

Selected Solutions to Homework # 4

1. # 4 in Appendix C: Let r be a real number, $r \neq 1$. Prove that for every $n \in \mathbb{Z}_+$,

$$1 + r + r^2 + \cdots + r^{n-1} = \frac{r^n - 1}{r - 1}.$$

Proof: We give a proof using induction. Let $r \in \mathbb{R}$ and $r \neq 1$. Let P_n be the statement that the above equation holds for the integer n . The statement P_1 reads $1 = \frac{r-1}{r-1}$, which is evidently true. Suppose that the statement P_n is true for some $n \geq 1$. We are to show that the statement P_{n+1} is true. Consider $1 + 2 + \cdots + r^{n-1} + r^n$. By grouping the first n terms and applying P_n , we have

$$1 + 2 + \cdots + r^{n-1} + r^n = \frac{r^n - 1}{r - 1} + r^n = \frac{r^n - 1}{r - 1} + \frac{r^n(r - 1)}{r - 1} = \frac{r^{n+1} - 1}{r - 1}.$$

Thus we have shown that P_{n+1} is a consequence of P_n . By the principle of mathematical induction, P_n is true for every integer $n \geq 1$. Q.E.D.

2. # 16 in Appendix C. This is the game known as the “Towers of Hanoi”. I will give a brief sketch of how to solve this problem.

To move one ring from the first peg to the third peg takes one move. Thus, the number of move to solve the puzzle in the case of one ring is 1. Since $2^n - 1 = 1$ if $n = 1$, the puzzle can be solved in $2^n - 1$ moves and it certainly cannot be solved in fewer moves if $n = 1$.

Suppose that it is true for some $n \geq 1$ that the number moves to solve the puzzle in the case of n rings is $2^n - 1$ and that it cannot be solved in fewer moves.

If we are presented with a puzzle with $n + 1$ rings, move the top n rings to the second peg. This takes $2^n - 1$ moves because moving n pegs to the second peg is equivalent to the problem of moving to the third peg. Next move the bottom ring from the first peg to the third. Finally, move the the stack of n rings from the second to the third peg; this last step takes $2^n - 1$ moves. All together, the solution took $(2^n - 1) + 1 + (2^n - 1) = 2^{n+1} - 1$ moves. Any solution to the puzzle must include a move which moves the bottom ring from the first peg to the third. By the rules of the game, this is only possible if all of the

other n pegs are on the second peg; so, no fewer moves were possible at this first stage. Certainly the middle stage of moving the bottom peg to the third peg in one move is optimal. Finally, the last stage of moving the n rings on the second peg to the third requires at least $2^n - 1$ moves (since this puzzle is equivalent to the original with n pegs). Therefore, our solution is optimal.

3. # 17 in section 1.2

If $a|c$ and $b|c$, must $ab|c$? No: take $a = b = c = 2$.

What if $(a, b) = 1$? Yes: write $c = ax = by$ for some $x, y \in \mathbb{Z}$. Since $(a, b) = 1$, there exists $m, n \in \mathbb{Z}$ such that $am + bn = 1$. Therefore, $cam + cbn = c$. Write $(c)am = (by)am$ and $(c)bn = (ax)bn$. Thus, $byam + axbn = c$. In other words, $ab(my + nx) = c$. Therefore, ab divides c .

4. # 20 in section 1.2

5. Use the Euclidean algorithm to compute the greatest common divisor of 2011 and 1492.

$$\begin{array}{rcl}
 2011 & = & 1492 * 1 + 519 \\
 1492 & = & 519 * 2 + 454 \\
 519 & = & 454 * 1 + 65 \\
 454 & = & 65 * 6 + 64 \\
 65 & = & 64 * 1 + 1 \\
 64 & = & 1 * 64 + 0
 \end{array}$$

Therefore, $(2011, 1492) = 1$. (The greatest common divisor is the last non-zero remainder.)

6. (Bonus) How does one construct positive integers a and b such that the Euclidean algorithm requires many steps before terminating? Give an example of a pair of integers which require at least 100 divisions in order to compute their greatest common divisor.

Hint: work backwards...