5.1 The Remainder and Factor Theorems; Synthetic Division

In this section you will learn to:

- understand the definition of a zero of a polynomial function
- use long and synthetic division to divide polynomials
- use the remainder theorem
- use the factor theorem

Example 1: Use long division to find the quotient and the remainder: $5593 \div 27$

<u>Steps for Long Division</u>:

1.

2.

3.

4.

Example 2: Use the "Steps for Long Division" to divide each of the polynomials below.

$x-5\overline{)\,x^2-2x-35}$

$(7-11x-3x^2+2x^3)\div(x-3)$

Example 3: Check your answer for the division problems in Example 2.

The Division Algorithm: If $f(x)$ and $d(x)$ are polynomials where $d(x) \neq 0$ and degree $d(x)$ < degree $f(x)$,

then

$\overline{)\qquad\qquad}$

$$f(x) = d(x) \cdot q(x) + r(x)$$

If $r(x) = 0$ then $d(x)$ and $q(x)$ are **factors** of $f(x)$.

Example 4: Perform the operation below. Write the remainder as a rational expression (remainder/divisor).

$$\frac{2x^5 - 8x^4 + 2x^3 + x^2}{2x^3 + 1}$$

<u>Synthetic Division</u> – Generally used for "short" division of polynomials when the divisor is in the form $\boldsymbol{x - c}$. (Refer to page 506 in your textbook for more examples.)

Example 5: Use both long and short (synthetic) division to find the quotient and remainder for the problem below.

$$(2x^3 - 11x + 7) \div (x - 3)$$

Example 6: Divide $\dfrac{x^3 + 8}{x + 2}$ using synthetic division.

Example 7: Factor $x^3 + 8$ over the real numbers. (Hint: Refer to Example 6.)

Remainder Theorem	Factor Theorem
If the polynomial $f(x)$ is divided by $(x - c)$, then the remainder is $f(c)$.	Let $f(x)$ be a polynomial. If $f(c) = 0$, then $(x - c)$ is a factor of $f(x)$. If $(x - c)$ is a factor of $f(x)$, then $f(c) = 0$. If $(x - c)$ is a factor of $f(x)$ or if $f(c) = 0$, then c is called a **zero** of $f(x)$.

Example 8: $f(x) = 3x^3 + 4x^2 - 5x + 7$. Find $f(-4)$ using

(a) synthetic division.

(b) the Remainder Theorem.

Example 9: Solve the equation $2x^3 - 3x^2 - 11x + 6 = 0$ given that -2 is a zero of $f(x) = 2x^3 - 3x^2 - 11x + 6$.

5.1 Homework Problems:

For Problems 1-5, use long division to find each quotient, $q(x)$, and remainder, $r(x)$.

1. $(x^2 - 2x - 15) \div (x - 5)$

2. $(x^3 + 5x^2 + 7x + 2) \div (x + 2)$

3. $(6x^3 + 7x^2 + 12x - 5) \div (3x - 1)$

4. $\dfrac{x^4 - 81}{x - 3}$

5. $\dfrac{18x^4 + 9x^3 + 3x^2}{3x^2 + 1}$

For Problems 6 – 11, divide using synthetic division.

6. $(2x^2 + x - 10) \div (x - 2)$

7. $(5x^3 - 6x^2 + 3x + 11) \div (x - 2)$

8. $(x^2 - 5x - 5x^3 + x^4) \div (5 + x)$

9. $\dfrac{x^7 + x^5 - 10x^3 + 12}{x + 2}$

10. $\dfrac{x^4 - 256}{x - 4}$

11. $\dfrac{x^5 - 2x^4 - x^3 + 3x^2 - x + 1}{x - 2}$

For Problems 12 – 16, use synthetic division and the Remainder Theorem to find the indicated function value.

12. $f(x) = x^3 - 7x^2 + 5x - 6;\ f(3)$

13. $f(x) = 4x^3 + 5x^2 - 6x - 4;\ f(-2)$

14. $f(x) = 2x^4 - 5x^3 - x^2 + 3x + 2;\ f\left(-\dfrac{1}{2}\right)$

15. $f(x) = 6x^4 + 10x^3 + 5x^2 + x + 1;\ f\left(-\dfrac{2}{3}\right)$

16. Use synthetic division to divide $f(x) = x^3 - 4x^2 + x + 6$ by $x + 1$. Use the result to find all zeros of *f*.

17. Solve the equation $2x^3 - 5x^2 + x + 2 = 0$ given that 2 is a zero of $f(x) = 2x^3 - 5x^2 + x + 2$.

18. Solve the equation $12x^3 + 16x^2 - 5x - 3 = 0$ given that $-\dfrac{3}{2}$ is a zero (root).

5.1 Homework Answers: **1.** $q(x) = x + 3$ **2.** $q(x) = x^2 + 3x + 1$ **3.** $q(x) = 2x^2 + 3x + 5$

4. $q(x) = x^3 + 3x^2 + 9x + 27$ **5.** $q(x) = 6x^2 + 3x - 1;\quad r(x) = -3x + 1$ **6.** $q(x) = 2x + 5$

7. $q(x) = 5x^2 + 4x + 11;\quad r(x) = 33$ **8.** $q(x) = x^3 - 10x^2 + 51x - 260;\quad r(x) = 1300$

9. $q(x) = x^6 - 2x^5 + 5x^4 - 10x^3 + 10x^2 - 20x + 40;\quad r(x) = -68$ **10.** $q(x) = x^3 + 4x^2 + 16x + 64$

11. $q(x) = x^4 - x^2 + x + 1;\quad r(x) = 3$ **12.** -27 **13.** -4 **14.** 1 **15.** $\dfrac{7}{9}$

16. $x^2 - 5x + 6;\quad x = -1, 2, 3$ **17.** $\left\{-\dfrac{1}{2}, 1, 2\right\}$ **18.** $\left\{-\dfrac{3}{2}, -\dfrac{1}{3}, \dfrac{1}{2}\right\}$

5.3 Roots of Polynomial Equations

In this section you will learn to:

- find zeros of polynomial equations
- solve polynomial equations with real and imaginary zeros
- find possible rational roots of polynomial equations
- understand properties of polynomial equatins
- use the Linear Factorization Theorem

> **Zeros of Polynomial Functions** are the values of x for which $f(x) = 0$.
>
> (Zero = Root = Solution = x-intercept (if the zero is a real number))

Example 1: Consider the polynomial that only has 3 and ½ as zeros.

(a) How many polynomials have such zeros?

(b) Find a polynomial that has a leading coefficient of 1 that has such zeros.

(c) Find a polynomial, with integer coefficients, that has such zeros.

> If the same factor ($x - r$) occurs k times, then the zero r is called a zero with **multiplicity *k*.**
>
> **Even Multiplicity** → Graph **touches** x-axis and turns around.
>
> **Odd Multiplicity** → Graph **crosses** x-axis.

Example 2: Find all of the (real) zeros for each of the polynomial functions below. Give the multiplicity of each zero and state whether the graph crosses the x-axis or touches (and turns at) the x-axis at each zero. Use this information and the Leading Coefficient Test to sketch a graph of each function

(a) $f(x) = x^3 + 2x^2 - 4x - 8$ (b) $f(x) = -x^4 + 4x^2$ (c) $g(x) = -x^4 + 4x^3 - 4x^2$

The Rational Zero Theorem: If $f(x) = a_n x^n + a_{n-1}x^{n-1} + \ldots a_2 x^2 + a_1 x + a_0$ has ***integer*** coefficients and $\frac{p}{q}$ (reduced to lowest terms) is a rational zero of f, then p is a factor of the constant term, a_0, and q is a factor of the leading coefficient, a_n.

Example 3: List all **possible** rational zeros of the polynomials below. (Refer to Rational Zero Theorem on
Page 1 of this handout.)

(a) $f(x) = -x^5 + 7x^2 - 12$ Possible Rational Zeros: ____________________________

(b) $p(x) = 6x^3 - 8x^2 - 8x + 8$ Possible Rational Zeros: ____________________________

Example 4: Find all zeros of $f(x) = 2x^3 - 5x^2 + x + 2$.

Example 5: Solve $x^4 - 8x^3 + 64x - 105 = 0$.

Linear Factorization Theorem:

If $f(x) = a_n x^n + a_{n-1}x^{n-1} + \ldots . a_2 x^2 + a_1 x + a_0$, where n ≥ 1 and $a_n \neq 0$, then
$f(x) = a_n(x - c_1)(x - c_2)\ldots(x - c_n)$, where $c_1, c_2, c_3, \ldots c_n$ are complex numbers.

Example 6: Find all complex zeros of $f(x) = 2x^4 + 3x^3 + 3x - 2$, and then write the polynomial $f(x)$ as a **product of linear factors**.

$f(x) =$ ____

Properties of Polynomial Equations:

Given the polynomial $f(x) = a_n x^n + a_{n-1}x^{n-1} + \ldots . a_2 x^2 + a_1 x + a_0$.

1. If a polynomial equation is of degree *n*, then counting multiple roots (multiplicities) separately, the equation has *n* roots.

2. If $a + bi$ is a root of a polynomial equation ($b \neq 0$), then the imaginary number $a - bi$ is also a root. In other words, imaginary roots, if they exist, occur in **conjugate pairs**.

Example 7: Find all zeros of $f(x) = x^4 - 4x^2 - 5$. (Hint: Use factoring techniques from Chapter 1.) Write $f(x)$ as a product of linear factors.

$f(x) =$ ____

Example 8: Find a third-degree polynomial function, $f(x)$, with real coefficients that has 4 and $2i$ as zeros and such that $f(-1) = 50$.

Step 1: Use the zeros to find the factors of $f(x)$.

Step 2: Write as a linear factorization, then expand/multiply.

Step 3: Use $f(-1) = 50$ to substitute values for x and $f(x)$.

Step 4: Solve for a_n.

Step 5: Substitute a_n into the equation for $f(x)$ and simplify.

Step 6: Use your calculator to check.

5.3 Homework Problems:

For Problems 1 – 4, use the Rational Zero Theorem to list all possible rational zeros for each function.

1. $f(x) = x^3 + 3x^2 - 6x - 8$
2. $f(x) = 2x^4 + 3x^3 - 11x^2 - 9x + 15$
3. $f(x) = 3x^4 - 11x^3 - 3x^2 - 6x + 8$
4. $f(x) = 4x^5 - 8x^4 - x + 2$

For Problems 5 – 8, find the zeros for the given functions.

5. $f(x) = x^3 - 2x^2 - 11x + 12$
6. $f(x) = 2x^3 - 5x^2 + x + 2$
7. $f(x) = 2x^3 + x^2 - 3x + 1$
8. $f(x) = x^3 - 4x^2 + 8x - 5$

For Problems 9 – 12, solve each of the given equations.

9. $x^3 - 2x^2 - 7x - 4 = 0$
10. $x^3 - 5x^2 + 17x - 13 = 0$
11. $2x^3 - 5x^2 - 6x + 4 = 0$
12. $x^4 - 2x^2 - 16x - 15 = 0$

For Problems 13-16, find an *nth* degree polynomial function, $f(x)$, with real coefficients that satisfies the given conditions.

13. $n = 3$; 1 and $5i$ are zeros; $f(-1) = -104$
14. $n = 4$; 2, -2, and i are zeros; $f(3) = -150$
15. $n = 3$; 6 and $-5 + 2i$ are zeros; $f(2) = -636$
16. $n = 4$; i and $3i$ are zeros; $f(-1) = 20$

5.3 Homework Answers: **1.** $\pm 1, \pm 2, \pm 4, \pm 8$ **2.** $\pm 1, \pm 3, \pm 5, \pm 15, \pm\frac{1}{2}, \pm\frac{3}{2}, \pm\frac{5}{2}, \pm\frac{15}{2}$

3. $\pm 1, \pm 2, \pm 4, \pm 8, \pm\frac{1}{2}, \pm\frac{2}{3}, \pm\frac{4}{3}, \pm\frac{8}{3}$ **4.** $\pm 1, \pm 2, \pm\frac{1}{2}, \pm\frac{1}{4}$ **5.** $-3, 1, 4$ **6.** $-\frac{1}{2}, 1, 2$

7. $\frac{1}{2}, \frac{-1 \pm \sqrt{5}}{2}$ **8.** $1, \frac{3 \pm i\sqrt{11}}{2}$ **9.** $\{-1, 4\}$ **10.** $\{1, 2 \pm 3i\}$ **11.** $\left\{\frac{1}{2}, 1 \pm \sqrt{5}\right\}$

12. $\{-1, 3, -1 \pm 2i\}$ **13.** $f(x) = 2x^3 - 2x^2 + 50x - 50$ **14.** $f(x) = -3x^4 + 9x^2 + 12$

15. $f(x) = 3x^3 + 12x^2 - 93x - 522$ **16.** $f(x) = x^4 + 10x^2 + 9$