Selected Solutions to Homework # 11

- #2 in 11.6: Find the equation of a tangent plane and the equation of a normal line to the surface

  $$x^2 + y^2 - z^2 = 18$$

  at the point $P(3, 5, -4)$.

  Let $f = x^2+y^2-z^2$. Then the surface is a level surface of f. Therefore, the gradient of f at P is normal to the surface. We compute this vector:

  $$\nabla f = \langle 2x, 2y, -2z\rangle; \qquad \nabla f(P) = \langle 6, 10, 8\rangle.$$

  The tangent plane at P has equation

  $$6(x-3) + 10(y-5) + 8(z+4) = 0.$$

  The normal line at P is described by the parametric equations:

  $$x = 3 + 6t, \quad y = 5 + 10t, \quad z = -4 + 8t.$$

- # 18 in 11.6: Find parametric equations for the line tangent to the curve given by the intersection of the surfaces

  $$x^2 + y^2 = 4 \text{ and } x^2 + y^2 - z = 0$$

  at the point $P(\sqrt{2}, \sqrt{2}, 4)$.

  The idea is to compute two normal vectors, and then compute their cross product to produce a vector which is tangent to both surfaces and, hence, tangent to their intersection.

  Let $f = x^2 + y^2$, and $g = x^2 + y^2 - z$. Compute their gradients and evaulate at P:

  $$\nabla f(P) = \langle 2\sqrt{2}, 2\sqrt{2}, 0\rangle, \qquad \nabla g(P) = \langle 2\sqrt{2}, 2\sqrt{2}, -1\rangle.$$

  The cross product of these two vectors is $(\pm 2\sqrt{2})\langle 1, -1, 0\rangle$ Thus, the following are parametric equations for the tangent line:

  $$x = \sqrt{2} + 2\sqrt{2}t, \quad y = \sqrt{2} - 2\sqrt{2}t, \quad z = 0.$$

- Estimate how much $f = e^x \cos(yz)$ will change as the point P moves from the origin a distance $ds = 0.1$ in the direction of $2\mathbf{i} + 2\mathbf{j} - 2\mathbf{k}$.

  We compute the differential using $df = \nabla f \cdot \mathbf{u} \cdot ds$ (see p. 629):

$$(e^x \cos(yz)\mathbf{i} - ze^x \sin(yz)\mathbf{j} - ye^x \sin(yz)\mathbf{k}) \cdot (\mathbf{i} + \mathbf{j} - \mathbf{k})(\frac{1}{2\sqrt{2}})(0.1)$$

  Plugging in the origin $(0, 0, 0)$, we have

$$df = \mathbf{i} \cdot (\mathbf{i} + \mathbf{j} - \mathbf{k})\frac{1}{20\sqrt{2}} = \frac{1}{20\sqrt{2}}$$

- # 32 in 11.6: Find the linearization of

$$f = (1/2)x^2 + xy + (1/4)y^2 + 3x - 3y + 4$$

  at $P(2, 2)$. Then find an upper bound for the error in the linear approximation.

  We compute the partial derivatives:

$$f_x = x + y + 3, \qquad f_y = x + (1/2)y - 3.$$

  The linearization is

$$f(2, 2) + f_x(2, 2)(x - 2) + f_y(2, 2)(y - 2) = 11 + 7(x - 2) + 0.$$

  The error in using the linearization to estimate values in the rectangle R (given in the problem as $|x - 2| \leq 0.1, |y - 2| \leq 0.1$) is bounded by the maximum value of the 2nd partial derivatives of f over R. We compute the 2nd derivatives:

$$f_{xx} = 1, \qquad , f_{xy} = 1, \qquad f_{yy} = 1/2.$$

  Thus, the error is less than or equal to $(1/2)M(|x-2|+|y-2|)^2$. Since we have just seen that $M = 1$ is sufficient. The error is less than or equal to $(1/2)(1)(.1 + .1)^2 = 0.02$.

- # 48 in 11.6. Around $(1, 0)$ is $f = x^2(y + 1)$ more sensitive to changes in x or changes in y? What ratio of dx to dy will make df equal to zero at $(1, 0)$?

  We can estimate the sensitivity to small changes in x and y by looking at the differential. The differential df is an estimate of the change in

f, and it, in turn, is expressed in terms of the small changes dx and dy. We compute

$$df = 2x(y+1)\,dx + x^2\,dy.$$

At $(1,0)$ we have $df = 4dx + dy$. Thus, a small change in x has approximately four times the effect as a small change in y.

(Note: You should be careful to say "approximately" and stipulate that this estimate only holds for x and y which are sufficiently close to $(1,0)$. The meaning of "sufficiently close" can be quantified by bouding the error as in the previous problem.)

We can see from the above calculation that if the ratio of dx to dy is $-1/4$, then $df = 0$.