated *module of $\mathcal{A}$-derivations* defined by

$$D(\mathcal{A}) = \{\theta \mid \theta \text{ a derivation and } \theta(Q) \in Q \cdot K[x]\}.$$

We say that $\mathcal{A}$ is a *free arrangement* if $D(\mathcal{A})$ is a free A-module. Terao first introduced free arrangements and proved the following fundamental theorem [29, 30]. A simpler proof was obtained with Solomon [25].

THEOREM 1.1. *If $\mathcal{A}$ is free then*

(1) *$D(\mathcal{A})$ has a homogeneous basis $\theta_1, \ldots, \theta_n$,*
(2) *the set*

$$\{d_1, \ldots, d_n\} = \{\deg \theta_1, \ldots, \deg \theta_n\}$$

depends only on $\mathcal{A}$,
(3) *the exponents of $\mathcal{A}$ are the nonnegative integers*

$$\exp(\mathcal{A}) = \{d_1 + 1, \ldots, d_n + 1\}.$$

Notice that if θ_i is presented in matrix format, then the corresponding exponent can be read off as the degree of one (any) of the entries.

In order to find such homogeneous bases, we use a result whose holomorphic version is due to Saito [24], and whose algebraic analogue comes from Terao [30] and Solomon-Terao [25]. Given any set of derivations $\theta_1, \ldots, \theta_n$, consider the rectangular matrix whose columns are the corresponding column vectors

$$\Theta = [\theta_1, \ldots, \theta_n] = [\theta_j(x_i)].$$

THEOREM 1.2. *Suppose $\theta_1, \ldots, \theta_n \in D(\mathcal{A})$ where $\mathcal{A}$ has defining form Q. Then the following conditions are equivalent:*

(1) $\det \Theta = cQ$ *where $c \in K$ is nonzero,*
(2) *$\mathcal{A}$ is free with basis $\theta_1, \ldots, \theta_n$.*

Thus, we can prove that an arrangement $\mathcal{A}$ is free by constructing homogeneous derivations that

(1) are in the submodule of $\mathcal{A}$-derivations and
(2) have the proper determinant.

If $\mathcal{A}$ is an arrangement, then we will use $\Theta(\mathcal{A})$ to denote the set of all matrices Θ corresponding to a basis of $D(\mathcal{A})$.

We can simplify the verification of the first of these two conditions as follows. Note that $Q \mid \theta(Q)$ if and only if $\alpha_H \mid \theta(Q)$ for all hyperplanes H in the arrangement, since the α_H are relatively prime. Furthermore, $\alpha_H \mid \theta(Q)$ is equivalent to $\alpha_H \mid \theta(\alpha_H)$, since θ is a derivation. Thus, we have our basic tool in the following result (see [19, Proposition 4.8]).

COROLLARY 1.1. *Let $\theta_1, \ldots, \theta_n$ be derivations and let $\mathcal{A}$ be an arrangement. Then the θ_i form a basis for $D(\mathcal{A})$ if*

1. $\alpha_H \mid \theta_i(\alpha_H)$ *for all* $H \in \mathcal{A}$, *and*
2. $\det\Theta = cQ$ *where* $c \in K$ *is nonzero.*

Many of our free arrangements will come from those which interpolate between two Coxeter arrangements. Certainly, any finite set $P \subseteq K^n$ of vectors gives rise to the arrangement whose hyperplane subspaces are the $H = p^\perp$ for $p \in P$. Here, orthogonal complement is being taken using the standard bilinear form on K^n with respect to the basis $\mathbf{e}_1, \ldots, \mathbf{e}_n$. Let R and S be root systems with $R \subset S$. By adding the roots of $S \backslash R$ to R one at a time, one obtains a sequence of subsets each of which determines a hyperplane arrangement. It turns out that these arrangements are often free and so the associated characteristic polynomials factor over the nonnegative integers. Zaslavsky [31] was the first to consider the family of hyperplane arrangements interpolating between the root systems D_n and B_n. These investigations were continued by Cartier [4], Orlik and Solomon [18], Orlik-Solomon-Terao [12, Example 2.6], Ziegler [34], and Hanlon [11]. Surprisingly, other interpolating families seem not to have been studied previously, even though they are related to the notion of inductive freeness.

2. Linear interpolations

In this section we will consider the cases where the number of roots in $S \backslash R$ is a linear function of the dimension of R. First, however, we must introduce our fundamental arrangements, which will be the three infinite families of Coxeter arrangements. It will be convenient in what follows to use the notation $\{\mathbf{v}_1, \ldots, \mathbf{v}_n\}^\perp = \{\mathbf{v}_1^\perp, \ldots, \mathbf{v}_n^\perp\}$. Now let

$$\mathcal{A}_{n-1} = \{\mathbf{e}_i - \mathbf{e}_j : 1 \le i < j \le n\}^\perp \tag{3}$$

and

$$\mathcal{B}_n = \mathcal{A}_{n-1} \cup \{\mathbf{e}_i + \mathbf{e}_j : 1 \le i < j \le n\}^\perp \cup \{\mathbf{e}_i : 1 \le i \le n\}^\perp \tag{4}$$

and

$$\mathcal{D}_n = \mathcal{B}_n \backslash \cup \{\mathbf{e}_i : 1 \le i \le n\}^\perp \tag{5}$$

The arrangements in (3), (4), and (5) are said to be *Coxeter arrangements of type* A, B, and D, respectively. If K is of characteristic 2 then $\mathcal{A}_{n-1} = \mathcal{D}_n$. To avoid this degeneracy, we assume in the rest of this paper that the characteristic of K is not 2.

To describe the matrices in $\Theta(\mathcal{A})$ for each case, we will need to define some derivations. Let

$$\mathbf{X}^d = \begin{bmatrix} x_1^d \\ x_2^d \\ \vdots \\ x_n^d \end{bmatrix} \qquad \text{and} \qquad \hat{\mathbf{X}}_n = \begin{bmatrix} \hat{x}_1 \\ \hat{x}_2 \\ \vdots \\ \hat{x}_n \end{bmatrix}$$

where $\hat{x}_i = x_1 x_2 \cdots x_{i-1} x_{i+1} \cdots x_{n-1} x_n$. We will sometimes use the abbreviation $\mathbf{X}^1 = \mathbf{X}$. Although our notation does not take the number of variables into account, we always assume it is n unless stated otherwise. This given, we have the following theorem (see Orlik [16]).

THEOREM 2.1. *The Coxeter arrangements (3)–(5) are free. They have defining forms, basis matrices, and exponents as follows.*

(1) *For type A,*

$$Q(\mathcal{A}_{n-1}) = \prod_{1 \le i < j \le n} (x_i - x_j)$$
$$[\mathbf{X}^0, \mathbf{X}^1, \mathbf{X}^2, \ldots, \mathbf{X}^{n-1}] \in \Theta(\mathcal{A}_{n-1})$$
$$\exp(\mathcal{A}_{n-1}) = \{0, 1, 2, \ldots, n-1\}.$$

(2) *For type B,*

$$Q(\mathcal{B}_n) = x_1 x_2 \cdots x_n \prod_{1 \le i < j \le n} (x_i^2 - x_j^2)$$
$$[\mathbf{X}^1, \mathbf{X}^3, \mathbf{X}^5, \ldots, \mathbf{X}^{2n-1}] \in \Theta(\mathcal{B}_n)$$
$$\exp(\mathcal{B}_n) = \{1, 3, 5, \ldots, 2n-1\}.$$

(3) *For type D,*

$$Q(\mathcal{D}_n) = \prod_{1 \le i < j \le n} (x_i^2 - x_j^2)$$
$$[\mathbf{X}^1, \mathbf{X}^3, \mathbf{X}^5, \ldots, \mathbf{X}^{2n-3}, \hat{\mathbf{X}}_n] \in \Theta(\mathcal{D}_n)$$
$$\exp(\mathcal{D}_n) = \{1, 3, 5, \ldots, 2n-3, n-1\}.$$

First we will interpolate between $\mathcal{A}_{n-1}$ and $\mathcal{A}_n$. Let

$$\mathcal{A}_{n,k} = \mathcal{A}_{n-1} \cup \{\mathbf{e}_1 - \mathbf{e}_{n+1}, \mathbf{e}_2 - \mathbf{e}_{n+1}, \ldots, \mathbf{e}_k - \mathbf{e}_{n+1}\}^{\perp}.$$

It will also be convenient to let $\mathbf{P}_d = \partial/\partial x_d$. So, as a column vector, $\mathbf{P}_d$ has a one in the dth position and zeros elsewhere. The next result should be compared with [12, Example 2.6] and [19, Example 4.55].

THEOREM 2.2. *The arrangements $\mathcal{A}_{n,k}$ are free with*

$$Q(\mathcal{A}_{n,k}) = c(x)Q(\mathcal{A}_{n-1}) \quad \textit{where } c(x) = \prod_{1\leq i\leq k}(x_i - x_{n+1})$$
$$[\mathbf{X}^0, \mathbf{X}, \mathbf{X}^2, \ldots, \mathbf{X}^{n-1}, c(x)\mathbf{P}_{n+1}] \in \Theta(\mathcal{A}_{n,k})$$
$$\exp(\mathcal{A}_{n,k}) = \{0, 1, 2, \ldots, n-1, k\}$$

where we view $\mathcal{A}_{n,0}$ as an arrangement in K^{n+1}.

Proof. The equation for Q is obvious. Also, the fact about the exponent set will follow immediately from our assertion about $\Theta(\mathcal{A}_{n,k})$. Thus, we will only prove the latter.

We first check the divisibility condition in Corollary 1.1. All the derivations of the form $\mathbf{X}^d$ are part of a basis for $D(\mathcal{A}_n)$. Thus, we automatically have $\alpha_H \mid \mathbf{X}^d(\alpha_H)$ for all $H \in \mathcal{A}_{n,k} \subseteq \mathcal{A}_n$. As for $\theta = c(x)\mathbf{P}_{n+1}$, there are two possibilities. If $\alpha_H = x_i - x_j$ where $i < j < n+1$, then $\theta(\alpha_H) = 0$ since only the $n+1$st entry of $\mathbf{P}_{n+1}$ is nonzero. So clearly $\alpha_H \mid \theta(\alpha_H)$ in this case. If $\alpha_H = x_i - x_{n+1}$ where $i \leq k$, then $\alpha_H \mid c(x)$. Thus, again, $\alpha_H \mid \theta(\alpha_H)$.

As far as the determinant criterion in Corollary 1.1, let Θ be the matrix in the statement of the theorem. Then we have

$$\begin{aligned}\det\, \Theta &= \det[\mathbf{X}^0, \mathbf{X}, \mathbf{X}^2, \ldots, \mathbf{X}^{n-1}, c(x)\mathbf{P}_{n+1}] \\ &= c(x)\, \det[\mathbf{X}^0, \mathbf{X}, \mathbf{X}^2, \ldots, \mathbf{X}^{n-1}, \mathbf{P}_{n+1}] \\ &= c(x)\, \det[\mathbf{X}^0, \mathbf{X}, \mathbf{X}^2, \ldots, \mathbf{X}^{n-1}] \\ &= \pm c(x)Q(\mathcal{A}_{n-1}). \quad \square\end{aligned}$$

It is instructive to make a table of the exponents for the interpolating arrangements of Theorem 2.2. Assume that $n = 3$. So $\mathcal{A}_{3,0}$ (which is $\mathcal{A}_2$ except for the ambient space) has $\exp(\mathcal{A}_{3,0}) = \{0, 1, 2, 0\}$ while $\mathcal{A}_{3,3}$ (which is exactly $\mathcal{A}_3$) has $\exp(\mathcal{A}_{3,3}) = \{0, 1, 2, 3\}$. For the intermediate steps, we have

k	$\exp(\mathcal{A}_{3,k})$
0	0, 1, 2, 0
1	0, 1, 2, *1*
2	0, 1, 2, *2*
3	0, 1, 2, *3*

where we have set the exponent that has changed in italic. The behavior of the changing exponent mirrors the fact that the last basis element of $\mathcal{A}_{n,k}$ is obtained from that of $\mathcal{A}_{n,k-1}$ by multiplying by $x_k - x_{n+1}$, which is of degree one. We encourage the reader to make such an exponent table for each theorem that follows.

Many of our proofs will use the same reasoning as Theorem 2.2. Thus, we will pass over the facts about the defining form and exponent multiset without mention. Furthermore, checking the divisibility condition often uses a few simple facts about derivations which we collect in the next lemma for easy reference.

LEMMA 2.1. *The following divisibility results hold.*

(1) *For all i, we have $x_i \mid \theta(x_i)$ if and only if x_i divides the ith entry of the column vector for θ.*
(2) *If there exists a polynomial $p(t)$ such that $\theta(x_i) = p(x_i)$ for all i, then we have $x_i - x_j \mid \theta(x_i - x_j)$ for all i, j.*
(3) *For all i, j, m we have $x_i + x_j \mid \mathbf{X}^{2m+1}(x_i + x_j)$.*
(4) *For all i and j, if $x_i + x_j$ divides the ith and jth entries of the column vector for θ then $x_i + x_j \mid \theta(x_i + x_j)$.*
(5) *For all $i, j \leq n$ we have $x_i \pm x_j \mid \hat{\mathbf{X}}_n(x_i \pm x_j)$.*

Proof. Statements 1 and 4 are obvious. For 2, merely note that $\theta(x_i - x_j) = p(x_i) - p(x_j)$. Finally, 3 and 5 follow by direct calculation. □

Note that for the multiset of exponents, it is really immaterial in which order we add the hyperplanes of $\mathcal{A}_n \backslash \mathcal{A}_{n-1}$. This is because the lattices of corresponding intermediate arrangements will be isomorphic. (Strictly speaking, $\mathcal{A}_{n-1}$ is not contained in $\mathcal{A}_n$ because of the difference in dimension. However, we will ignore such facts whenever it does no harm to do so.) In fact for an arbitrary order, the only change needed in the statement of the previous theorem is to let $c(x)$ be the product of the linear forms forms for the hyperplanes added so far. This will be true when interpolating between $\mathcal{B}_{n-1}$ and $\mathcal{B}_n$, or between $\mathcal{D}_n$ and $\mathcal{B}_n$, but not in any of the other cases we consider.

To go from $\mathcal{B}_{n-1}$ to $\mathcal{B}_n$, take any linear ordering of the hyperplanes of $\mathcal{B}_n \backslash \mathcal{B}_{n-1}$, say $H_1, H_2, \ldots, H_{2n-1}$. Then define

$$\mathcal{B}_{n,k} = \mathcal{B}_{n-1} \cup \{H_1, H_2, \ldots, H_k\}.$$

The proof of the next theorem is so similar to that of Theorem 2.2 that it will be omitted.

THEOREM 2.3. *The arrangements $\mathcal{B}_{n,k}$ are free with*

$$Q(\mathcal{B}_{n,k}) = c(x)Q(\mathcal{B}_{n-1}) \quad \text{where } c(x) = \textstyle\prod_{1\leq i\leq k}\alpha_{H_i}$$
$$[\mathbf{X}, \mathbf{X}^3, \mathbf{X}^5, \ldots, \mathbf{X}^{2n-3}, c(x)\mathbf{P}_n] \in \Theta(\mathcal{B}_{n,k})$$
$$\exp(\mathcal{B}_{n,k}) = \{1, 3, 5, \ldots, 2n-3, k\}$$

where we view $\mathcal{B}_{n,0}$ as an arrangement in K^n.

However, when we interpolate between $\mathcal{D}_{n-1}$ and $\mathcal{D}_n$, order does make a difference. In fact, if one considers the arrangement

$$\mathcal{A} = \mathcal{D}_3 \cup \{\mathbf{e}_1 + \mathbf{e}_4, \mathbf{e}_1 - \mathbf{e}_4\}^{\perp}$$

then $\chi(\mathcal{A}, t)$ can be easily calculated using deletion and restriction [19]. One finds that $\chi(\mathcal{A}, t) = (t-1)(t-3)(t^2 - 4t + 5)$ which does not factor over the integers. Thus, $\mathcal{A}$ cannot be free by Theorem 1.1. So we will add the hyperplanes of $\mathcal{D}_n \backslash \mathcal{D}_{n-1}$ by putting in all those of the form $(\mathbf{e}_i - \mathbf{e}_n)^{\perp}$ first, followed by all of those looking like $(\mathbf{e}_i + \mathbf{e}_n)^{\perp}$. If we do this, then the second half of the interpolation will be exactly like part of one of our interpolations from $\mathcal{A}_{n-1}$ to $\mathcal{D}_n$. For this reason, we will postpone the details until Section 4.

For the last interpolation of this section we will go between the arrangements $\mathcal{D}_n$ and $\mathcal{B}_n$. Take an arbitrary order $H_1, \ldots, H_n$ of $\mathcal{B}_n \backslash \mathcal{D}_n$ and let

$$\mathcal{DB}_{n,k} = \mathcal{D}_n \cup \{H_1, H_2, \ldots, H_k\}.$$

The proof of the following theorem again closely follows the model of Theorem 2.2. and so is left to the reader.

THEOREM 2.4. *The arrangements $\mathcal{DB}_{n,k}$ are free with*

$$Q(\mathcal{DB}_{n,k}) = c(x)Q(\mathcal{D}_n) \quad \text{where} c(x) = \textstyle\prod_{1\le i\le k} \alpha_{H_i}$$

$$[\mathbf{X}, \mathbf{X}^3, \mathbf{X}^5, \ldots, \mathbf{X}^{2n-3}, c(x)\hat{\mathbf{X}}_n] \in \Theta(\mathcal{DB}_{n,k})$$

$$\exp(\mathcal{DB}_{n,k}) = \{1, 3, 5, \ldots, 2n-3, n+k-1\}.$$

3. Determinantal identities

In this section we collect various determinantal formulas needed in the sequel. At first we proved them directly. Later we learned from Bernard Leclerc that they followed from some classical identities for alternants. We would like to thank him for bringing Garbieri's theorem (Theorem 3.1) to our attention and supplying bibliographical information.

Let L be a commutative ring with unity and consider $f_1(t), \ldots, f_n(t) \in L[t]$. Let

$$m = \max\{n, 1 + \deg f_1, 1 + \deg f_2, \ldots, 1 + \deg f_n\}$$

so that $m \geq n$ and each f_i has at most m nonzero coefficients. Now write

$$f_i(t) = \sum_{j=1}^{m} c_{ij} t^{j-1}$$

where $c_{ij} \in L$ for all i, j. If $x_1, \ldots, x_m$ are indeterminates over $\mathbf{Z}$, then we define an $n \times m$ matrix (classically called the *alternant* of $f_1, \ldots, f_n$ if $n = m$)

$$D(f;x) = [f_i(x_j)]_{n\times m}.$$

Notice that we will use a subscript outside of the brackets to give the dimensions of a matrix, if necessary. Also, if B is a submatrix of C and we write $B = [b_{ij}]_{n\times m}$, then we are assuming $1 \le i \le n$ and $1 \le j \le m$, regardless of B's position within C.

If $n = m$, then the determinant of $D(f;\, x)$ is denoted $\Delta(f;\, x)$, considered as an element of $L[x_1, \dots, x_n]$. In the special case when $f_i(t) = t^{i-1}$, $1 \le i \le n$, we get the matrix $D(x)$ and the familiar Vandermonde determinant $\Delta(x)$.

We will need to use *circulant* matrices, which are those of the form

$$\begin{bmatrix} a_1 & a_2 & a_3 & \cdots & a_m \\ a_m & a_1 & a_2 & \cdots & a_{m-1} \\ a_{m-1} & a_m & a_1 & \cdots & a_{m-2} \\ \vdots & \vdots & \vdots & & \vdots \\ a_{m-n+2} & a_{m-n+3} & a_{m-n+4} & \cdots & a_{m-n+1} \end{bmatrix} \overset{\text{def}}{=} [[a_1,\, a_2,\, a_3,\, \cdots,\, a_m]]_{n\times m}.$$

Also let $e_j(d) = e_j(x_1, \dots, x_d)$ be the jth elementary symmetric function in the first d indeterminates $x_1, \dots, x_d$. Note that $e_j(d) = 0$ if $j < 0$ or $j > d$. It will be convenient to consider the corresponding generating functions

$$E_d(t) = \prod_{i=1}^{d}(t + x_i) = \sum_{j=0}^{d} e_j(d)t^{d-j} \tag{6}$$

and

$$\bar{E}_d(t) = \prod_{i=1}^{d}(t - x_i) = \sum_{j=0}^{d}(-1)^j e_j(d)t^{d-j}. \tag{7}$$

We will repeatedly use the fact that $\bar{E}_d(x_i) = 0$ for $i \le d$.

Let C denote the $m \times m$ matrix with block form

$$C = \left[\frac{A}{B}\right]$$

where

$$A = [c_{ij}]_{n\times m}$$

and

$$B = [[(-1)^n e_n(n),\, (-1)^{n-1}e_{n-1}(n), \cdots,\, e_0(n),\, 0,\, \cdots,\, 0]]_{(m-n)\times m}. \tag{8}$$

Note that B encodes the coefficients of $\bar{E}_n$ and that $B = \emptyset$ if $m = n$.

We are now ready to state the main determinantal identity of this section. The following theorem is due to Garbieri [10]; see also [14, 15]. For completeness, we give a proof from [15].

THEOREM 3.1. *Use the preceding notation and assume that L is an integral domain. Then we have*

$$\Delta(f;\, x) = \det\ C \cdot \Delta(x)$$

in $L[x_1, \ldots, x_n]$.

Proof. Recall that $m \geq n$ and let $y_1, y_2, \ldots, y_{m-n}$ be a set of auxiliary variables. (The set is empty if $m = n$.) Consider the auxiliary functions

$$g_i(t) = t^{i-1}\bar{E}_n(t)$$

where $1 \leq i \leq m - n$. Then

$$C \cdot D(x,\, y) = \left[\begin{array}{c|c} D(f;\, x) & D(f;\, y) \\ \hline D(g;\, x) & D(g;\, y) \end{array}\right]$$

Notice that $D(g;\, x)$ is the zero matrix since $g_i(x_j) = 0$ for $1 \leq j \leq n$. Therefore, taking determinants, we get

$$\begin{aligned} \det\ C \cdot \Delta(x,\, y) &= \Delta(f;\, x) \cdot \Delta(g;\, y) \\ &= \Delta(f;\, x)\bar{E}_n(y_1)\bar{E}_n(y_2)\cdots\bar{E}_n(y_{m-n})\Delta(y). \end{aligned}$$

Since $\Delta(x,\, y) = \Delta(x)\Delta(y)\bar{E}_n(y_1)\bar{E}_n(y_2)\cdots\bar{E}_n(y_{m-n})$, we finally obtain the desired $\Delta(f;\, x) = \det\ C \cdot \Delta(x)$ because $L[x_1, \ldots, x_n]$ has no zero divisors. □

To compute the determinants of the basis matrices in the next section, we need some corollaries of this result. Consider the column vector corresponding to equation (6), namely

$$\mathbf{E}_d = \begin{bmatrix} E_d(x_1) \\ E_d(x_2) \\ \vdots \\ E_d(x_n) \end{bmatrix}$$

Then we have the following determinant.

COROLLARY 3.1. *If $1 \leq l \leq n$, then*

$$\det[\mathbf{X}^0, \mathbf{X}^2, \ldots, \mathbf{X}^{2l-2}, \mathbf{E}_l, \ldots, \mathbf{E}_{n-1}] = (-1)^{c(n,l)}\Delta(x) \prod_{1\leq i<j\leq l} (x_i + x_j)$$

where

$$c(n, l) = \binom{l}{2} + \sum_{i=0}^{2l-2-n} (n + i - 1)$$

Proof. The matrix in question is the transpose of the matrix $D(f; x)$ where

$$f_i(t) = t^{2i-2}, \quad i = 1, \ldots, l,$$
$$f_i(t) = E_{i-1}(t), i = l+1, \ldots, n.$$

So $[\mathbf{X}^0, \mathbf{X}^2, \ldots, \mathbf{X}^{2l-2}, \mathbf{E}_l, \ldots, \mathbf{E}_{n-1}] = \Delta(f; x)$.

Let $m = \max\{2l-1, n\}$. Then, by Theorem 3.1,

$$\Delta(f; x) = \Delta(x) \cdot \det \begin{bmatrix} A_1 \\ \hline A_2 \\ \hline B \end{bmatrix}_{m \times m} \tag{9}$$

where

$$A_1 = \begin{bmatrix} 1 & 0 & 0 & 0 & 0 & 0 & \cdots \\ 0 & 0 & 1 & 0 & 0 & 0 & \cdots \\ 0 & 0 & 0 & 0 & 1 & 0 & \cdots \\ & & & \vdots & & & \end{bmatrix}_{l \times m}$$
$$A_2 = [e_{l+i-j}(l+i-1)]_{(n-l) \times m}$$

and B is an in equation (8). Expanding about the rows in A_1 we obtain

$$\Delta(f; x) = (-1)^{c(n,l)} \Delta(x) \det[e_{l+i-2j}(v_i)]_{(m-l) \times m} \tag{10}$$

where

$$v_i = \begin{cases} l+i-1 & \text{for } 1 \le i \le n-l \\ n & \text{for } \quad i > n-l. \end{cases}$$

Using the identity $e_k(v+1) = e_k(v) + x_{v+1} e_{k-1}(v)$ and elementary row operations, we get

$$\det[e_{l+i-2j}(v_i)] = \det[e_{l+i-2j}(l)].$$

The right-hand side is equal to the Schur function $s_{l-1,l-2,\ldots,1}(x_1, \ldots, x_l)$ by the Naegelsbach identity (the dual of the Jacobi-Trudi identity), see [13, equation (3.5)] for details. From the bialternant formula for Schur functions, we obtain

$$s_{l-1,l-2,\ldots,1}(x_1, \ldots, x_l) = \prod_{1 \le i < j \le l} (x_i + x_j) \tag{11}$$

Plugging this value back into equation (10), we are done. □

For our second application of Theorem 3.1, let

$$F_d(t) = \frac{E_{d+1}(t) - e_{d+1}(d+1)}{t} = \sum_{j=0}^{d} e_j(d+1) t^{d-j}$$

and consider the column vector

$$\mathbf{F}_d = \begin{bmatrix} F_d(x_1) \\ F_d(x_2) \\ \vdots \\ F_d(x_n) \end{bmatrix}$$

The proof of the following corollary is similar to that of the previous one, and so is omitted.

COROLLARY 3.2. *If* $2 \le l \le n$, *then*

$$\det[\mathbf{X}, \mathbf{X}^3, \ldots, \mathbf{X}^{2l-3}, \mathbf{F}_{l-1}, \mathbf{F}_l, \ldots, \mathbf{F}_{n-1}] = (-1)^{d(n,l)} \Delta(x) \prod_{1 \le i < j \le l} (x_i + x_j) \quad (12)$$

where

$$d(n, l) = \binom{l}{2} + \sum_{i=0}^{2l-3-n} (n + i)$$

Note that we can also consider

$$\bar{\mathbf{F}}_d = \begin{bmatrix} F_d(-x_1) \\ F_d(-x_2) \\ \vdots \\ F_d(-x_n) \end{bmatrix}$$

It follows from our method of proof that if an arbitrary $\mathbf{F}_d$ in equation (12) is replaced by $\bar{\mathbf{F}}_d$, then the right-hand side is multiplied by ± 1.

For our next pair of corollaries, we will need to recall some facts about supersymmetric Schur functions. Define certain *supersymmetric functions*, $s_j(\mathbf{x}; \mathbf{y})$, in the variable sets $\mathbf{x} = \{x_1, \ldots, x_k\}$ and $\mathbf{y} = \{y_1, \ldots, y_l\}$, as the coefficients of the generating function

$$\sum_{j \ge 0} s_j(\mathbf{x}; \mathbf{y}) t^j = \frac{(1 + y_1 t)(1 + y_2 t) \cdots (1 + y_l t)}{(1 - x_1 t)(1 - x_2 t) \cdots (1 - x_k t)}. \quad (13)$$

If $\lambda = (\lambda_1, \lambda_2, \ldots, \lambda_r)$ is a partition, then the corresponding *super Schur function* is

$$s_\lambda(\mathbf{x}; \mathbf{y}) = \det[s_{\lambda_i + i - j}(\mathbf{x}; \mathbf{y})]_{r \times r}.$$

Thus $s_{(j)}(\mathbf{x}; \mathbf{y}) = s_j(\mathbf{x}; \mathbf{y})$.

We need the following factorization formula for super Schur functions.

THEOREM 3.2. *Suppose* λ *contains the* $k \times l$ *rectangle* $\rho = (l^k)$. *Let* α *and* β *be the partitions consisting of rows* $1, \ldots, k$ *and* $k+1, \ldots, r$ *from* λ/ρ, *respectively. Then*

$$s_\lambda(\mathbf{x};\mathbf{y}) = s_\alpha(\mathbf{x})s_{\beta'}(\mathbf{y})\prod_{i,j}(x_i + y_j)$$

where the prime denotes conjugation.

This result can be derived in various ways. It follows from a formula discovered by A.N. Sergeev which is a supersymmetric analog of the fact that the ordinary Schur functions can be expressed either as a Jacobi-Trudi determinant or as a quotient of alternants (see Pragacz [20], Bergeron-Garsia [3] or Pragacz-Thorup [21]). The factorization formula was originally proved by Berele and Regev [2] for the hook Schur functions. Once one knows that they coincide with super Schur functions, Theorem 3.2 follows. A bijective proof of Theorem 3.2 was given by Remmel [22] using the hook definition.

We will need a special case of Theorem 3.2. Replace $y_1, \ldots, y_l$ by $x_{k+1}, \ldots, x_n$. Let $s_\lambda(k; n)$ denote the resulting Schur supersymmetric function. Combining the previous theorem with equation (11), we immediately obtain the next result.

LEMMA 3.1. *If* $\lambda = (n-1, n-2, \ldots, n-k)$ *then*

$$s_\lambda(k; n) = \prod_{\substack{i<j\le n\\ i\le k}} (x_i + x_j).$$

To state what our basis matrices look like, we need to introduce the elementary and complete homogeneous symmetric functions in squares of variables

$$\tilde{e}_j(d) = e_j(x_1^2, x_2^2, \ldots, x_d^2)$$
$$\tilde{h}_j(d) = h_j(x_1^2, x_2^2, \ldots, x_d^2).$$

For the elementaries, we also need the corresponding generating function

$$\tilde{E}_d(t) = \prod_{i=1}^{d}(t^2 - x_i^2) = \sum_{j=0}^{d}(-1)^j\tilde{e}_j(d)t^{2d-2j}$$

and the column vector

$$\tilde{\mathbf{E}}_d = \begin{bmatrix} \tilde{E}_d(x_1) \\ \tilde{E}_d(x_2) \\ \vdots \\ \tilde{E}_d(x_n) \end{bmatrix}.$$

Note that the first d entries of $\tilde{\mathbf{E}}_d$ are zero, and that the subscript is only half of the degree of the polynomials which are its entries.

There is another useful expression for the complete homogeneous supersymmetric functions in this context.

LEMMA 3.2. *We have*

$$s_j(k; n) = \sum_{i=0}^{j} \tilde{h}_i(k) e_{j-2i}(n)$$

Proof. Take equation (13) with $y_1, \ldots, y_l$ replaced by $x_{k+1}, \ldots, x_n$, and multiply the top and bottom by $(1 + x_1 t)(1 + x_2 t) \cdots (1 + x_k t)$. The resulting generating function for $s_j(k; n)$ is

$$\frac{(1 + x_1 t)(1 + x_2 t) \cdots (1 + x_n t)}{(1 - x_1^2 t^2)(1 - x_2^2 t^2) \cdots (1 - x_k^2 t^2)}.$$

Extracting the coefficient of t^j yields the desired result. □

Finally, we will need an orthogonality result for the elementary and complete homogeneous symmetric functions. Since its proof is similar to that of the usual orthogonality relations, we will omit the demonstration.

LEMMA 3.3. *Suppose r, s, k are constants satisfying $r > s - k \geq 0$. Then*

$$\sum_{i=0}^{r} (-1)^i e_i(s) h_{r-i}(k) = 0.$$

Note that the lemma clearly still holds if the ordinary symmetric functions are replaced by those in squares of variables.

To state the next corollary, let $\lceil \cdot \rceil$ and $\lfloor \cdot \rfloor$ denote the ceiling and floor functions (round up and round down), respectively.

COROLLARY 3.3. *Fix k, n with $0 \leq k \leq n$. Let $u = \lceil \frac{n+k}{2} \rceil$ and $w = \lfloor \frac{n+k}{2} \rfloor$. Then*

$$\det[\mathbf{X}^0, \mathbf{X}^2, \ldots, \mathbf{X}^{2u-2}, \mathbf{X}\tilde{\mathbf{E}}_k, \mathbf{X}\tilde{\mathbf{E}}_{k+1}, \ldots, \mathbf{X}\tilde{\mathbf{E}}_{w-1}]$$
$$= (-1)^{f(n,k)} \Delta(x) \prod_{\substack{i<j\leq n \\ i \leq k}} (x_i + x_j)$$

where

$$f(n, k) = \binom{u}{2} + k(w - k) + \sum_{i=0}^{k-1} (n + i - 1)$$
$$\equiv \binom{u-k}{2} \pmod 2.$$

Proof. Using Theorem 3.1 in the usual way, we see that our determinant has the form of the right side of equation (9). Here, $m = n + k$, A_1 is a matrix of ones and zeros, A_2 is a checkerboard pattern of zeros and elementary symmetric functions in squares, and B is as in equation (8). Expanding about the rows in A_1 shows that, up to sign, our determinant is equal to $\Delta(x)\det C'$ where

$$C' = \left[\frac{A'}{B'}\right]_{w\times w}$$

with

$$A' = [(-1)^{i-j+k}\widetilde{e}_{i-j+k}(i+k-1)]_{(w-k)\times w}$$
$$B' = [e_{n+i-2j}(n)]_{k\times w}.$$

We will now use elementary column operations to modify the jth column of C' for j starting at 1 and ending at k. Specifically, add to column j multiples of the $j+1$st through wth columns, with the coefficient of column $j+l$ being $\widetilde{h}_l(k)$. The resulting matrix has the block form

$$C'' = \left[\begin{array}{c|c} A'' & A''' \\ \hline B'' & B''' \end{array}\right]$$

where $A''_{(w-k)\times k}$ is the zero matrix (by Lemma 3.3), $A'''_{(w-k)\times(w-k)}$ is lower triangular with $\widetilde{e}_0 = 1$ on the diagonal, and $B''_{k\times k}$ is the matrix whose determinant defines the super Schur function $s_{n-1,n-2,\dots,n-k}(k;n)$ (by Lemma 3.2). Expanding around the first $w-k$ rows of C'' and using Lemma 3.1 completes the proof. □

The proof of the next corollary follows the same lines as the previous demonstration, so it is left to the reader. In the proof, it is helpful to note that $\widehat{x}_i = F_{n-1}(-x_i)$.

COROLLARY 3.4. *Fix k, n with $0 \le k \le n$. Let $u = \lceil\frac{n+k}{2}\rceil$ and $w = \lfloor\frac{n+k}{2}\rfloor$. Then*

$$\det[\mathbf{X}^1, \mathbf{X}^3, \dots, \mathbf{X}^{2u-3}, \widetilde{\mathbf{E}}_k, \widetilde{\mathbf{E}}_{k+1}, \dots, \widetilde{\mathbf{E}}_{w-1}, \widehat{\mathbf{X}}_n] = (-1)^{g(n,k)}\Delta(x)\prod_{\substack{i<j\le n\\ i\le k}}(x_i+x_j)$$

where

$$g(n,k) = \binom{u}{2} + k(w-k) + \sum_{i=0}^{k-2}(n+i) \equiv \binom{u-k}{2} + n - 1 \pmod 2.$$

We note that all of these corollaries can be proved directly (with no reference to Garbieri's formula or the theory of symmetric functions) by using leading coefficient arguments. While these proofs are more elementary, they are also more complicated.

4. Nonlinear interpolations

When interpolating between $\mathcal{A}_{n-1}$ and $\mathcal{B}_n$ or $\mathcal{A}_{n-1}$ and $\mathcal{D}_n$, the order in which the hyperplanes are added matters. For example, consider

$$\mathcal{A} = \mathcal{A}_3 \cup \{\mathbf{e}_1 + \mathbf{e}_2, \mathbf{e}_2 + \mathbf{e}_3, \mathbf{e}_3 + \mathbf{e}_4\}^{\perp}.$$

Then deletion and restriction shows that $\chi(\mathcal{A}, t) = (t-1)(t-3)(t^2-5t+7)$ where the quadratic term does not have integral zeros. Thus, by Theorem 1.1, the arrangement $\mathcal{A}$ cannot possibly be free. This example indicates that intermediate arrangements for some orderings might not be free.

Consider the following set of ordered pairs, which we will list in a triangular array:

$$T = \left\{ \begin{array}{cccc} (1,2) & (1,3) & \cdots & (1,n) \\ & (2,3) & \cdots & (2,n) \\ & & \ddots & \vdots \\ & & & (n-1,n) \end{array} \right\}. \tag{14}$$

It will also be convenient to consider the sets

$$T_c = T \cup \{(0, l) : 2 \le l \le n+1\} \quad \text{and} \quad T_r = T \cup \{(k, k) : 1 \le k \le n\}.$$

Note that $(\mathbf{e}_i + \mathbf{e}_j)^{\perp} \in \mathcal{D}_n \backslash \mathcal{A}_{n-1}$ if and only if $(i, j) \in T$.

We can add the $(\mathbf{e}_i + \mathbf{e}_j)^{\perp}$ *by columns* where we read each column of T from top to bottom, starting with the leftmost column and moving right. More precisely, put a total order on T_c (and hence on T) by defining

$$(i, j) \le_c (k, l) \iff j < l, \quad \text{or} \quad j = l \text{ and } i \le k. \tag{15}$$

We can also add these hyperplanes *by rows* where we read each row of T from left to right, starting with the top row and moving down. Define a total order on T_r by letting

$$(i, j) \le_r (k, l) \iff i < k, \quad \text{or} \quad i = k \text{ and } j \le l. \tag{16}$$

We will start by considering column interpolations, doing one ending in $\mathcal{B}_n$ first and then one ending in $\mathcal{D}_n$. When ending in $\mathcal{B}_n$, we initially add $\{\mathbf{e}_1, \ldots, \mathbf{e}_n\}^{\perp}$ and then the elements corresponding to the elements of (14). So for $(k, l) \in T_c$, let

$$\mathcal{AB}^c_{n,k,l} = \mathcal{A}_{n-1} \cup \{\mathbf{e}_1, \ldots, \mathbf{e}_n\}^{\perp} \cup \{\mathbf{e}_i + \mathbf{e}_j : (i, j) \in T \text{ and } (i, j) \le_c (k, l)\}^{\perp}.$$

Note that for $2 \leq l \leq n+1$ the arrangements $\mathcal{AB}^c_{n,0,l}$ and $\mathcal{AB}^c_{n,l-2,l-1}$ contain the same hyperplanes and so are equal. The reason for having two names for the same arrangement is this. In the case that our interpolation ends at the bottom of a column of T, we actually have two bases for the arrangement. Note that, in Theorem 4.1, the matrices for the cases $(k, l) = (0, l)$ and $(k, l) = (l-2, l-1)$ are different even though the arrangements are the same. Thus, that theorem actually gives two elements of $\Theta(\mathcal{AB}^c_{n,0,l}) = \Theta(\mathcal{AB}^c_{n,l-2,l-1})$.

To describe the basis matrices, we will use the following notation. If $A = [a_{ij}]_{n\times m}$ and $B = [b_{ij}]_{n\times m}$ are matrices of the same dimensions, then the *Hadamard product* of A and B is

$$AB = [a_{ij}b_{ij}]_{n\times m}.$$

Note that we have already been tacitly using this notation, since $\mathbf{X}^m$ is just the Hadamard product of $\mathbf{X}$ with itself m times. We will use juxtaposition for both ordinary and Hadamard product, but no confusion will result. We will only use the latter on column vectors, where the former is not defined.

THEOREM 4.1. *The arrangements $\mathcal{AB}^c_{n,k,l}$ are free with*

$$Q(\mathcal{AB}^c_{n,k,l}) = Q(\mathcal{A}_{n-1})x_1x_2\cdots x_n \prod_{\substack{(i,j)\in T\\(i,j)\leq_c(k,l)}} (x_i + x_j)$$

$$[\mathbf{X}^1, \mathbf{X}^3, \ldots, \mathbf{X}^{2l-3}, c(x)\mathbf{XE}_{l-1}, \mathbf{XE}_l, \mathbf{XE}_{l+1}\mathbf{XE}_{n-1}] \in \Theta(\mathcal{AB}^c_{n,k,l})$$

where $c(x) = (x_1 + x_l)(x_2 + x_l)\cdots(x_k + x_l)$

$$\exp(\mathcal{AB}^c_{n,k,l}) = \{1, 3, \ldots, 2l-3, k+l, l+1, l+2, \ldots, n\}.$$

Before proving this theorem we would like to give the reader a bit of intuition about what is going on. We start with the matrix

$$\Theta_0 = [\mathbf{XE}_0, \ldots, \mathbf{XE}_{n-1}] \tag{17}$$

for $\mathcal{A}_{n-1} \cup \{\mathbf{e}_1, \ldots, \mathbf{e}_n\}^\perp$. (This is easily obtained by using elementary column operations on the matrix for $\mathcal{A}_{n-1}$ in Theorem 2.1, and then multiplying the jth column by x_j.) We wish to end at the matrix

$$\Theta_1 = [\mathbf{X}^1, \mathbf{X}^3, \ldots, \mathbf{X}^{2n-1}] \tag{18}$$

for $\mathcal{B}_n$. Now consider what is happening when we are adding hyperplanes corresponding to column l of our triangle (14), i.e., when we have the general Θ as given in the statement of the theorem. At this point the first $l-1$ columns have been changed into those that we want for Θ_1. The last $n-l$ columns have not been touched, and so are still in the form found in Θ_0. And the lth column itself is being modified, multiplying it by the equations of the hyperplanes being added. When we reach the hyperplane corresponding to the last entry in the lth column of the array (14), the lth column of our basis matrix is equal to

$(x_1 + x_l)(x_2 + x_l) \cdots (x_{l-1} + x_l) \cdot \mathbf{XE}_{l-1}$. We now exchange it for $\mathbf{X}^{2l-1}$ (so that we get another basis of the same arrangement) and continue the process with the $(l + 1)$st column.

Proof of Theorem 4.1. Note that $x_i + x_j \mid E_d(x_i)$ and $x_i + x_j \mid E_d(x_j)$ for all $i < j \leq d$. Also, $x_i + x_l \mid c(x)$ for all $i \leq k$. Thus, the divisibility criterion is easily checked using Lemma 2.1.

As for the determinant, let Θ be the matrix in the statement of the theorem. In $\det\Theta$ we can factor $c(x)$ out of the lth column and x_i out of row i for $1 \leq i \leq n$. This gives

$$\begin{aligned}\det\Theta &= x_1 x_2 \cdots x_n c(x) \det[\mathbf{X}^0, \mathbf{X}^2, \mathbf{X}^{2l-4}, \mathbf{E}_{l-1}, \mathbf{E}_l, \ldots, \mathbf{E}_{n-1}] \\ &= \pm Q(\mathcal{AB}^c_{n,k,l})\end{aligned}$$

by Corollary 3.1 (with l replaced by $l - 1$). □

Now we will interpolate from A_{n-1} to D_n by columns. For $(k, l) \in T_c$, let

$$\mathcal{AD}^c_{n,k,l} = \mathcal{A}_{n-1} \cup \{\mathbf{e}_i + \mathbf{e}_j : (i, j) \in T \text{ and } (i, j) \leq_c (k, l)\}^{\perp}.$$

Using the same convention as before for $\leq_c$, we obtain the following result.

THEOREM 4.2. *The arrangements $\mathcal{AD}^c_{n,k,l}$ are free with*

$$Q(\mathcal{AD}^c_{n,k,l}) = Q(\mathcal{A}_{n-1}) \prod_{\substack{(i,j)\in T \\ (i,j)\leq_c(k,l)}} (x_i + x_j)$$

$$[\mathbf{X}^1, \mathbf{X}^3, \ldots, \mathbf{X}^{2l-5}, c(x)\mathbf{F}_{l-2}, \mathbf{F}_{l-1}, \mathbf{F}_l, \ldots, \mathbf{F}_{n-1}] \in \Theta(\mathcal{AD}^c_{n,k,l})$$

where $c(x) = (x_1 + x_l)(x_2 + x_l) \cdots (x_k + x_l)$

$$\exp(\mathcal{AD}^c_{n,k,l}) = \{1, 3, \ldots, 2l - 5, k + l - 2, l - 1, l, \ldots, n - 1\}.$$

Proof. For divisibility, it suffices to prove that $x_i + x_j \mid \mathbf{F}_d(x_i + x_j)$ for $i < j \leq d + 1$. All the rest of the cases are taken care of by Lemma 2.1. Recall that

$$F_d(t) = \frac{E_{d+1}(t) - e_{d+1}(d+1)}{t}.$$

So

$$\begin{aligned}\mathbf{F}_d(x_i + x_j) &= F_d(x_i) + F_d(x_j) \\ &= 2\prod_{\substack{s=1 \\ s\neq i}}^{d+1} (x_i + x_s) - \hat{x}_i + 2\prod_{\substack{s=1 \\ s\neq j}}^{d+1} (x_j + x_s) - \hat{x}_j \qquad (19)\end{aligned}$$

where $\hat{x}_i$ and $\hat{x}_j$ are the ith and jth components of $\hat{\mathbf{X}}_{d+1}$, respectively. Thus, $x_i + x_j \mid \hat{x}_i + \hat{x}_j$ (Lemma 2.1, part 5), as well as the products in (19). This finishes the proof of divisibility.

The determinant condition is an immediate consequence of Corollary 3.2, replacing l by $l-1$ and factoring out $c(x)$ as usual. □

Using the remark following Corollary 3.2, we can construct more bases in $\Theta(\mathcal{AD}^c_{n,k,l})$ by replacing an arbitrary $\mathbf{F}_d$ in Theorem 4.2 by $\bar{\mathbf{F}}_d$.

We can now finish the description of the interpolation between D_{n-1} and D_n begun in Section 2. Let $H_1, H_2, \ldots, H_{2n-1}$ be any linear order of $\mathcal{D}_n \backslash \mathcal{D}_{n-1}$ such that all hyperplanes of the form $(\mathbf{e}_i - \mathbf{e}_n)^\perp$ come before all those of the form $(\mathbf{e}_i + \mathbf{e}_n)^\perp$. Define

$$\mathcal{D}_{n,k} = \mathcal{D}_{n-1} \cup \{H_1, H_2, \ldots, H_k\}.$$

Recall that $\mathbf{P}_n$ is the column vector for the derivation $\partial/\partial x_n$.

THEOREM 4.3. *The arrangements $\mathcal{D}_{n,k}$ are free with*

$$Q(\mathcal{D}_{n,k}) = c(x)Q(\mathcal{D}_{n-1}) \quad \textit{where } c(x) = \Pi_{1\le i\le k}\alpha_{H_i}$$

$$[\mathbf{X}^1, \mathbf{X}^3, \ldots, \mathbf{X}^{2n-5}, \mathbf{F}_{n-2}, c(x)\mathbf{P}_n] \in \Theta(\mathcal{D}_{n,k}) \quad \textit{for } 0 \le k \le n-1$$

$$[\mathbf{X}^1, \mathbf{X}^3, \ldots, \mathbf{X}^{2n-5}, d(x)\mathbf{F}_{n-2}, \mathbf{F}_{n-1}] \in \Theta(\mathcal{D}_{n,k}) \quad \textit{for } n \le k \le 2n-2$$

$$\textit{where } d(x) = \Pi_{n\le i\le k}\alpha_{H_i}$$

$$\exp(\mathcal{D}_{n,k}) = \begin{cases} \{1, 3, \ldots, 2n-5, n-2, k\} & \textit{for } 0 \le k \le n-1 \\ \{1, 3, \ldots, 2n-5, k-1, n-1\} & \textit{for } n \le k \le 2n-2 \end{cases}$$

where we view $\mathcal{D}_{n,0}$ as an arrangement in K^n.

Proof. We will only do the case $0 \le k \le n-1$, since the second half is covered by the previous theorem. (Although we required a precise order for the hyperplanes $(\mathbf{e}_i + \mathbf{e}_n)^\perp$ in that theorem, it is clear that the proof goes through for any ordering of these hyperplanes.) We have already checked all the necessary divisibility results. Taking the determinant of Θ, the matrix in the theorem, we obtain

$$\begin{aligned}\det\Theta &= \det[\mathbf{X}^1, \mathbf{X}^3, \ldots, \mathbf{X}^{2n-5}, \mathbf{F}_{n-2}, c(x)\mathbf{P}_n]_{n\times n} \\ &= c(x)\det[\mathbf{X}^1, \mathbf{X}^3, \ldots, \mathbf{X}^{2n-5}, \mathbf{F}_{n-2}]_{(n-1)\times(n-1)} \\ &= \pm c(x)Q(\mathcal{D}_{n-1})\end{aligned}$$

by Theorem 4.2 again. □

Now we can consider row interpolations. For $(k, l) \in T_r$, let

$$\mathcal{AB}^r_{n,k,l} = \mathcal{A}_{n-1} \cup \{\mathbf{e}_1, \ldots, \mathbf{e}_n\}^\perp \cup \{\mathbf{e}_i + \mathbf{e}_j : (i, j) \in T \text{ and } (i, j) \le_r (k, l)\}^\perp.$$

THEOREM 4.4. *Fix k, n with $1 \le k \le n$. Let $u = \lceil \frac{n+k-1}{2} \rceil$ and $w = \lfloor \frac{n+k-1}{2} \rfloor$. Then arrangements $\mathcal{AB}^r_{n,k,l}$ are free with*

$$Q(\mathcal{AB}^r_{n,k,l}) = Q(\mathcal{A}_{n-1})x_1x_2\cdots x_n \prod_{\substack{(i,j)\in T \\ (i,j)\le_r(k,l)}} (x_i + x_j)$$

$$[\mathbf{X}^1, \mathbf{X}^3, \ldots, \mathbf{X}^{2u-1}, c(x)\mathbf{X}^2\widetilde{\mathbf{E}}_{k-1}, \mathbf{X}^2\widetilde{\mathbf{E}}_k, \mathbf{X}^2\widetilde{\mathbf{E}}_{k+1}, \ldots, \mathbf{X}^2\widetilde{\mathbf{E}}_{w-1}] \in \Theta(\mathcal{AB}^r_{n,k,l})$$

where $c(x) = (x_k + x_{k+1})(x_k + x_{k+2})\cdots(x_k + x_l)$

$$\exp(\mathcal{AB}^r_{n,k,l}) = \{1, 3, \ldots, 2u-1, k+l, 2k+2, 2k+4, \ldots, 2w\}.$$

Proof. Recall that $\widetilde{E}_d(x_j) = 0$ for $j \le d$, and also that $x_i + x_j \mid \widetilde{E}_d(x_j)$ for $1 \le i \le d < j$. Also, $x_k + x_j \mid c(x)$ for all $k < j \le l$. So the divisibility criterion follows readily from Lemma 2.1.

The determinantal condition comes from Corollary 3.3 via manipulations similar to those in the proof of Theorem 4.1. □

For the final interpolation of this section, consider $(k, l) \in T_r$ and let

$$\mathcal{AD}^r_{n,k,l} = \mathcal{A}_{n-1} \cup \{\mathbf{e}_i + \mathbf{e}_j : (i,j) \in T \quad \text{and} \quad (i,j) \le_r (k,l)\}^{\perp}.$$

The reader can easily supply a demonstration of the next theorem based on the previous proofs of this section. In particular, the determinant condition follows from Corollary 3.4.

THEOREM 4.5. *Fix k, n with $1 \le k \le n$. Let $u = \lceil \frac{n+k-1}{2} \rceil$ and $w = \lfloor \frac{n+k-1}{2} \rfloor$. Then arrangements $\mathcal{AD}^r_{n,k,l}$ are free with*

$$Q(\mathcal{AD}^r_{n,k,l}) = Q(\mathcal{A}_{n-1}) \prod_{\substack{(i,j)\in T \\ (i,j)\le_r(k,l)}} (x_i + x_j)$$

$$[\mathbf{X}^1, \mathbf{X}^3, \ldots, \mathbf{X}^{2u-3}, c(x)\widetilde{\mathbf{E}}_{k-1}, \widetilde{\mathbf{E}}_k, \widetilde{\mathbf{E}}_{k+1}, \ldots, \widetilde{\mathbf{E}}_{w-1}, \hat{\mathbf{X}}_n] \in \Theta(\mathcal{AD}^r_{n,k,l})$$

where $c(x) = (x_k + x_{k+1})(x_k + x_{k+2})\cdots(x_k + x_l)$

$$\exp(\mathcal{AD}^r_{n,k,l}) = \{1, 3, \ldots, 2u-3, k+l-2, 2k, 2k+2, \ldots, 2w-2, n-1\}.$$

5. Complex reflection arrangements

In this section we will consider some generalizations of Coxeter arrangements and corresponding interpolations. For this, we will need to specialize our field to the complex numbers, **C**. Now fix an integer $s \ge 2$ and let $\zeta \in \mathbf{C}$ be a primitive sth root of unity, e.g, $\zeta = e^{2\pi i/s}$.

Define arrangements

$$\mathcal{B}_n(s) = \{\mathbf{e}_i - \zeta^a \mathbf{e}_j : 1 \le i < j \le n,\ 1 \le a \le s\}^\perp \cup \{\mathbf{e}_i : 1 \le i \le n\}^\perp$$

and

$$\mathcal{D}_n(s) = \mathcal{B}_n(s) \backslash \cup \{\mathbf{e}_i : 1 \le i \le n\}^\perp.$$

These arrangements consist of the complex reflecting hyperplanes for some of the finite unitary reflection groups of monomial matrices [17]. Note that $\mathcal{B}_n(2) = \mathcal{B}_n$ and $\mathcal{D}_n(2) = \mathcal{D}_n$. In fact, when $s = 2$ Theorems 5.3 and 5.4 yield new bases for the $\mathcal{A}_{n-1}$ to $\mathcal{B}_n$ interpolating arrangements that are valid over any field, not just $\mathbf{C}$. The lattices $L(\mathcal{B}_n(s))$ are isomorphic to the Dowling lattices [6, 7].

Collecting the usual information about these arrangements yields the following theorem [16].

THEOREM 5.1. *The arrangements $\mathcal{B}_n(s)$ and $\mathcal{D}_n(s)$ are free. They have defining forms, basis matrices, and exponents as follows.*

(1) *For $\mathcal{B}_n(s)$:*

$$Q(\mathcal{B}_n(s)) = x_1 x_2 \cdots x_n \prod_{1 \le i < j \le n} (x_i^s - x_j^s)$$
$$[\mathbf{X}, \mathbf{X}^{s+1}, \mathbf{X}^{2s+1}, \ldots, \mathbf{X}^{(n-1)s+1}] \in \Theta(\mathcal{B}_n(s))$$
$$\exp(\mathcal{B}_n(s)) = \{1, s+1, 2s+1, \ldots, (n-1)s+1\}.$$

(2) *For $\mathcal{D}_n(s)$:*

$$Q(\mathcal{D}_n(s)) = \prod_{1 \le i < j \le n} (x_i^s - x_j^s)$$
$$[\mathbf{X}, \mathbf{X}^{s+1}, \mathbf{X}^{2s+1}, \ldots, \mathbf{X}^{(n-2)s+1}, \hat{\mathbf{X}}_n^{s-1}] \in \Theta(\mathcal{D}_n(s))$$
$$\exp(\mathcal{D}_n(s)) = \{1, s+1, 2s+1, \ldots, (n-2)s+1, (n-1)(s-1)\}.$$

Interpolating between $\mathcal{B}_n(s)$ and itself is order independent. So take any linear ordering of $\mathcal{B}_n(s) \backslash \mathcal{B}_{n-1}(s)$, say $H_1, H_2, \ldots, H_{(n-1)s+1}$, and define

$$\mathcal{B}_{n,k}(s) = \mathcal{B}_{n-1}(s) \cup \{H_1, H_2, \ldots, H_k\}.$$

The proof of the next theorem is like many others, and so is safely left to the reader.

THEOREM 5.2. *The arrangements $\mathcal{B}_{n,k}(s)$ are free with*

$$Q(\mathcal{B}_{n,k}(s)) = c(x) Q(\mathcal{B}_{n-1}) \quad \textit{where } c(x) = \Pi_{1 \le i \le k} \alpha_{H_i}$$
$$[\mathbf{X}, \mathbf{X}^{s+1}, \ldots, \mathbf{X}^{(n-2)s+1}, c(x)\mathbf{P}_n] \in \Theta(\mathcal{B}_{n,k}(s))$$
$$\exp(\mathcal{B}_{n,k}(s)) = \{1, s+1, 2s+1, \ldots, (n-2)s+1, k\}$$

where we view $\mathcal{B}_{n,0}(s)$ as an arrangement in $\mathbf{C}^n$.

Since $\mathcal{A}_{n-1} \subset \mathcal{B}_n(s)$, we may also interpolate between these arrangements. We will again use our triangle (14). However, now the entry (i, j) is to be interpreted as a list

$$\mathbf{e}_i - \zeta\mathbf{e}_j,\ \mathbf{e}_i - \zeta^2\mathbf{e}_j,\ \ldots,\ \mathbf{e}_i - \zeta^{s-1}\mathbf{e}_j. \tag{20}$$

For the column version, choose $(k, l) \in T_c$ and consider

$$\mathcal{AB}^c_{n,k,l,b}(s) = \mathcal{A}_{n-1} \cup \{\mathbf{e}_1, \ldots, \mathbf{e}_n\}^\perp \cup \left\{ \bigcup_{\substack{(i,j)\in T \\ (i,j)<_c(k,l) \\ 1\le a<s}} (\mathbf{e}_i - \zeta^a\mathbf{e}_j)^\perp \right\} \cup \left\{ \bigcup_{1\le a\le b} (\mathbf{e}_k - \zeta^a\mathbf{e}_l)^\perp \right\}$$

so that the parameter b denotes how many terms of the list (20) to take. Note that when $(k, l) = (0, l)$, these arrangements are only defined for $b = 0$. For divisibility considerations, it is useful to note that

$$x_i - \zeta^p x_j \mid \mathbf{X}^{sd+1}(x_i - \zeta^p x_j) \tag{21}$$

for all i, j, p and d (a generalization of part 3 of Lemma 2.1).

THEOREM 5.3. *The arrangements $\mathcal{AB}^c_{n,k,l,b}(s)$ are free with*

$$Q(\mathcal{AB}^c_{n,k,l,b}(s)) = Q(\mathcal{A}_{n-1})x_1x_2\cdots x_n \prod_{\substack{(i,j)\in T \\ (i,j)<_c(k,l)}} \frac{x_i^s - x_j^s}{x_i - x_j} \cdot \prod_{1\le p\le b} (x_k - \zeta^p x_l)$$

$$[\mathbf{X}, \mathbf{X}^{s+1}, \ldots, \mathbf{X}^{(l-2)s+1}, c(x)\mathbf{X}\bar{\mathbf{E}}_{l-1}, \mathbf{X}\bar{\mathbf{E}}_l, \ldots, \mathbf{X}\bar{\mathbf{E}}_{n-1}] \in \Theta(\mathcal{AB}^c_{n,k,l,b}(s))$$

where $c(x) = \Pi_{i<k}\frac{x_i^s - x_l^s}{x_i - x_l}\Pi_{1\le a\le b}(x_k - \zeta^a x_l)$, *and*

$$\exp(\mathcal{AB}^c_{n,k,l,b}(s)) = \{1, s+1, \ldots, (1-2)s+1, (k-1)(s-1)+b+l, l+1, \ldots, n\}.$$

Proof. Throughout the proof, keep in mind that the matrix Θ in the statement of the theorem has the block form

$$\Theta = \left[\begin{array}{c|c} A & 0 \\ \hline B & C \end{array}\right] \tag{22}$$

where A has dimensions $(l-1)\times(l-1)$ and C is lower triangular. This is because of the fact that $\bar{\mathbf{E}}_d(x_i) = 0$ for all $i \le d$. Thus Lemma 2.1 and equation (21) cover all the divisibility cases.

As for the determinant, we use the fact that in (22), A is the matrix for $\mathcal{B}_{l-1}(s)$ and C is triangular. Then we have

$$\det\Theta = \det A \cdot \det C$$

$$= \det \Theta(\mathcal{B}_{l-1}(s)) \cdot C_{1,1}C_{2,2}C_{3,3}\cdots$$
$$= \pm x_1 \cdots x_{l-1} \prod_{i<j<l} (x_i^s - x_j^s) \cdot c(x) x_l \cdots x_n \prod_{\substack{i<j \\ l \le j \le n}} (x_i - x_j)$$
$$= \pm Q(\mathcal{AB}^c_{n,k,l,b}(s)). \square$$

For the row interpolations it will be convenient to set $\bar{k} = k-1$ and $\bar{s} = s-1$. For $(k, l) \in T_r$, consider the arrangements

$$\mathcal{AB}^T_{n,k,l,b}(s) = \mathcal{A}_{n-1} \cup \{\mathbf{e}_1, \ldots, \mathbf{e}_n\}^\perp \cup \left\{ \bigcup_{\substack{(i,j)\in T \\ (i,j)<_r(k,l) \\ 1\le a<s}} (\mathbf{e}_i - \zeta^a \mathbf{e}_j)^\perp \right\} \cup \left\{ \bigcup_{1\le a\le b} (\mathbf{e}_k - \zeta^a \mathbf{e}_l)^\perp \right\}.$$

When $(k, l) = (k, k)$ then we must take $b = 0$. Also, define the polynomial

$$E_{d.\bar{s}}(t) = \prod_{i\le d} \frac{t^s - x_i^s}{t - x_i}$$

with corresponding column vector $\mathbf{E}_{d.\bar{s}}$.

THEOREM 5.4. *The arrangements $\mathcal{AB}^T_{n,k,l,b}(s)$ are free with*

$$Q(\mathcal{AB}^T_{n,k,l,b}(s)) = Q(\mathcal{A}_{n-1}) x_1 x_2 \cdots x_n \prod_{\substack{(i,j)\in T \\ (i,j)<_r(k,l)}} \frac{x_i^s - x_j^s}{x_i - x_j} \prod_{1\le a\le b} (x_k - \zeta^a x_l)$$

$$[\mathbf{X}, \ldots \mathbf{X}^{\bar{k}s+1}, \mathbf{X}\mathbf{E}_{k\cdot\bar{s}}\bar{\mathbf{E}}_k, \ldots, \mathbf{X}\mathbf{E}_{k\cdot\bar{s}}\bar{\mathbf{E}}_{l-2}, \mathbf{c}\mathbf{X}\mathbf{E}_{\bar{k}\cdot\bar{s}}\bar{\mathbf{E}}_{l-1}, \mathbf{X}\mathbf{E}_{\bar{k}\cdot\bar{s}}\bar{\mathbf{E}}_l, \ldots, \mathbf{X}\mathbf{E}_{\bar{k}\cdot\bar{s}}\bar{\mathbf{E}}_{n-1}] \in \Theta(\mathcal{AB}^T_{n,k,l,b}(s))$$

where $\mathbf{c}$ *is the column vector corresponding to $c(t) = \Pi_{1\le a\le b}(x_k - \zeta^a t)$, and*

$$\exp(\mathcal{AB}^T_{n,k,l,b}(s)) = \{1, \ldots, \bar{k}s + 1, k\bar{s} + k + 1, \ldots, k\bar{s} + l - 1, \bar{k}\bar{s} + b + l, \bar{k}\bar{s} + l + 1, \ldots, \bar{k}\bar{s} + n\}$$

where the first (respectively, second and third) set of ellipsis in R refers to all integers with remainder 1 on division by s (respectively, all integers) between the given bounds.

Proof. Divisibility is verified in the usual manner. Also, the matrix Θ in the theorem has the block form (22), where A has dimensions $k \times k$ and C is lower triangular. Thus

$$\det \Theta = \det A \cdot \det C$$

$$= \det \Theta(\mathcal{B}_k(s)) \cdot C_{1,1}C_{2,2}C_{3,3} \cdots$$
$$= \pm x_1 \cdots x_n \prod_{i<j\leq k} (x_i^s - x_j^s)$$
$$\cdot c(x_l) \prod_{k<j<l} E_{k\cdot\bar{s}}(x_j) \prod_{l\leq j\leq n} E_{\bar{k}\cdot\bar{s}}(x_j) \prod_{\substack{i<j\\ k<j\leq n}} (x_i - x_j)$$
$$= \pm Q(\mathcal{AB}^r_{n,k,l,b}(s)). \square$$

Unfortunately, it is not possible to interpolate from $\mathcal{A}_{n-1}$ for $\mathcal{D}_n(s)$ for $s > 2$ because the arrangements are not always free. For example, let

$$\mathcal{A} = \{\mathbf{e}_1 - \zeta^a\mathbf{e}_2 : 1 \leq a \leq s\}^\perp \cup \{\mathbf{e}_1 - \zeta^a\mathbf{e}_3 : 1 \leq a \leq s\}^\perp \cup \{(\mathbf{e}_2 - \mathbf{e}_3)^\perp\}.$$

Then, using deletion and restriction, we find

$$\chi(\mathcal{A}, t) = (t-1)[t^2 - 2s \cdot t + (s^2 + s - 2)].$$

The discriminant of the quadratic factor is $(2s)^2 - 4(s^2 + s - 2) = 4(2 - s)$. Thus, the characteristic polynomial can only have real roots for $s \leq 2$, which is the case considered in Section 4.

6. The Möbius function of $L(\mathcal{DB}_{n,k})$

Hanlon [11] computed the characteristic polynomials of the arrangements $\mathcal{DB}_{n,k}$ by explicitly calculating the Möbius function for the corresponding lattice. The purpose of this section is to go the other way. That is, we will show how our computations of various basis matrices and the associated characteristic polynomials can lead to a complete description of the Möbius function.

Recall that if $\mathcal{A}$ is an arrangement in K^n, then $\hat{1}$ denotes the maximal element of $L = L(\mathcal{A})$. The value $\mu(L) = \mu(\hat{1})$ is easy to compute from $\chi(\mathcal{A}, t)$: It is just the coefficient of $t^{\dim \hat{1}}$. In all the lattices we have considered, this is always the coefficient of the smallest power of t that appears in $\chi(\mathcal{A}, t)$, i.e., the product of the negatives of the nonzero elements of $\exp(\mathcal{A})$. To obtain the rest of the Möbius values, it will be convenient to use Zaslavsky's theory of signed graphs [31, 32]. Any graph theory terms which are not defined can be found described in the text of Chartrand and Lesniak [5]. Undefined terms and unproven results from lattice theory can be looked up in Stanley's book [27].

Each element of $L(\mathcal{DB}_{n,k})$ will be encoded using a graph, G, on the labeled vertex set $[n] = \{1, 2, \ldots, n\}$. The edges of G will be of three types:

- a positive edge between vertices i and j, denoted ij^+,
- a negative edge between vertices i and j, denoted ij^-,
- a half edge with only one endpoint i, denoted i^h.

The edges ij^+, ij^- and i^h correspond to the roots $\mathbf{e}_i - \mathbf{e}_j$, $\mathbf{e}_i + \mathbf{e}_j$ and $\mathbf{e}_i$, respectively. (In the general theory there are also *loops* which are edges with two endpoints at the same vertex i, corresponding to the root $2\mathbf{e}_i$.) The reason for the choice of signs will be explained shortly.

To characterize the graphs which appear in $L(\mathcal{DB}_{n,k})$, we need some notation. For any $V \subseteq [n]$, let K_V^+ (respectively, K_V^-) denote the *signed complete graph* consisting of all positive (respectively, all negative) edges between vertices of V. Similarly, let $K_{V,W}^+$ ($K_{V,W}^-$) denote the *complete bipartite graph* consisting of all positive (respectively, all negative) edges between vertex sets V and W. In using this notation, we tacitly assume that $V \cap W = \emptyset$. Finally, let $K_V^{\pm(k)}$ be the complete signed graph, i.e., the one that has all edges of both signs between vertices in V together with all half edges on $V \cap [k]$.

THEOREM 6.1. *The lattice $L(\mathcal{DB}_{n,k})$ is isomorphic to the lattice of subgraphs G of $K_{[n]}^{\pm(k)}$ such that each component of G is of the form*

(1) $K_V^+ \cup K_W^+ \cup K_{V,W}^-$, *or*
(2) $K_V^{\pm(k)}$.

Furthermore, there can be at most one component of type 2.
If edge e corresponds to the vector $\mathbf{e}$, then the isomorphism of the preceding theorem is obtained by sending G to $\cap_{e \in G} \mathbf{e}^\perp$. The reason for our choice of edge signs is so that the components of type 1 will be balanced (every cycle has positive sign if we multiply the signs of its edges) Also, if a component of type 2 exists, then it is unbalanced (some cycle corresponds to a negative product).

Now for $G \in L(\mathcal{DB}_{n,k})$, let L_G be the lattice of all elements of $L(\mathcal{DB}_{n,k})$ less than or equal to G. Also, suppose that G has components $G_1, G_2, \ldots, G_k$. Then from the preceding theorem it is clear that we have an isomorphism

$$L_G \cong L_{G_1} \times L_{G_2} \times \cdots \times L_{G_k}$$

where $\times$ denotes the cross product of partially ordered sets. But it is well known that in such a situation we have

$$\mu(G) = \mu(L_G) = \mu(L_{G_1})\mu(L_{G_2}) \cdots \mu(L_{G_k}).$$

For the unbalanced component, we have already done the work. Using Theorem 2.4, we immediately have the following corollary.

COROLLARY 6.1. *If G in $L(\mathcal{DB}_{n,k})$ has the form $K_V^{\pm} \cup H$, where $|V| = v$ and $|H| = h$ is a set of half edges then*

$$\mu(G) = (-1)^v 1 \cdot 3 \cdots (2v-3)(v+h-1).$$

To finish our characterization, we need to find $\mu(G)$ where G is a balanced component. This can be done using the computation of the Möbius function for the partition lattice (see Rota [23]). However, we will accomplish this task using an interpolating family. Suppose $\mathcal{A}$ and $\mathcal{A}'$ are arrangements in K^p and K^q, respectively. Then there is a *product arrangement* in K^{p+q} defined by

$$\mathcal{A} \times \mathcal{A}' = \{H \oplus K^q : H \in \mathcal{A}\} \cup \{K^p \oplus H' : H' \in \mathcal{A}'\}.$$

Notice that if we have basis matrices $\Theta \in \Theta(\mathcal{A})$ and $\Theta' \in \Theta(\mathcal{A}')$, then we have a basis matrix $\Theta \oplus \Theta' \in \Theta(\mathcal{A} \times \mathcal{A}')$ with the block form

$$\Theta \oplus \Theta' = \left[\begin{array}{c|c} \Theta & 0 \\ \hline 0 & \Theta' \end{array}\right].$$

Also, we can add scaler multiples of the lower right block to the lower left one without disturbing the entries in the upper left. These will be the types of basis elements that we will use to interpolate between the cross product of two arrangements of type A and the arrangement of a balanced component. Specifically, fix p and q, and define

$$\bar{\mathbf{X}}^d = \begin{cases} \mathbf{X}^d & \text{for } d \text{ odd} \\ \begin{bmatrix} x_1^d \\ \vdots \\ x_p^d \\ -x_{p+1}^d \\ \vdots \\ -x_{p+q}^d \end{bmatrix} & \text{for } d \text{ even} \end{cases} \tag{23}$$

Also, let

$$\bar{E}_{p,d}(t) = (t - x_{p+1})(t - x_{p+2}) \cdots (t - x_d)$$

with $\bar{E}_{p,d}(t) = 1$ for $d \leq p$. The corresponding column vector is

$$\bar{\mathbf{E}}_{p,d} = \begin{bmatrix} 0 \\ \vdots \\ 0 \\ \bar{E}_{p,d}(x_{p+1}) \\ \vdots \\ \bar{E}_{p,d}(x_{p+q}) \end{bmatrix}$$

where the first p components are zero (and possibly others because of the roots of $\bar{E}_{p,d}$.

Now, let $\mathcal{A}_{p-1}$ and $\mathcal{A}_{q-1}$ be type A arrangements in the spaces generated by $\mathbf{e}_1, \ldots, \mathbf{e}_p$ and $\mathbf{e}_{p+1}, \ldots, \mathbf{e}_{p+q}$, respectively. Consider the following set of ordered pairs, which we will list in a rectangular array:

$$S = \left\{ \begin{matrix} (1, p+1) & (1, p+2) & \cdots & (1, p+q) \\ (2, p+1) & (2, p+2) & \cdots & (2, p+q) \\ \vdots & \vdots & & \vdots \\ (p, p+1) & (p, p+2) & \cdots & (p, p+q) \end{matrix} \right\} \tag{24}$$

It will also be convenient to consider

$$S_c = S \cup \{(0, l) : p+1 \leq l \leq p+q+1\}$$

Hyperplanes indexed by the elements of S can be added either by rows or columns. But by symmetry (interchanging p and q), we get the same lattices either way, so we will only do the latter. Let $\leq_c$ denote the total order (15) on the set S_c. Now for $(k, l) \in S_c$, consider

$$\mathcal{A}^{\times}_{p,q,k,l} = (\mathcal{A}_{p-1} \times \mathcal{A}_{q-1}) \cup \{e_i + e_j : (i, j) \in S \quad \text{and} \quad (i, j) \leq_c (k, l)\}^{\perp}.$$

Note that when $k = p$ and $l = p+q$ then $\mathcal{A}^{\times}_{p,q,k,l}$ is the arrangement corresponding to a graphical component $K^+_V \cup K^+_W \cup K^-_{V,W}$ where $|V| = p$ and $|W| = q$.

THEOREM 6.2. *Let $n = p + q$. Then the arrangements $\mathcal{A}^{\times}_{p,q,k,l}$ are free with*

$$Q(\mathcal{A}^{\times}_{p,q,k,l}) = Q(\mathcal{A}_{p-1})Q(\mathcal{A}_{q-1}) \prod_{\substack{(i,j)\in T \\ (i,j)\leq_c(k,l)}} (x_i + x_j)$$

$$[\bar{\mathbf{X}}^0, \bar{\mathbf{X}}^1, \ldots, \bar{\mathbf{X}}^{l-2}, c(x)\bar{\mathbf{E}}_{p,l-1}, \bar{\mathbf{E}}_{p,l}, \bar{\mathbf{E}}_{p,l+1}, \ldots, \bar{\mathbf{E}}_{p,n-1}] \in \Theta(\mathcal{A}^{\times}_{p,q,k,l})$$

where $c(x) = (x_1 + x_l)(x_2 + x_l) \ldots (x_k + x_l)$

$$\exp(\mathcal{A}^{\times}_{p,q,k,l}) = \{0, 1, \ldots, l-2, k+l-p-1, l-p, l-p+1, \ldots, q-1\}.$$

Proof. To check divisibility there are two cases. Suppose first that the derivation is of the form $\bar{\mathbf{X}}^d$. Then we have divisibility by $x_i - x_j$ because either $i, j \leq p$ or $i, j > p$. In both situations $\bar{\mathbf{X}}^d(x_i - x_j) = 0$ from the definition (23). Divisibility by $x_i + x_j$ following because $i \leq p < j$. So

$$\bar{\mathbf{X}}^d(x_i + x_j) = \begin{cases} x_i^d + x_j^d \text{ for } d \text{ odd} \\ x_i^d - x_j^d \text{ for } d \text{ even} \end{cases}$$

which immediately implies what we want.

Now consider those derivations containing a factor of $\bar{\mathbf{E}}_{p,d}$. For divisibility by $x_i - x_j$, note that for $i, j \leq p$ all the entries are zero. And for $i, j > p$ the entries are linear combinations of the columns for $\mathcal{A}_{q-1}$, so we have the desired result in either case. Finally, look at the $x_i + x_j$. Divisibility is automatic when $j \leq p$ since then the relevant entries are zero. The only other situation is $c(x)\bar{E}_{l-1}(x_i + x_l)$, and then $x_i + x_l$ is a factor of $c(x)$.

Note that our matrices Θ have the block triangular form (22). In this case A is $(l-1) \times (l-1)$ and C is lower triangular. By induction on q, we can assume that A is a basis matrix for $\mathcal{A}^{\times}_{p,l-p-1,p,l-p-1}$. Hence

$$\begin{aligned}\det\Theta &= \det A \cdot \det C = \det\Theta(\mathcal{A}^{\times}_{p,l-p-1,p,l-p-1}) \cdot C_{1,1}C_{2,2}C_{3,3}\cdots \\ &= \pm \prod_{\substack{i<j\le p \\ \text{or} \\ p+1\le i<j\le l-1}} (x_i - x_j) \prod_{i\le p<j\le l-1} (x_i + x_j)\cdot c(x) \prod_{\substack{p+1\le i<j \\ j\ge l}} (x_i - x_j) \\ &= \pm Q(\mathcal{A}^{\times}_{p,q,k,l}). \square\end{aligned}$$

Now if $k = p$ and $l = p + q = n$, then $\exp(\mathcal{A}^{\times}_{p,q,k,l}) = \{0, 1, 2, \ldots, n-1\}$. Thus, we have the remaining Möbius values.

COROLLARY 6.2. *If* $G = K_V^+ \cup K_W^+ \cup K_{V,W}^-$ *where* $|V| + |W| = p + q = n$, *then* $\mu(G) = (-1)^{n-1}(n-1)!$

Of course, this is not the most efficient way to compute the Möbius function. But we have gained a lot of extra information along the way.

7. Comments

There are other methods for proving factorization of the characteristic polynomial, χ. First of all, it would be interesting to compute the Möbius function directly for the various families introduced above. Then one could prove that their characteristic polynomials factor directly as Hanlon [11] did for $\mathcal{DB}_{n,k}$. Zaslavsky [33] also computed the characteristic polynomials and Möbius values for these lattices using coloring techniques.

Theorem 1.1 gives an algebraic explanation of the factorization of χ. Stanley [26] developed the notion of a supersolvable lattice to combinatorially explain this phenomenon. Bennett and Sagan [1] have developed a generalization of the notion of supersolvability. It can be use to combinatorially prove factorization of $\chi(\mathcal{DB}_{n,k}, t)$, even though the lattices are not supersolvable for $0 < k \le n-2$, or for $k = 0$ and $n \ge 4$. This method should extend to the other nonsupersolvable cases under consideration as well.

It is natural to ask which subarrangements of Coxeter arrangements are free. An answer for subarrangements of $\mathcal{A}_{n-1}$ is as follows. Recall that each subset of hyperplanes in an arrangement of type A, B or D can be considered as a signed graph. Those contained in $\mathcal{A}_{n-1}$ are just graphs with no negative edges or half edges. Stanley [26] gave the following criterion to test supersolvability in this case.

THEOREM 7.1. *Let $\mathcal{A} \subseteq \mathcal{A}_{n-1}$ have graph G. Then $\mathcal{A}$ is supersolvable if and only if there is a sequence*

$$\emptyset = G_0 \subset G_1 \subset \ldots \subset G_n = G$$

of induced subgraphs of G such that

1. *G_i has i vertices for $1 \leq i \leq n$,*
2. *if v_i is the vertex of $G_i \backslash G_{i-1}$, then the subgraph of G_i induced by v_i and its neighbors in G_i is complete.*

Fulkerson and Gross [9] showed that the two conditions in the previous theorem are equivalent to G being *chordal*, i.e., every cycle of G has a chord. Previously, Jambu and Terao [12] demonstrated that any supersolvable arrangement is free. It is easy to see that the converse is also true for subarrangements $\mathcal{A} \subseteq \mathcal{A}_{n-1}$. (If $\mathcal{A}$'s graph is not chordal, then it has an induced cycle. This corresponds to a localization of the arrangement which is not uniform, hence $\mathcal{A}$ is not free.) It would be interesting to characterize the free subarrangements of other Coxeter arrangements. We should note that recently Edelman and Reiner [8] have been able to characterize the free arrangements lying between $\mathcal{A}_{n-1}$ and $\mathcal{B}_n$.

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