3.1 Functions and Function Notation

In this section you will learn to:

- find the domain and range of relations and functions
- identify functions given ordered pairs, graphs, and equations
- use function notation and evaluate functions
- use the Vertical Line Test (VLT) to identify functions
- apply the difference quotient

Domain – set of all first components (generally *x*) of the ordered pairs.

Range – set of all second components (generally *y*) of the ordered pairs.

Relation – any set of ordered pairs**.**

Function – a correspondence from a first set, called the **domain**, to a second set, called the **range**, such that each element in the domain corresponds to *exactly one* element in the range.

Example 1: Graph the following relation representing a student's scores for the first four quizzes:
{(Quiz #1, 20), (Quiz #2, 15), (Quiz #3, 20), (Quiz #4, 12)}

Is this relation a function? __________

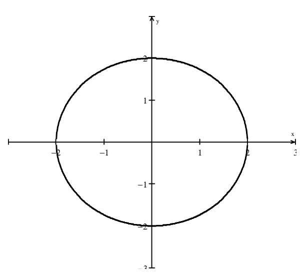
Find the domain. ______________________________

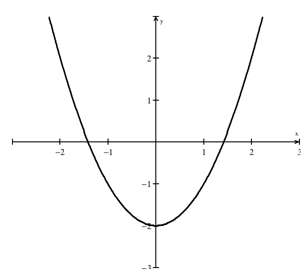
Find the range. ______________________________

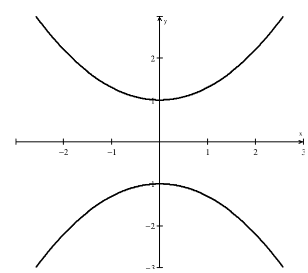
If the point (Quiz #2, 20) is added, is the relation still a

function? Explain:____

Example 2: Find the domain and range of each relation and determine whether the relation is a function.







Function? ______ Function? ______ Function? ______

Domain: _________________ Domain: _________________ Domain: __________________

Range: __________________ Range: __________________ Range: ___________________

Example 3: Use the definition of a function to determine if each of the sets of ordered pairs is a function.

{(1, 2), (3, 4), (4, 5), (5, 5)}

Function? _______

Domain: ________

Range: _________

{(2, 1), (4, 3), (5, 4), (5, 5)}

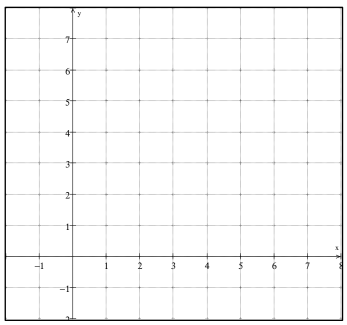
Function? _______

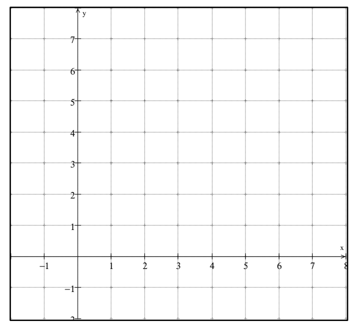
Domain: ________

Range: _________

Vertical Line Test for Functions – (VLT) If any vertical line intersects a graph in more than one point, the graph does not define y as a function of x.

Example 4: Plot the ordered pairs in Example 3 and use the Vertical Line Test to determine if the relation is a function.



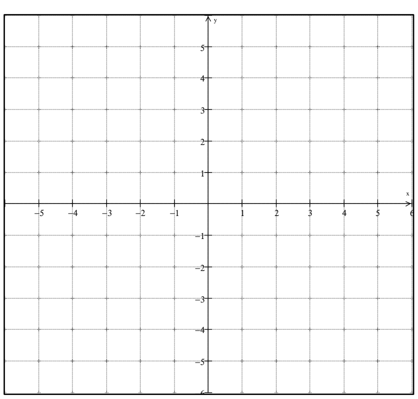


<u>Is the Equation a Function</u>?

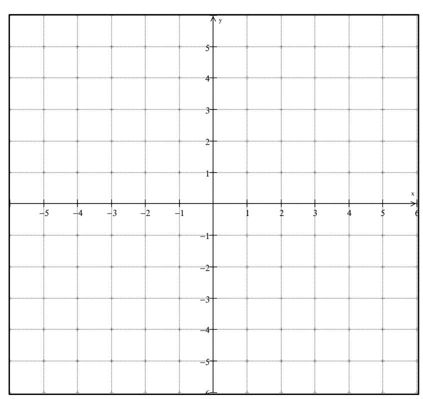
(When solving an equation for y in terms of x, if two or more values of y can be obtained for a given x, then the equation is NOT a function. It is a relation.)

Example 5: Solve the equations for y to determine if the equation defines a function. Also sketch a graph for each equation.

$x^2 + y = 4$



$y^2 + x = 4$



<u>Finding the Domain of a Function</u>: Determine what numbers are allowable inputs for *x*. This set of numbers is call the **domain**.

Example 6: Find the domain, using interval notation, of the function defined by each equation.

$y = 2x + 7$

D: ____________________

$y = \sqrt{3x - 5}$

D: ____________________

$y = \dfrac{x}{x + 3}$

D: ____________________

$y = |x| + 5$

D: ____________________

$y = \sqrt[3]{x - 10}$

D: ____________________

$y = \dfrac{x + 1}{x^2 - 5x - 6}$

D: ____________________

<u>Function Notation/Evaluating a Function</u>: The notation $y = f(x)$ provides a way of denoting the value of *y* (the **dependent variable**) that corresponds to some input number *x* (the **independent variable**).

Example 7: Given $f(x) = x^2 - 2x - 3$, evaluate and simplify

$f(0) =$

$f(-2) =$

$f(a) =$

$f(-x) =$

$f(x + 2) =$

$f(x) - f(-x) =$

Example 8: A company produces tote bags. The fixed costs for producing the bags are $12,000 and the variable costs are $3 per tote bag.

Write a function that describes the total cost, C, of producing b bags. ______________________

Find $C(200)$. __________

Find the cost of producing 625 tote bags. ___________________

Definition of Difference Quotient: $\dfrac{f(x+h)-f(x)}{h}$ where $h \neq 0$

The difference quotient is important when studying calculus. The difference quotient can be used to find quantities such as velocity of a guided missile or the rate of change of a company's profit or loss.

Example 9: Find and simplify the **difference quotient** for the functions below.

$f(x) = -2x - 3$

$f(x) = -3x^2 - 2x + 5$

3.1 Homework Problems

1. Determine whether each equation defines y to be a function of x.

   (a) $y = -3$ (b) $y + 9x^2 - 2 = 0$ (c) $y^2 - 4x = 3$

   (d) $x + y^3 = 27$ (e) $|x| + y = 7$ (f) $|x + y| = 7$

2. Find the domain of each function using interval notation.

   (a) $f(x) = 3x + 5$ (b) $f(x) = x^2 - 9x + 5$ (c) $f(x) = \sqrt{x - 3}$

   (d) $f(x) = \sqrt{3 - 2x}$ (e) $f(x) = \sqrt{x^2 - 2x - 3}$ (f) $f(x) = \dfrac{x + 5}{5x + 10}$

   (g) $f(x) = \sqrt[3]{3 - x}$ (h) $f(x) = \sqrt{x^2 - 9}$

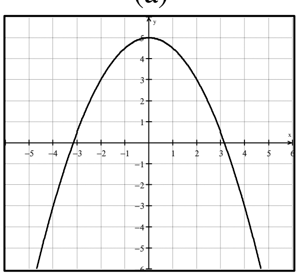
3. Let the function f be defined by $y = 2x^2 - 3x - 5$. Find each of the following:

   (a) $f(0)$ (b) $f(-1)$ (c) $f(k)$ (d) $f(-x)$

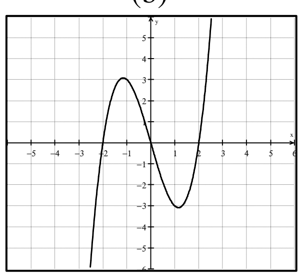
   (e) $f(3x)$ (f) $f(x - 1)$ (g) $f(x^2)$ (h) $f(-x) - f(x)$

4. Refer to the graphs of the relations below to determine whether each graph defines y to be a function of x. Then find the domain and range of each relation.

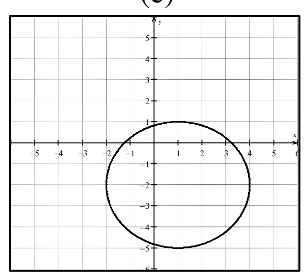
(a)



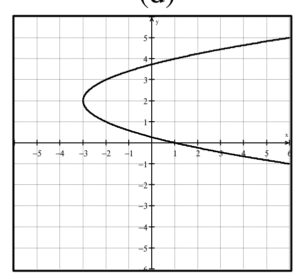
(b)



(c)



(d)



5. Evaluate the difference quotient for each function.

   (a) $f(x) = 5x$ (b) $f(x) = 6x + 8$ (c) $f(x) = x^2$

   (d) $f(x) = x^2 - 4x + 3$ (e) $f(x) = 2x^2 + x - 1$ (f) $f(x) = -2x^2 + 5x + 7$

6. Amy is purchasing t-shirts for her softball team. A local company has agreed to make the shirts for \$9 each plus a graphic arts fee of \$85. Write a linear function that describes the cost, C, for the shirts in terms of q, the quantity ordered. Then find the cost of order 20 t-shirts.

7. The cost, C, of water is a linear function of g, the number of gallons used. If 1000 gallons cost \$4.70 and 9000 gallons cost \$14.30, express C as a function of g.

8. If 50 U.S. dollars can be exchanged for 69.5550 Euros and 125 U.S. dollars can be exchanged for 173.8875 Euros, write a linear function that represents the number of Euros, E, in terms of U.S. dollars, D.

9. The Fahrenheit temperature reading (F) is a linear function of the Celsius reading (C). If $C = 0$ when $F = 32$ and the readings are the same at -40°, express F as a function of C.

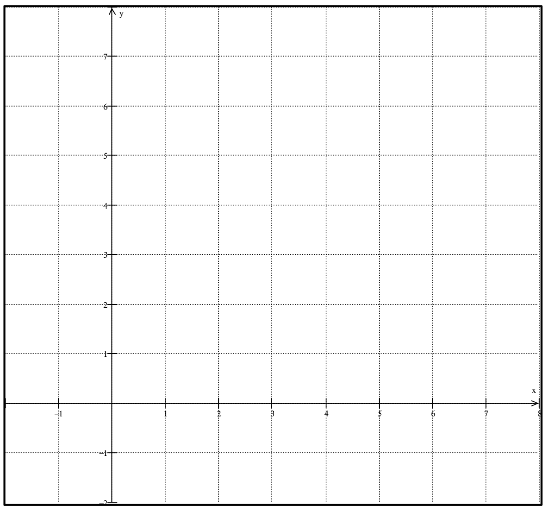
3.1 Homework Answers: 1. (a) function; (b) function; (c) not a function; (d) function; (e) function; (f) not a function **2.** (a) $(-\infty, \infty)$; (b) $(-\infty, \infty)$; (c) $[3, \infty)$; (d) $\left(-\infty, \frac{3}{2}\right]$; (e) $(-\infty, -1] \cup [3, \infty)$; (f) $(-\infty, -2) \cup (-2, \infty)$; (g) $(-\infty, \infty)$; (h) $(-\infty, -3] \cup [3, \infty)$ **3.** (a) -5; (b) 0; (c) $2k^2 - 3k - 5$; (d) $2x^2 + 3x - 5$; (e) $18x^2 - 9x - 5$; (f) $2x^2 - 7x$; (g) $2x^4 - 3x^2 - 5$; (h) $6x$ **4.** (a) function; D: $(-\infty, \infty)$; R: $(-\infty, 5]$; (b) function; D: $(-\infty, \infty)$; R: $(-\infty, \infty)$; (c) not a function; D:[-2, 4]; R:[-5, 1]; (d) not a function; D: $[-3, \infty)$; R: $(-\infty, \infty)$ **5.** (a) 5; (b) 6; (c) $2x + h$; (d) $2x + h - 4$; (e) $4x + 2h + 1$; (f) $-4x - 2h + 5$ **6.** $C(q) = 9q + 85$; \$265 **7.** $C(g) = \frac{3}{2500}g + \frac{7}{2}$ **8.** $E(D) = 1.3911D$ **9.** $F(C) = \frac{9}{5}C + 32$

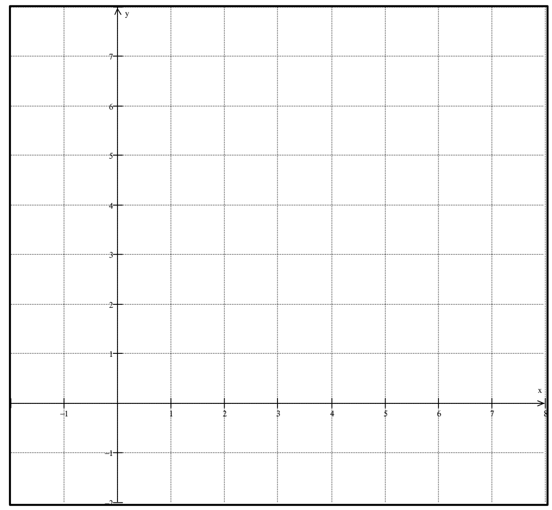
3.2 Quadratic Functions

In this section you will learn to:

- recognize the characteristics of quadratics functions
- find the vertex of a parabola
- graph quadratic functions
- apply quadratic functions to real world problems
- solve maximum and minimum problems

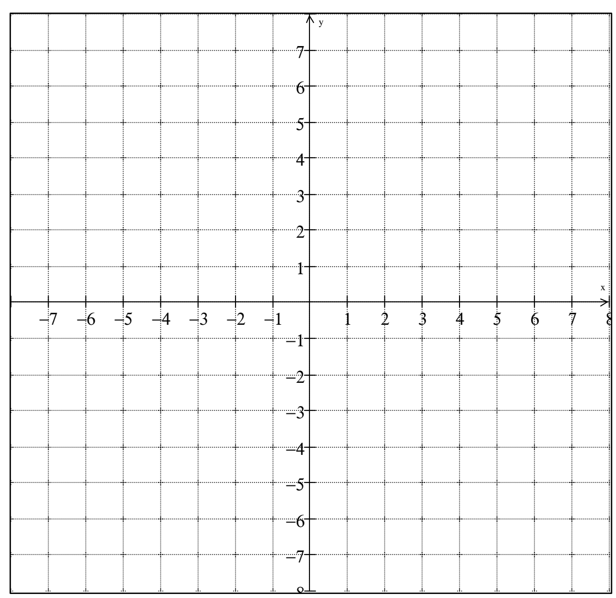
<u>Graphs of Quadratic Functions</u>:





The **Standard Form of a Quadratic Function** is $y = f(x) = a(x-h)^2 + k$, where $a \neq 0$	
Its graph is a parabola with vertex at **(*h, k*).**	If a > 0, then the parabola opens up.
Its graph is symmetric to line $x = h$	If a < 0, then the parabola opens down.

Example 1: Graph the quadratic function $f(x) = -(x+2)^2 + 3$.



Steps:

1. Opens up or down?
   ($a > 0$ or $a < 0$)

2. Find vertex (h, k).
   Find the domain.
   Find the range.

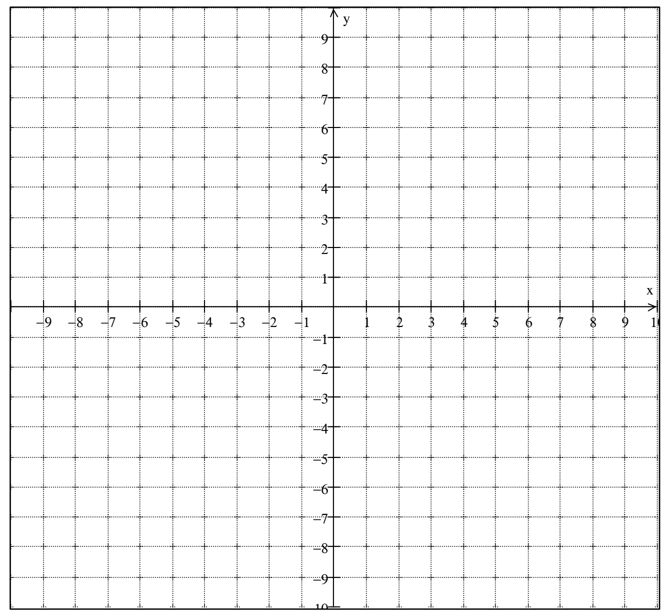
3. Find x-intercepts.
   (Let $y = 0$.)

4. Find y-intercept.
   (Let $x = 0$.)

5. Graph the parabola.
   Plot intercepts, vertex and
   additional point(s). (Use line/axis of symmetry.)

The **General Form of a Quadratic Function** is $y = f(x) = ax^2 + bx + c$, where $a \neq 0$	
Graph is a parabola with vertex at $\left(-\frac{b}{2a}, f\left(-\frac{b}{2a}\right)\right)$ or $\left(-\frac{b}{2a}, c - \frac{b^2}{4a}\right)$.	If a > 0, then the parabola opens up.
	If a < 0, then the parabola opens down.
Graph is symmetric to the line $x = -\frac{b}{2a}$.	*y*-intercept is (0, *c*).

Example 2: Graph the quadratic function $f(x) = x^2 - 2x - 8$.



Steps:

1. Opens up or down?
   ($a > 0$ or $a < 0$)

2. Find vertex (*h, k*).

   Domain:

   Range:

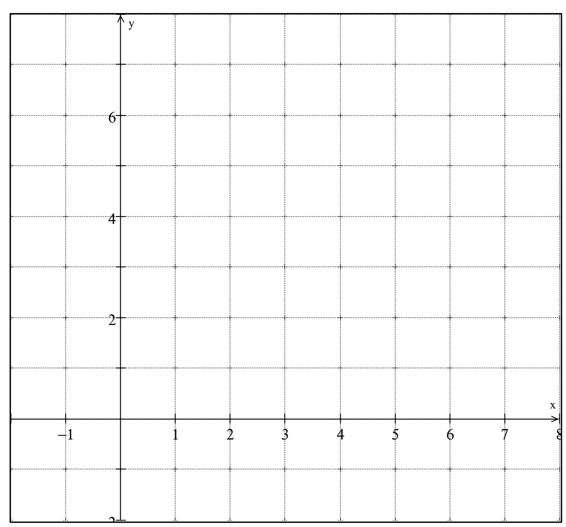
   Eq. of line of symmetry:

3. Find *x*-intercepts.
   (Let $y = 0$.)

4. Find *y*-intercept.
   (Let $x = 0$.)

5. Graph the parabola.
   Plot intercepts, vertex and additional point(s).
   (Use line/axis of symmetry.)

Example 3: For the parabola defined by $f(x) = x^2 - 6x + 11$, find



(a) the coordinates of the vertex.

(b) the *x*- and *y*-intercepts.

(c) the domain and range.

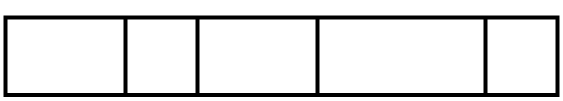
(d) Sketch the graph of *f*.

Example 4: Write an equation in standard form of the parabola that has vertex (5, 4) and passes through the point $(-2, 151)$.

Example 5: You have 400 feet of fencing to enclose a rectangular plot. Find the length and width of the plot that will maximize the area. What is the largest area that can be enclosed?

Example 6: A rectangular plot is to be fenced off and divided into two parts/plots on land that borders the river with each part bordering the river. If you do not fence the side along the river, find the length and width of the plot that will maximize the area if you have 400 feet of fence. What is the largest area that can be enclosed?

Example 7: You have 600 feet of fencing to enclose five animal pens as shown below. Find the length and width of the outer dimensions that will maximize the area. What is the largest area that can be enclosed?



Example 8: A rocket is shot up vertically close to the edge on the top of a 300-foot cliff. The quadratic function $h(t) = -16t^2 + 128t + 300$ models the rocket's height above the ground, $h(t)$, in feet, t seconds after it is launched.

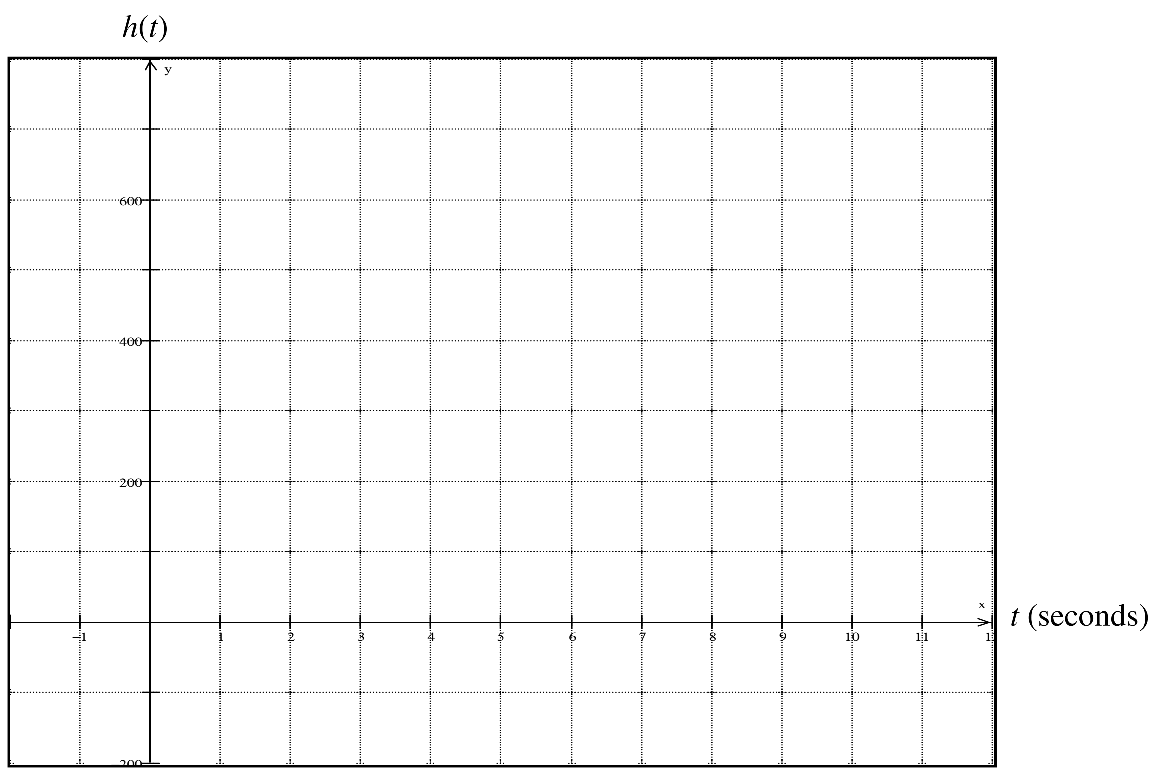
(a) How many seconds does it take the rocket to reach its maximum height?

(b) What is its maximum height?

(c) Find $h(0)$. What does this mean?

(d) When does the rocket hit the ground?

(e) Graph this quadratic function.

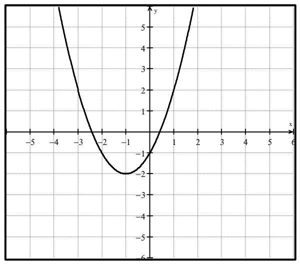


3.2 Homework Problems:

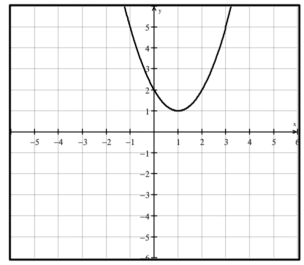
Match each of the equations below with its graph.

1. $y = x^2 - 2x + 1$
2. $y = (x+1)^2 - 2$
3. $y = x^2 - 2x + 2$
4. $y = -x^2 - 1$

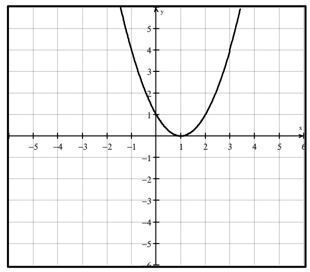
(a)



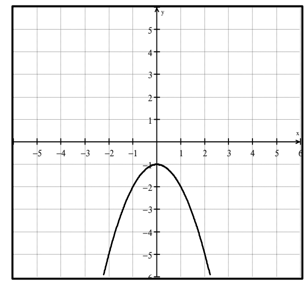
(b)



(c)



(d)



5. Find the vertex of each parabola.

   (a) $f(x = 2(x-3)^2 - 1$ (b) $f(x) = -3(x+1)^2 + 5$ (c) $f(x) = 2x^2 - 8x + 3$

   (d) $f(x) = -2x^2 + 8x - 1$ (e) $f(x) = x^2 + 3x - 10$ (f) $f(x) = 5 - 4x - x^2$

6. Consider the quadratic function $f(x) = (x-1)^2 - 4$.

   (a) Find the coordinates of the vertex for the parabola.
   (b) Find the equation for the axis of symmetry.
   (c) Find the *x*- and *y*-intercept(s).
   (d) Identify the function's domain and range.

7. Consider the quadratic function $f(x) = -2x^2 - 12x + 3$.

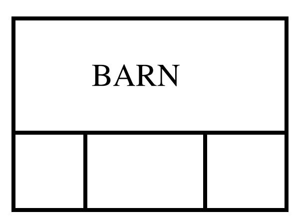
   (a) Find the coordinates of the vertex for the parabola.
   (b) Find the equation for the axis of symmetry.
   (c) Identify the function's domain and range.

8. Write an equation in standard form of the parabola that has the characteristics below.

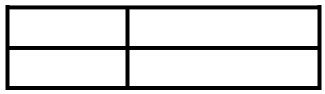
   (a) vertex at (1, -8); passing through the point (3, 12)
   (b) vertex at (5, 2); passing through the point (8, -25)

9. A rectangular plot is to be fenced off on all four sides and divided into two parts/plots. Find the length and width of the plot that will maximize the area if you have 200 feet of fence. What is the largest area that can be enclosed?

10. A rectangular plot is to be fenced off and divided into three parts/plots on land that borders a barn. If you do not fence the side along the barn, find the length and width of the plot that will maximize the area if you have 200 feet of fence. What is the largest area that can be enclosed?



11. You have 300 feet of fencing to enclose four garden plots as shown below. Find the length and width of the outer dimensions that will maximize the area. What is the largest area that can be enclosed?



12. A rain gutter is made from sheets of aluminum that are 20 inches wide by turning up the edges to form right angles. Determine the depth of the gutter that will maximize its cross-section area and allow the greatest amount of water to flow. What is the maximum cross-sectional area?

13. The path of a basketball thrown from the free throw line can be modeled by the quadratic function $f(x) = -0.06x^2 + 1.5x + 6$, where x is the horizontal distance (in feet) from the free throw line and $f(x)$ is the height (in feet) of the ball. Find the maximum height of the basketball. If the ball thrown is an air ball, how far from the free throw line will the ball land? (Round to nearest tenths.)

14. A rocket is shot up vertically from a 10-foot platform. The quadratic function $h(t) = -16t^2 + 256t + 10$ models the rocket's height above the ground, $h(t)$, in feet, t seconds after it is launched. (Round to nearest tenths.)

    (a) How many seconds does it take the rocket to reach its maximum height?
    (b) What is its maximum height?
    (c) Find $h(0)$.
    (d) How long will it take the rocket to hit the ground?

15. The annual yield per apple tree is fairly constant at 320 pounds when the number of trees per acre is 50 or fewer. For each additional tree over 50, the annual yield per tree for all trees on the acre decreases by 4 pounds due to overcrowding. Find the number of trees that should be planted on an acre to produce the maximum yield. How many pounds is the maximum yield?

3.2 Homework Answers: **1.** c **2.** a **3.** b **4.** d **5.** (a) (3, -1); (b) (-1, 5); (c) (2, -5); (d) (2, 7); (e) $\left(-\frac{3}{2}, -\frac{49}{4}\right)$; (f) (-2, 9) **6.** (a) (1, -4); (b) x = 1; (c) x-intercepts: 1, 3; y-intercept: -3; (d) D: $(-\infty, \infty)$; R: $[-4, \infty)$ **7.** (a) (-3, 21); (b) x = -3; (c) D: $(-\infty, \infty)$; R: $(-\infty, 21]$;

8. (a) $y = 5(x-1)^2 - 8$; (b) $y = -3(x-5)^2 + 2$ **9.** $33\frac{1}{3}$ feet by 50 feet; $1666\frac{2}{3}$ square feet

10. 25 feet by 100 feet; 2500 square feet **11.** 50 feet by 50 feet; 2500 square feet

12. 5 inches; 50 square inches **13.** 15.38 feet; 28.51 feet **14.** (a) 8 seconds; (b) 1034 feet;

(c) 10; (d) 16.04 seconds **15.** 65 trees; 16,900 pounds

3.3 Polynomial and Other Functions

In this section you will learn to:

- understand characteristics of polynomial functions
- find intervals on which a function is increasing, decreasing, or constant
- find the relative maximum or minimum of a function
- determine whether a function is even, odd, or neither
- graph and evaluate piecewise functions

A **polynomial function of degree *n*** is defined by $f(x) = a_n x^n + a_{n-1}x^{n-1} + . \quad . \quad . \quad . \quad . + a_2 x^2 + a_1 x + a_0$,
where n is a **nonnegative integer,** and $a_n, a_{n-1}, a_{n-2}, \quad . \quad . \quad . a_2, a_1, a_0$ are **real numbers** and $a_n \neq 0$.

a_n is called the **leading coefficient.**

a_0 is called the **constant term.**

The **degree** of the polynomial is ***n***

Graphs of polynomials are smooth (rounded curves) and continuous (no breaks).

A polynomial of degree n has **at most *n*-1** turning points (graph changes direction).

Recall: $y = f(x) = c$ degree = ______ ____________________________

$y = f(x) = mx + b$ degree = ______ ____________________________

$y = f(x) = ax^2 + bx + c$ degree = ______ ____________________________

$y = f(x) = ax^3 + bx^2 + cx + d$ degree = ______ ____________________________

Example 1: Determine which functions are polynomial functions. For those that are, identify the degree. For those that are not, explain why they are not polynomial functions.

(a) $f(x) = 5x^{\frac{2}{3}} + 3x^2 - 7$ Yes No ____

(b) $g(x) = 10$ Yes No ____

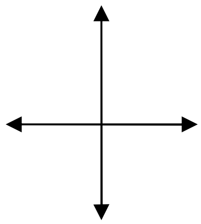
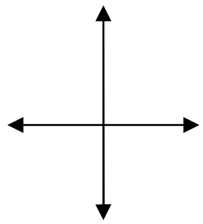
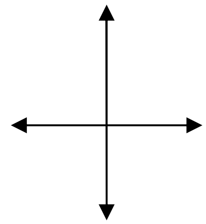
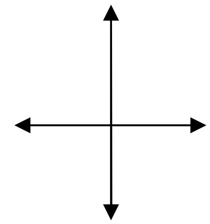
(c) $h(x) = x\sqrt{7} + \pi x^3$ Yes No ____

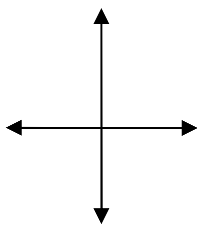
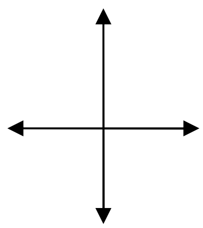
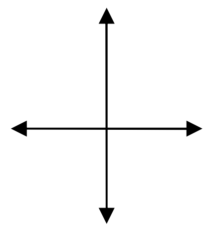
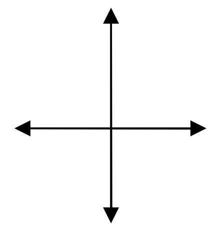
(d) $f(x) = \dfrac{3x^2 + 5}{x}$ Yes No ____

(e) $g(x) = |x|$ Yes No ____

(f) $h(x) = \dfrac{3x^2 + 5}{2}$ Yes No ____

End Behavior of a Polynomial (what happens to the graph of the function to the far left ($x \to -\infty$) and far right ($x \to \infty$)) and **Leading Coefficient (a_n) Test**

Degree (n) is Even	**Degree (n) is Even**	**Degree (n) is Odd**	**Degree (n) is Odd**
$\mathbf{a_n > 0}$	$\mathbf{a_n < 0}$	$\mathbf{a_n > 0}$	$\mathbf{a_n < 0}$
			

Think: $y = x^2$ 	Think: $y = -x^2$ 	Think: $y = x^3$ 	Think: $y = -x^3$ 

Example 2: Without using a calculator, determine the end behavior of the following.

(a) $f(x) = x^5 - x^3 - 1$

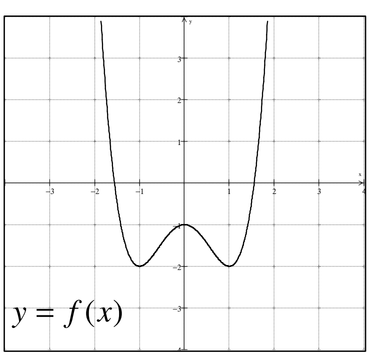
(b) $f(x) = -4x^4 - 3x^2 + 7$

(c) $f(x) = -5(x-3)^2(x+2)^3$

(d) $f(x) = -x^3 + 8x^4 + 4x^2 + 2$

Relative Maximum/Minimum: The point(s) at which a function changes its increasing or decreasing behavior. These points are also called **turning points**.
A function is **increasing** if the y values increase on the graph of f from left to right.
A function is **decreasing** if the y values decrease on the graph of f from left to right.
A function is **constant** if the y values remain unchanged on the graph of f from left to right.

Example 3:



f has a **relative minimum(s)** at ____________.

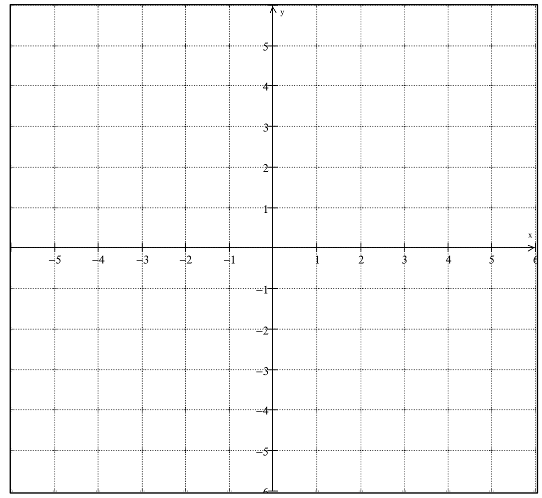
The **relative minimum(s)** of f are ___________.

f has a **relative maximum(s)** at ____________.

The **relative maximum(s)** f are ___________.

On which intervals is f increasing? _____________ decreasing? ______________ constant? ___________

Example 4: Use the following steps to graph to graph $f(x) = x^4 - 4x^2$.



<u>Steps for Graphing a Polynomial Function:</u>

1. Use **Leading Coefficient Test** to determine End Behavior.

2. Find the ***x*-intercept(s)**. Let $f(x) = 0$.

3. Find the ***y*-intercept**. Let $x = 0$.

4. Determine where the graph is above or below x-axis.

5. Plot a few points and draw a smooth, continuous graph.

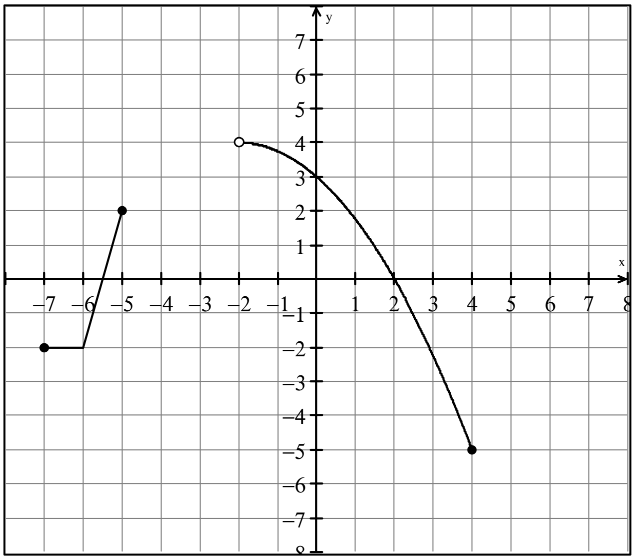
6. Use # of **turning points** to check graph accuracy.

Even Functions	**Odd Functions**
Note: If $f(x) \neq f(-x)$ and $f(x) \neq -f(-x)$, then $f(x)$ is "**neither**" odd nor even.	

Example 5: Determine whether each of the functions below is even, odd, or neither.

$f(x) = x^3 - 3x$ ________ $f(x) = x^4 - 2x^2 + 1$ ________ $f(x) = x^3 - x^2 - 6x$ ________

Example 6: Refer to the graph below to answer/find the following:



(a) Is the graph a function? _______

(b) Is this a graph of a polynomial function? ___

(c) domain _________

(d) range __________

(e) $f(0)$ = ___________

(f) x-intercept(s) ___________

(g) y-intercept(s) ___________

(h) increasing (interval) ____________

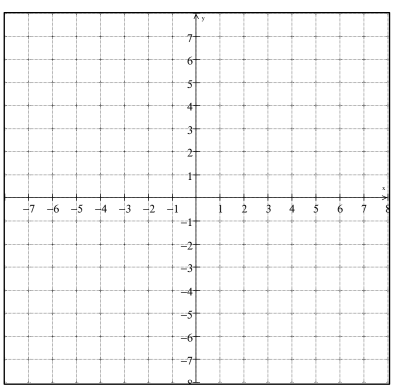
(i) decreasing (interval) ____________

(j) constant (interval) ___________

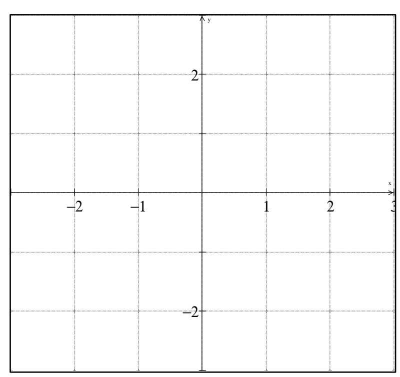
(k) For what value(s) of x does $f(x) = -5$? _____

Piecewise Function: A function that is defined by two (or more) equations over a specified domain.

Example 7: Graph $f(x) = \begin{cases} -x, & x < 0 \\ x^2 - 3, & x \geq 0 \end{cases}$



Example 8: Graph $f(x) = \begin{cases} 1, & \text{if } x > 0 \\ 0, & \text{if } x = 0 \\ -1, & \text{if } x < 0 \end{cases}$



Example 9: Given $f(x) = \begin{cases} 2x, & \text{if } x \geq 5 \\ x^2 - 3, & \text{if } x < 5 \end{cases}$, evaluate each of the following:

(a) $f(0) - f(5)$

(b) $5f(-3) - [f(6)]^2$

3.3 Homework Problems:

1. Determine which functions are polynomial functions.

   (a) $f(x) = x^5 - \sqrt{3}x^2$ (b) $g(x) = x^{-2} + 8x^{-1} - 9$ (c) $h(x) = 2.5x^3 - \pi x^2 + 2$

   (d) $g(x) = 6x^7 + \dfrac{1}{x}$ (e) $f(x) = x^{\frac{1}{2}} - 5$ (f) $h(x) = \dfrac{3x^3 + 2x^2}{x^5}$

2. Use the Leading Coefficient Test to determine the end behavior of the graph of *f*.

   (a) $f(x) = -x^4 - x^2$ (b) $f(x) = 7x^3 - 4x^2$ (c) $f(x) = x^8$

   (d) $f(x) = 9 - x^3$ (e) $f(x) = (x-2)^2(x+3)^3$ (f) $f(x) = -2x(x+3)^2(x-5)$

3. Consider the graph of the function $f(x) = x^4 - 9x^2$.

   (a) Use Leading Coefficient Test to determine the end behavior of the function.
   (b) Find the x-intercept(s).
   (c) Find the y-intercept.
   (d) For what intervals is the graph above the x-axis?

4. Consider the graph of the function $f(x) = 6x^2 + x^3 - x^4$.

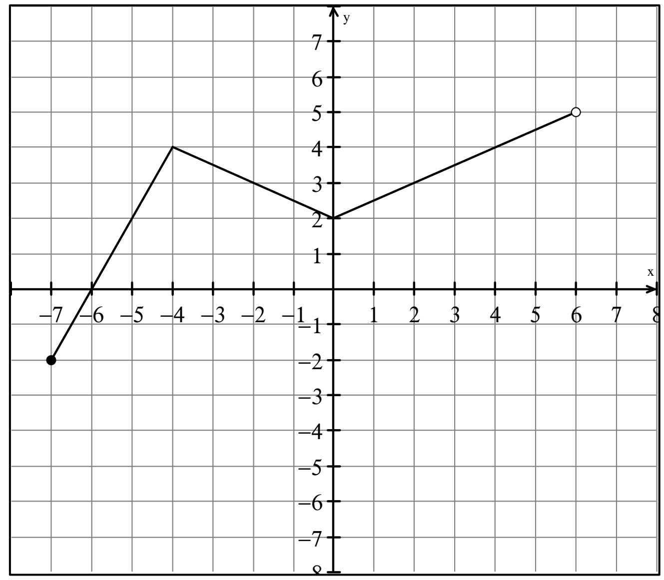
   (a) Use Leading Coefficient Test to determine the end behavior of the function.
   (b) Find the x-intercept(s).
   (c) Find the y-intercept.
   (d) For what intervals is the graph above the x-axis?

5. Determine whether each function is even, odd, or neither.

   (a) $f(x) = x^4 + 5x^2$ (b) $g(x) = -5x^3 - 3x^2 + 7$ (c) $h(x) = x^5 + 2x^3 - x$

   (d) $g(x) = 5x^2 + 6$ (e) $f(x) = -3$ (f) $f(x) = x^3 - 1$

6. Refer to the graph of f below to determine each of the following:
(Use **interval notation** whenever possible.)



(a) the domain of f
(b) the range of f
(c) x-intercept(s)
(d) y-intercept(s)
(e) interval(s) on which f is increasing
(f) interval(s) on which f is decreasing
(g) values of x for which $f(x) < 0$
(h) number(s) at which f has a relative maximum
(i) relative maximum of f
(j) $f(-2)$
(k) value(s) for which $f(x) = 0$
(l) values for which $f(x) = 2$
(m) Is f even, odd, or neither?

7. Given the piecewise function $f(x) = \begin{cases} x & \text{if } x < 0 \\ x^2 & \text{if } x \geq 0 \end{cases}$, evaluate the following:

(a) $f(0)$ (b) $f(10)$ (c) $f(-3) + f(5)$ (d) $-3f(-1) \cdot f(2)$ (e) $7 - f(-5)$

8. Given the piecewise function $f(x) = \begin{cases} 0 & \text{if } x < -2 \\ x+2 & \text{if } -2 \leq x \leq 2 \\ \sqrt{x} & \text{if } x > 2 \end{cases}$, evaluate the following:

(a) $f(-4) + f(0) + f(4)$ (b) $3f(-2) - 5f(2)$ (c) $2[f(9)]^2$ (d) $\dfrac{f(100) + f(1)}{13 + f(-10)}$

3.3 Homework Answers: 1. (a) polynomial; (b) not a polynomial; (c) polynomial; (d) not a polynomial; (e) not a polynomial; (f) not a polynomial **2.** (a) falls right and left; (b) falls left and rises right; (c) rises right and left; (d) rises left and falls right; (e) falls left and rises right; (f) falls left and right **3.** (a) rises left and right; (b) -3, 0, 3; (c) 0; (d) $(-\infty, -3)$ and $(3, \infty)$ **4.** (a) falls left and right; (b) 0, 3, -2; (c) 0; (d) (-2, 0) and (0, 3) **5.** (a) even; (b) neither; (c) odd; (d) even; (e) even; (f) neither **6.** (a) [-7, 6); (b) [-2, 5); (c) -6; (d) 2; (e) (-7, -4) and (0, 6); (f) (-4, 0); (g) [-7, -6); (h) x = -4; (i) 4; (j) 3; (k) x = -6; (l) x = -5, 0; (m) neither **7.** (a) 0; (b) 100; (c) 22; (d) 12; (e) 12 **8.** (a) 4; (b) -20; (c) 18; (d) 1

3.4 Transformation of Graphs of Functions

In this section you will learn to:

- recognize and graph common functions
- use vertical and horizontal shifts to graph functions
- use reflections about the *x*- and *y*- *axes* to graph functions
- use vertical and horizontal stretching/shrinking to graph functions
- graph functions using a sequence of transformations

Common Function/Graph	Domain	Range	Inc/Dec	Even/Odd
Constant Function: $f(x) = c$				
Identity Function: $f(x) = x$				
Absolute Value Function: $f(x) = \lvert x \rvert$				
Standard Quadratic Function: $f(x) = x^2$				
Square Root Function $f(x) = \sqrt{x}$				
Standard Cubic Function $f(x) = x^3$				
Cube Root Function $f(x) = \sqrt[3]{x}$				

Transformation Procedures (for $c > 0$)

Horizontal Shift	$y = f(x+c)$ $y = f(x-c)$	Moves graph c units to the **left**. Moves graph c units to the **right**.
Vertical Shift	$y = f(x)+c$ $y = f(x)-c$	Moves graph c units **up**. Moves graph c units **down**.
Reflection	$y = f(-x)$ $y = -f(x)$	Reflects graph about the ***y*-axis**. Reflects graph about the ***x*-axis**.
Horizontal Stretch/Shrink	$y = f(cx)$	**Shrinks** horizontally when $c > 1$. **Stretches** horizontally when $0 < c < 1$.
Vertical Stretch/Shrink	$y = cf(x)$	**Stretches** vertically when $c > 1$. **Shrinks** vertically when $0 < c < 1$.

Transformation Sequence:

Horizontal Shift	**H**
Stretch	**S**
Reflection	**R**
Vertical Shift	**V**

Graphing Using a Sequence of Transformations

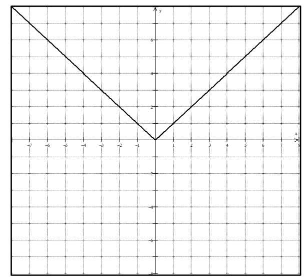
Transformations of functions can be performed graphically or using a "table" method as illustrated in the example below.

Example: Consider the standard absolute value function, $f(x) = |x|$. Use transformations of this graph to sketch $g(x) = -\left|\frac{1}{2}x + 3\right| - 1$.

Table Method: **Graph:**

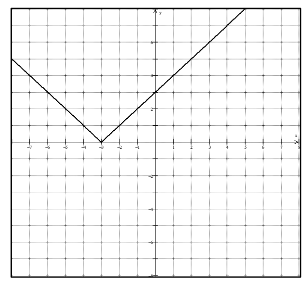
For the first set of coordinates, pick some ordered pairs on the graph of *f*. (Remember, you are starting with the graph of f and transforming this to the graph of g.)

x	$f(x) = \|x\|$
-2	2
-1	1
0	0
1	1
2	2



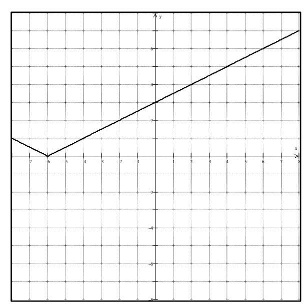
Horizontal Shift: Shift the graph 3 units to the left by subtracting 3 from each *x*-coordinate.

x	$\|x+3\|$
-5	2
-4	1
-3	0
-2	1
-1	2



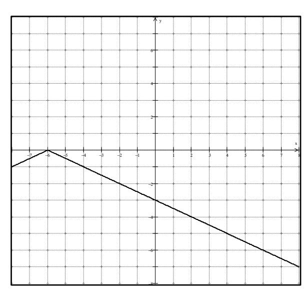
Stretch (horizontal): Stretch the graph horizontally by multiplying each *x*-coordinate by 2.

x	$\left\|\frac{1}{2}x+3\right\|$
-10	2
-8	1
-6	0
-4	1
-2	2



Reflection: Reflect the graph about the *x*-axis by multiplying each *y*-coordinate by -1.

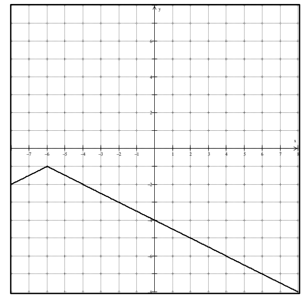
x	$-\left\|\frac{1}{2}x+3\right\|$
-10	-2
-8	-1
-6	0
-4	-1
-2	-2



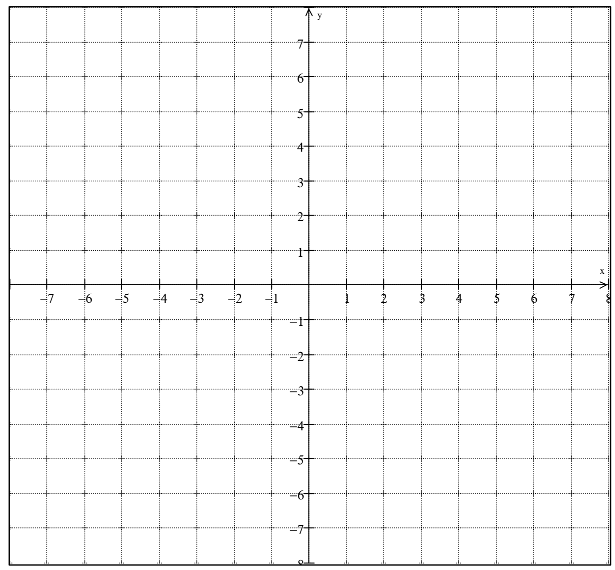
Vertical shift: Finally shift the graph 1 unit downward by subtracting 1 from each y-coordinate.

Use the coordinates in this last table to sketch the graph of $g(x)$.

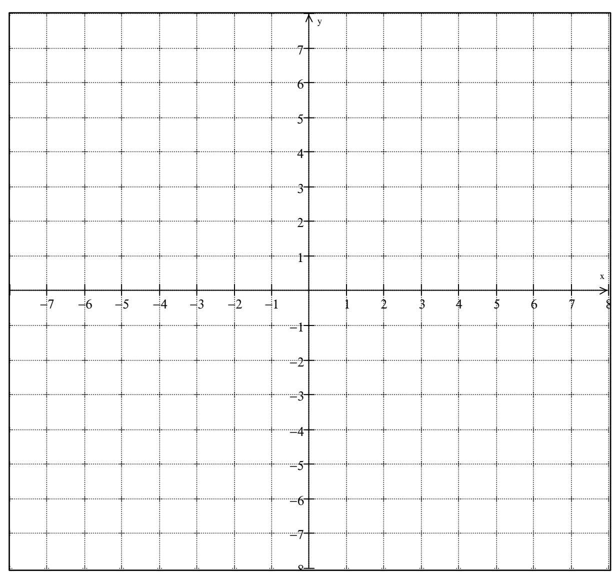
x	$g(x) = -\left\lvert \frac{1}{2}x + 3 \right\rvert - 1$
-10	-3
-8	-2
-6	-1
-4	-2
-2	-3



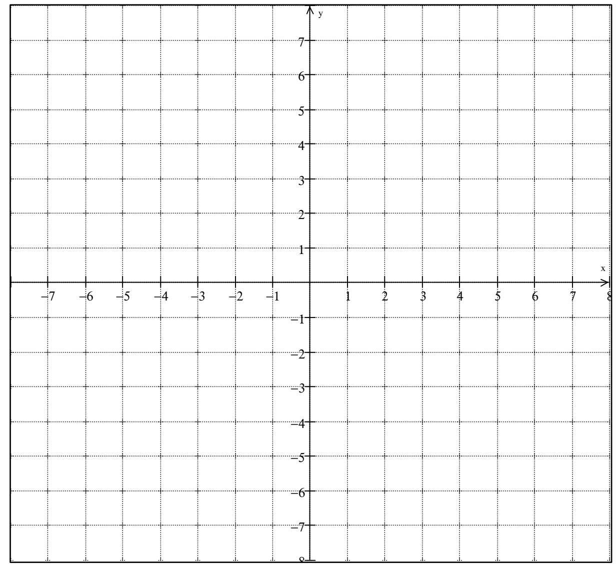
Example 1: Use the graph of $f(x) = x$, to graph $g(x) = f(x+2) - 3$.



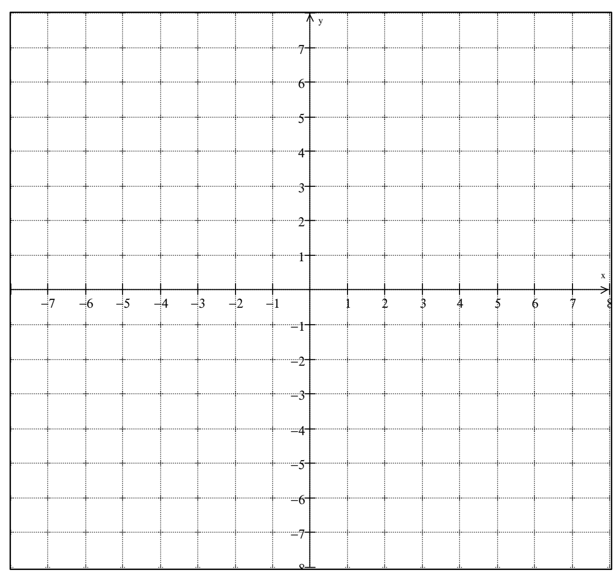
Example 2: Use the graph of $y = \sqrt{x}$, to graph $h(x) = \sqrt{-x+1}$.



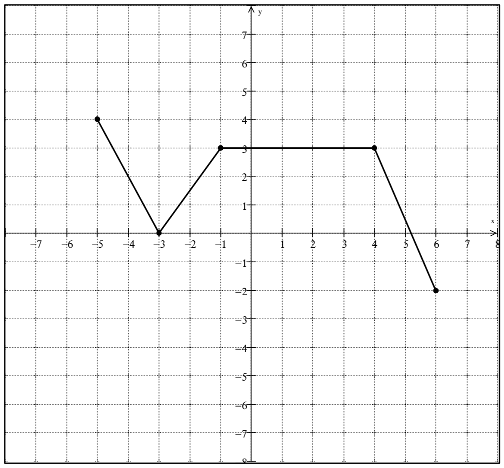
Example 3: Use the graph of $f(x) = |x|$ to graph $g(x) = -\left|\frac{1}{2}x - 3\right| - 4$.



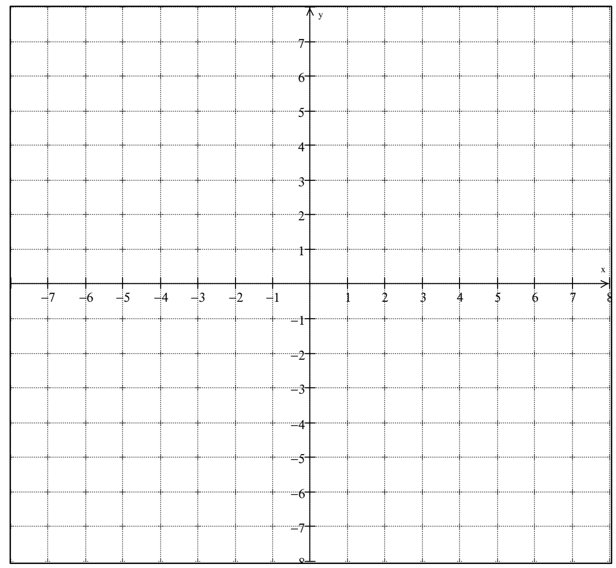
Example 4: Use the graph of $f(x) = x^2$ to graph $g(x) = -\frac{1}{2}f(-x+2) + 4$.



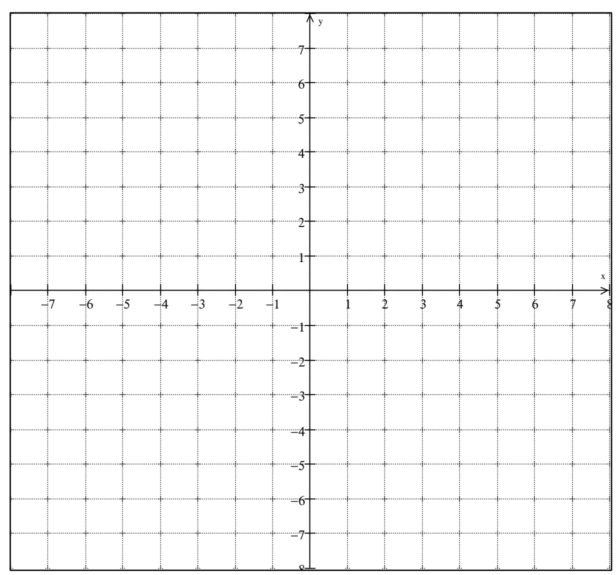
Example 5: Use the graph of the function f below to sketch the graphs of the given functions g and h.



$g(x) = f(x+2)+3$



$h(x) = -2f(x-1)+2$



3.4 Homework Problems:

1. The graph of the function, g, is a transformation of $f(x) = x^3$. Describe each transformation and the sequence in which each transformation should be performed.

   (a) $g(x) = x^3 + 2$ (b) $g(x) = 2x^3 - 3$ (c) $g(x) = (3x-1)^3$

   (d) $g(x) = -3(x+5)^3 - 7$ (e) $g(x) = 5 + \left(-\frac{1}{2}x - 3\right)^3$ (f) $g(x) = -2 - (-x+5)^3$

2. The graph of each function, g, is a transformation of the graph of $f(x) = |x|$. Graph g using a series of transformations and then check the graph of g using the "graph" and "table" feature on your graphing calculator.

   (a) $g(x) = |x-3| - 2$ (b) $g(x) = -\left|\frac{1}{2}x + 3\right|$ (c) $g(x) = 3|-x+2|$ (d) $g(x) = -|1-x| + 4$

3. The graph of each function, g, is a transformation of the graph of $f(x) = \sqrt{x}$. Graph g using a series of transformations and then check the graph of g using the "graph" and "table" feature on your graphing calculator.

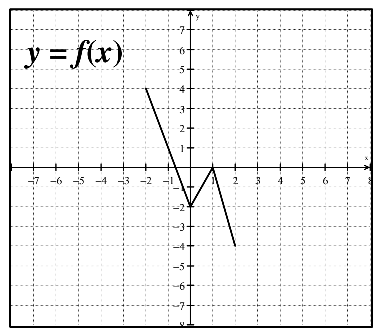
   (a) $g(x) = \sqrt{x} - 2$ (b) $g(x) = -\sqrt{x}$ (c) $g(x) = -2\sqrt{x-1}$

4. The graph of g is a transformation of f. Use the graph of f to graph g.

   (a) $g(x) = f(x+1) - 3$

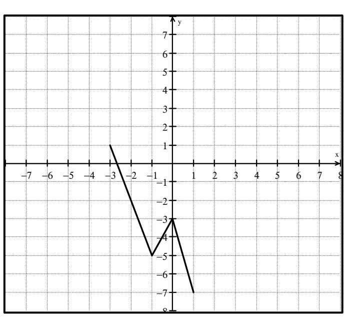
   (b) $g(x) = 2f(-x-4)$

   (c) $g(x) = -f\left(\frac{1}{2}x + 1\right) + 3$

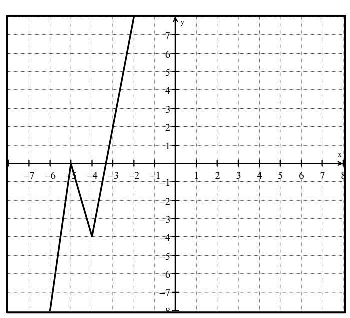


3.4 Homework Answers: 1. (a) vertical shift of 2 units up; (b) vertical stretch by a factor of 2; vertical shift of 3 units down; (c) horizontal shift of 1 unit right; horizontal stretch by a factor of 1/3; (d) horizontal shift of 5 units left; vertical stretch by a factor of 3; reflection over the *x*-axis; vertical shift of 7 units down; (e) horizontal shift of 3 units right; horizontal stretch by a factor of 2; reflection over the *y*-axis; vertical shift of 5 units up; (f) horizontal shift of 5 units left; reflection over the *x*- and *y*-axes; vertical shift of 2 units down

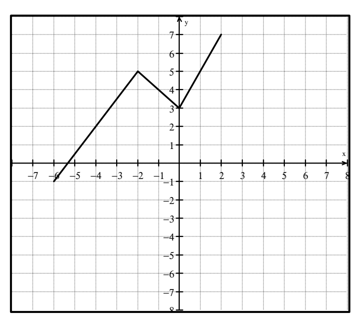
4. (a)



(b)



(c)



3.5 Rational Functions

In this section you will learn to:

- find the domain of a rational function
- find vertical and horizontal asymptotes of rational functions
- use transformations to graph rational functions
- use arrow notation

A **Rational Function, *f*,** is of the form $f(x) = \dfrac{p(x)}{q(x)}$, where p and q are polynomials and $q(x) \neq 0$ for all x.
The **domain** of f is the set of all real numbers, except the values that make the denominator 0. $(q(x) \neq 0)$

Example 1: Find the domain for each of the rational functions below. Write the domain in interval notation.

(a) $f(x) = \dfrac{x-5}{x-3}$ Domain: ______________________

(b) $g(x) = \dfrac{1}{x}$ Domain: ______________________

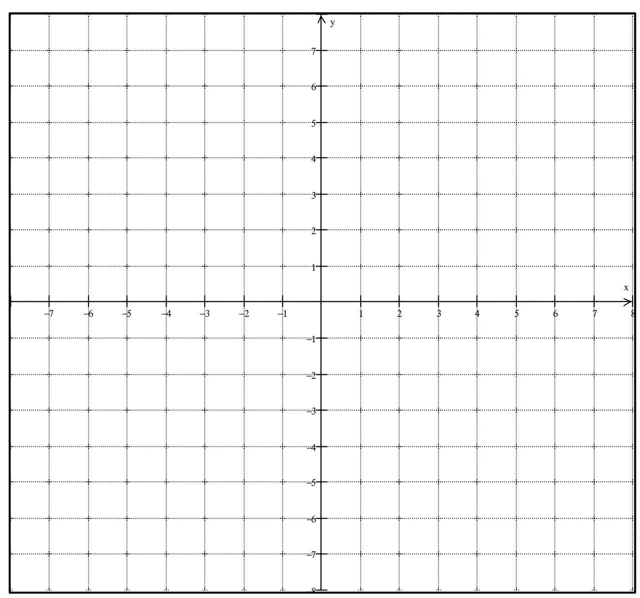
(c) $h(x) = \dfrac{x^2-9}{x^2-4}$ Domain: ______________________

(d) $f(x) = \dfrac{x+4}{x^2+4}$ Domain: ______________________

Example 2: Complete the tables for the **reciprocal function** $f(x) = \dfrac{1}{x}$**.** Graph the function and then find its domain and range.

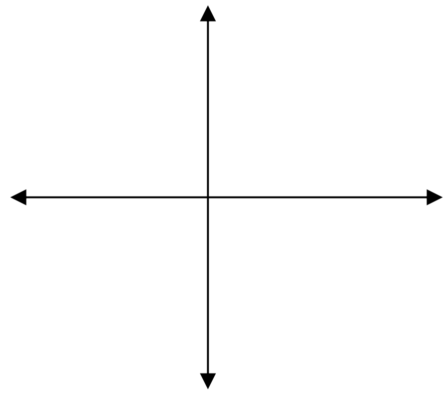
x	0	1	2	3	4	$\frac{1}{2}$	$\frac{1}{3}$	$\frac{1}{4}$
***f*(*x*)**								

x	0	-1	-2	-3	-4	$-\frac{1}{2}$	$-\frac{1}{3}$	$-\frac{1}{4}$
***f*(*x*)**								



Domain: ______________________ **Range:** ______________________

Arrow Notation	
$x \to a^+$	x approaches a from the right
$x \to a^-$	x approaches a from the left
$x \to \infty$	x approaches infinity (increases without bound)
$x \to -\infty$	x approaches negative infinity (decreases without bound)
$f(x) \to b$	$f(x)$ approaches b from above or below



Vertical Asymptote: $x = a$ is a vertical asymptote of f if $f(x)$ increases or decreases without bound as $x \to a$.

Horizontal Asymptote: $y = b$ is a horizontal asymptote of f if $f(x)$ approaches b as x increases or decreases without bound.

Example 3: Graph $f(x) = \dfrac{1}{x^2}$.

Then complete each of the following:

Vertical Asymptote at _____________

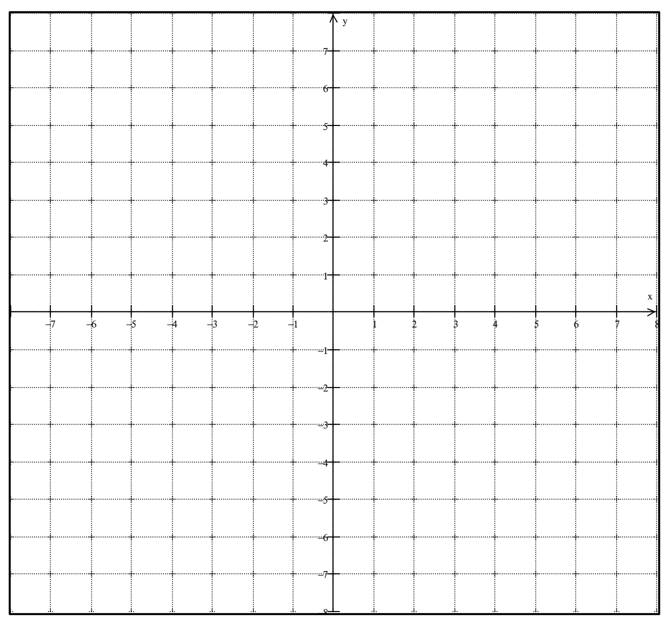
Horizontal Asymptote at _____________

As $x \to 0^+$, $f(x) \to$ ________ .

As $x \to 0^-$, $f(x) \to$ ________ .

As $x \to \infty$, $f(x) \to$ ________ .

As $x \to -\infty$, $f(x) \to$ ________ .



Domain of f: ___________________________________

Range of f: ___________________________________

Is this function even, odd, or neither? ______________

Finding Vertical and Horizontal Asymptotes	
Vertical	**Horizontal**
If $f(x)=\dfrac{p(x)}{q(x)}$ is a reduced rational function and a is a zero of $q(x)$, then the **vertical asymptote(s)** of the graph of f is (are) $\boldsymbol{x=a}$ (zeros of the denominator)	Given the rational function $f(x)=\dfrac{a_n x^n + a_{n-1}x^{n-1}+..+.a_0}{b_m x^m + b_{m-1}x^{m-1}+...+b_0}$ (where n = degree of numerator and m = degree of denominator), then the **horizontal asymptote** of the graph of f is $y=0$ if $n<m$ $\quad$ $y=\dfrac{a_n}{b_m}$ if $n=m$ $\quad$ No H. A. if $n>m$

Example 4: Find the **vertical asymptotes**, if any, of each rational function.

(a) $f(x)=\dfrac{x-5}{x-3}$ ____________ (b) $g(x)=\dfrac{1}{x}$ ____________ (c) $h(x)=\dfrac{x^2-9}{x^2-4}$ ____________

(d) $y=\dfrac{x+4}{x^2+4}$ ____________ (e) $f(x)=\dfrac{x-3}{x(x^2-5x-6)}$ ____________

Refer to Example 1 (a)- (d). For these problems, how does the **domain** of these functions relate to the **vertical asymptotes** in Example 4 (a)-(d)?

____

Example 5: Find the **horizontal asymptotes**, if any, of each rational function.

(a) $y=\dfrac{1}{x}$ ____________ (b) $f(x)=\dfrac{1}{x^2}$ ____________ (c) $g(x)=\dfrac{3x^2}{x^2}$ ____________

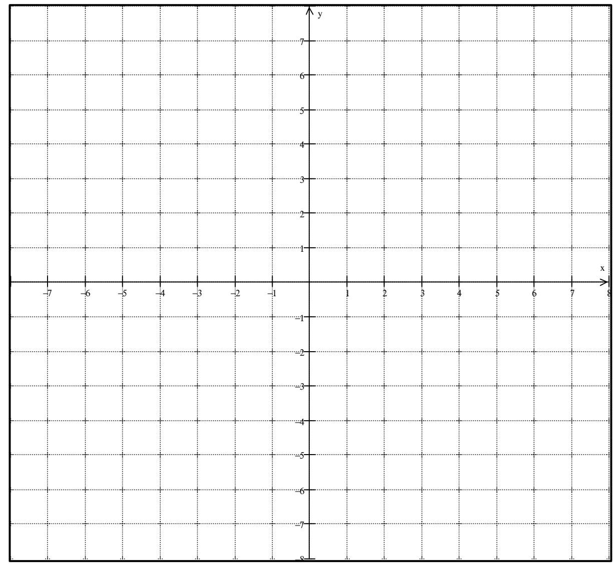
(d) $g(x)=\dfrac{3x^2}{5-x^2}$ ____________ (e) $h(x)=\dfrac{x^3+x^2}{x-1}$ ____________ (f) $y=\dfrac{2x-1}{3x+8}$ ____________

(g) $y=\dfrac{2x^2-1}{3x+8}$ ____________ (h) $y=\dfrac{2x-1}{3x^2+8}$ ____________ (i) $h(x)=\dfrac{2x+5}{x(x-8)}$ ____________

(j) $g(x)=\dfrac{2-3x^3-7x^5}{-2x^5-8x^2}$ ____________

Example 6: Use the graph of $f(x) = \dfrac{1}{x}$ to graph $g(x) = \dfrac{1}{x-4} - 1$.

The graph (including asymptotes) of $g(x)$
has shifted _____ unit(s) to the _________
and _____ unit(s) _________.



Equations of Asymptotes: ________ ________

As $x \to 4^+$, $g(x) \to$ ________ .

As $x \to 4^-$, $g(x) \to$ ________ .

As $x \to \infty$, $g(x) \to$ ________ .

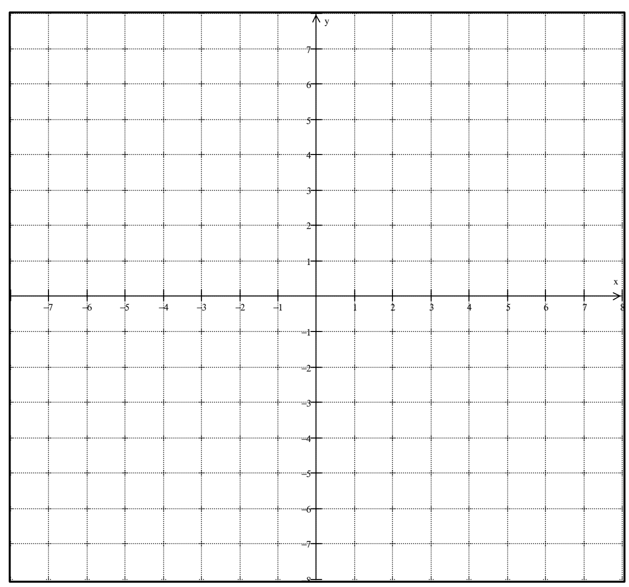
As $x \to -\infty$, $g(x) \to$ ________ .

Domain of g: ______________________

Range of g: ______________________

Example 7: Use the graph of $f(x) = \dfrac{1}{x^2}$ to graph $g(x) = \dfrac{1}{(x+2)^2} - 3$.

The graph (including asymptotes) of $g(x)$
has shifted _____ unit(s) to the _________
and _____ unit(s) _________.



Equations of Asymptotes: ________ ________

As $x \to -2^+$, $g(x) \to$ ________ .

As $x \to -2^-$, $g(x) \to$ ________ .

As $x \to \infty$, $g(x) \to$ ________ .

As $x \to -\infty$, $g(x) \to$ ________ .

Domain of g: ______________________

Range of g: ______________________

3.5 Homework Problems:

1. Find the domain of each rational function.

   (a) $\dfrac{3x}{x-2}$ (b) $\dfrac{2x^2}{x^2+4x-12}$ (c) $\dfrac{x-3}{x^2-100}$ (d) $\dfrac{x-5}{x^2+25}$ (e) $\dfrac{x-2}{x(5x-3)}$ (f) $\dfrac{x-3}{x^3-5x^2-6x}$

2. Use the graph of $f(x)=\dfrac{1}{x^2}$ to graph $g(x)=\dfrac{1}{(x+4)^2}-5$. Refer to your graph to answer the questions.

   (a) Describe the transformation of *f* to *g*. (b) Find the equations of any horizontal and vertical asymptotes.

   (c) As $x \to -4^+$, $g(x) \to$ _______ . (d) As $x \to -4^-$, $g(x) \to$ _______ .

   (e) As $x \to \infty$, $g(x) \to$ _______ . (f) As $x \to -\infty$, $g(x) \to$ _______ .

3. Use the graph of $f(x)=\dfrac{1}{x}$ to graph $g(x)=\dfrac{1}{(x-3)}+2$. Refer to your graph to answer the questions.

   (a) Describe the transformation of *f* to *g*. (b) Find the equations of the horizontal and vertical asymptotes.

   (c) As $x \to 3^+$, $g(x) \to$ _______ . (d) As $x \to 3^-$, $g(x) \to$ _______ .

   (e) As $x \to \infty$, $g(x) \to$ _______ . (f) As $x \to -\infty$, $g(x) \to$ _______ .

4. Find the vertical asymptotes, if any, of each rational function.

   (a) $\dfrac{3x}{x-2}$ (b) $\dfrac{2x^2}{x^2+4x-12}$ (c) $\dfrac{x-3}{x^2-100}$ (d) $\dfrac{x-5}{x^2+25}$ (e) $\dfrac{x-2}{x(5x-3)}$ (f) $\dfrac{x-3}{x^3-5x^2-6x}$

5. Find the horizontal asymptote, if any, of each rational function.

   (a) $\dfrac{3x}{x-2}$ (b) $\dfrac{2x^3}{x^2+4x-12}$ (c) $\dfrac{x-3}{x^3}$ (d) $\dfrac{3x-5}{5x+25}$ (e) $\dfrac{x^2-2}{x(5-3x)}$ (f) $\dfrac{x-3}{x^3-5x^2-6x}$

3.5 Homework Answers: 1. (a) $(-\infty, 2)\cup(2,\infty)$; (b) $(-\infty,-6)\cup(-6,2)\cup(2,\infty)$; (c) $(-\infty,-10)\cup(-10,10)\cup(10,\infty)$; (d) $(-\infty,\infty)$; (e) $(-\infty,0)\cup\left(0,\frac{3}{5}\right)\cup\left(\frac{3}{5},\infty\right)$; (f) $(-\infty,-1)\cup(-1,0)\cup(0,6)\cup(6,\infty)$ **2.** (a) horizontal shift 4 units left; vertical shift 5 units down; (b) $y=-5$; $x=-4$; (c) ∞; (d) ∞; (e) -5; (f) -5 **3.** (a) horizontal shift 3 units right; vertical shift 2 units up; (b) $y=2$; $x=3$; (c) ∞; (d) $-\infty$; (e) 2; (f) 2 **4.** (a) $x=2$; (b) $x=-6$; $x=2$; (c) $x=-10$; $x=10$; (d) no V.A.; (e) $x=0$; $x=\frac{3}{5}$; (f) $x=-1$; $x=0$; $x=6$ **5.** (a) $y=3$; (b) no H.A.; (c) $y=0$; (d) $y=\frac{3}{5}$; (e) $y=-\frac{1}{3}$; (f) $y=0$

3.6 Combination of Functions; Composite Functions

In this section you will learn to:

- find the domain of functions
- combine (+, -, ×, ÷) functions
- form and evaluate composite functions

The **domain of a function*** is the set of "allowable" values for *x*. You must <u>exclude</u> the following from the domain:

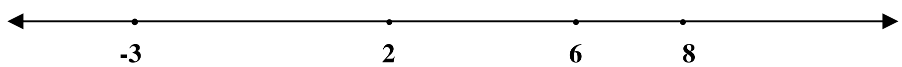
(a) real numbers that cause **division by zero**
(b) real numbers that result in a **square root of a negative number**

(**Tip:** Use a number line to determine the domain when using interval notation.)

*where the function does not model data or verbal conditions

Recall: The **intersection** of sets A and B, **A ∩ B**, is the set of elements common to **both** A and B.

Example: If A = (2, 8] and B = [-3, 6], then A ∩ B = ________



Example 1: Find the domain for each of the functions. Write the domain using interval notation.

(a) $f(x) = 3x - 2$

(b) $g(x) = x^2 + 5x - 2$

(c) $h(x) = \dfrac{1}{x-2}$

(d) $f(x) = \dfrac{6x}{x^2 - 49}$

(e) $g(x) = \sqrt{2x+6}$

(f) $h(x) = \dfrac{x-2}{x^2 + x - 12}$

(g) $f(x) = \dfrac{\sqrt{x-2}}{x-5}$

Combination of Functions (+, - ,×, ÷): Given two functions f and g. 1. **Sum:** $(f+g)(x) = f(x) + g(x)$ 2. **Difference:** $(f-g)(x) = f(x) - g(x)$ 3. **Product:** $(fg)(x) = f(x) \cdot g(x)$ 4. **Quotient:** $\left(\frac{f}{g}\right)(x) = \frac{f(x)}{g(x)}$, where $g(x) \neq 0$	**Note:** For +, -, ×, and ÷ of f and g, the domain of the combination **must be common to both** f and g: **domain of f ∩ domain of g**

Example 2: If $f(x) = \sqrt{x-3}$ and $g(x) = \sqrt{5-x}$, find each of the following including their domains.

(a) $(f+g)(x) =$ ______________________

Domain: ______________________

(b) $(f-g)(x) =$ ______________________

Domain: ______________________

(c) $(fg)(x) =$ ______________________

Domain: ______________________

(d) $\left(\frac{f}{g}\right)(x) =$ ______________________

Domain: ______________________

Example 3: If $f(x) = x - 5$ and $g(x) = x^2 - 2x - 8$, find each of the following including their domains.

(a) $(f+g)(x) =$ ____________________

Domain: ____________________

(b) $(f-g)(x) =$ ____________________

Domain: ____________________

(c) $(fg)(x) =$ ____________________

Domain: ____________________

(d) $\left(\frac{f}{g}\right)(x) =$ ____________________

Domain: ____________________

Example 4: Using the functions form Example 3, compute the following:

(a) $f(3) =$ ________

(b) $g(3) =$ ________

(c) $(f+g)(3) =$ ________

(d) $(fg)(3) =$ ________

(e) $\left(\frac{f}{g}\right)(3) =$ ________

Example 5: If $f(x) = 6 - \frac{1}{x}$ and $g(x) = \frac{1}{x}$, find $\left(\frac{f}{g}\right)(x)$ and list its domain.

Composition of Functions: The composition of the function f with g is denoted by $f \circ g$ and is defined by the equation

$$(f \circ g)(x) = f\big(g(x)\big) = f\big[g(x)\big]$$

The **domain of the composite function** $f \circ g$ is the set of all x such that

- x is in the domain of g
- $g(x)$ is in the domain of f. (Think "beginning/end" or "inside/outside")

Example 6: If $f(x) = 3x - 4$ and $g(x) = x^2 + 6$, evaluate each of the following including their domains.

(a) $f \circ g$

(b) $g \circ f$

Example 7: If $f(x) = \dfrac{2}{x-1}$ and $g(x) = \dfrac{3}{x}$, evaluate each of the following. Be sure to simplify your answer and to write the domain in interval notation.

(a) $f \circ g$

(b) $g \circ f$

Example 8: If $f(x) = x^2 + 5$ and $g(x) = \sqrt{3-x}$, find $f \circ g$ and the domain of the composite function.

Example 9: Evaluate the indicated function without finding an equation for the function given:

$$f(x) = x+1 \qquad g(x) = -2x \qquad h(x) = x^2 - 3$$

(a) $f(g(0))$ (b) $(h \circ g)(4)$ (c) $f(f(10))$ (d) $f(g[h(2)])$

3.6 Homework Problems

For Problems 1-9, find the domain for each function.

1. $f(x) = 3x - 4$
2. $f(x) = \dfrac{3}{x-5}$
3. $f(x) = x^2 + 3x - 10$
4. $h(x) = \dfrac{x}{x^2 + 3x - 10}$
5. $f(x) = \dfrac{1}{x+2} - \dfrac{1}{x-3}$
6. $g(x) = \sqrt{x-3}$
7. $f(x) = \dfrac{\sqrt{x-3}}{x-5}$
8. $h(x) = \dfrac{4}{\dfrac{3}{x} - 1}$
9. $g(x) = \dfrac{1}{x-5} + \sqrt{x-2}$

For Problems 10 – 14, find (a) $f + g$, (b) $f - g$, (c) fg, (d) f/g and also the domain for each function.

10. $f(x) = 2x + 3$
    $g(x) = x - 1$
11. $f(x) = 2x^2 - x - 3$
    $g(x) = x + 1$
12. $f(x) = \sqrt{x}$
    $g(x) = x - 4$
13. $f(x) = 2 + \dfrac{1}{x}$
    $g(x) = \dfrac{1}{x}$
14. $f(x) = \sqrt{x+4}$
    $g(x) = \sqrt{x-1}$

For Problems 15 – 18, find (a) $(f \circ g)(x)$, (b) $(g \circ f)(x)$, and (c) $(f \circ g)(2)$.

15. $f(x) = 2x$
    $g(x) = x + 7$
16. $f(x) = 4x - 3$
    $g(x) = 5x^2 - 2$
17. $f(x) = 4 - x$
    $g(x) = 2x^2 + x + 5$
18. $f(x) = \sqrt{x}$
    $g(x) = x - 1$

For Problems 19 - 21, find (a) $(f \circ g)(x)$, (b) $(g \circ f)(x)$, and also the domain for each composition.

19. $f(x) = \dfrac{2}{x+3}$
    $g(x) = \dfrac{1}{x}$
20. $f(x) = \dfrac{x}{x+1}$
    $g(x) = \dfrac{4}{x}$
21. $f(x) = \sqrt{x}$
    $g(x) = x - 2$

22. Find all values of x satisfying the given conditions: $f(x) = 2x - 5,\ g(x) = x^2 - 3x + 8,$ and $(f \circ g)(x) = 7$.

23. Evaluate the indicated function without finding an equation for the function.

    $f(x) = 2x - 5$ $\qquad$ $g(x) = 4x - 1$ $\qquad$ $h(x) = x^2 + x + 2$

    (a) $(f \circ g)(0)$ $\qquad$ (b) $(g \circ f)(0)$ $\qquad$ (c) $g(f(6))$

    (d) $g(g(-2))$ $\qquad$ (e) $g(f[h(1)])$ $\qquad$ (f) $f(g[h(1)])$

3.6 Homework Answers: 1. $(-\infty,\infty)$ **2.** $(-\infty,5)\cup(5,\infty)$ **3.** $(-\infty,\infty)$ **4.** $(-\infty,-5)\cup(-5,2)\cup(2,\infty)$ **5.** $(-\infty,-2)\cup(-2,3)\cup(3,\infty)$ **6.** $[3,\infty)$ **7.** $[3,5)\cup(5,\infty)$ **8.** $(-\infty,0)\cup(0,3)\cup(3,\infty)$ **9.** $[2,5)\cup(5,\infty)$ **10.** (a) $3x+2$; Domain: $(-\infty,\infty)$; (b) $x+4$; Domain: $(-\infty,\infty)$; (c) $2x^2+x-3$; Domain: $(-\infty,\infty)$; (d) $\dfrac{2x+3}{x-1}$; Domain: $(-\infty,1)\cup(1,\infty)$ **11.** (a) $2x^2-2$; Domain: $(-\infty,\infty)$; (b) $2x^2-2x-4$; Domain: $(-\infty,\infty)$; (c) $2x^3+x^2-4x-3$; Domain: $(-\infty,\infty)$; (d) $2x-3$; Domain: $(-\infty,-1)\cup(-1,\infty)$ **12.** (a) $\sqrt{x}+x-4$; Domain: $[0,\infty)$; (b) $\sqrt{x}-x+4$; Domain: $[0,\infty)$; (c) $\sqrt{x}(x-4)$; Domain: $[0,\infty)$; (d) $\dfrac{\sqrt{x}}{x-4}$; Domain: $[0,4)\cup(4,\infty)$

13. (a) $\dfrac{2x+2}{x}$; Domain: $(-\infty,0)\cup(0,\infty)$; (b) 2; Domain: $(-\infty,0)\cup(0,\infty)$; (c) $\dfrac{2x+1}{x^2}$; Domain: $(-\infty,0)\cup(0,\infty)$; (d) $2x+1$; Domain: $(-\infty,0)\cup(0,\infty)$; **14.** (a) $\sqrt{x+4}+\sqrt{x-1}$; Domain: $[1,\infty)$; (b) $\sqrt{x+4}-\sqrt{x-1}$; Domain: $[1,\infty)$; (c) $\sqrt{x^2+3x-4}$; Domain: $[1,\infty)$; (d) $\dfrac{\sqrt{x+4}}{\sqrt{x-1}}$; Domain: $(1,\infty)$ **15.** (a) $2x+14$; (b) $2x+7$; (c) 18 **16.** (a) $20x^2-11$; (b) $80x^2-120x+43$; (c) 69 **17.** (a) $-2x^2-x-1$; (b) $2x^2-17x+41$; (c) -11

18. (a) $\sqrt{x-1}$; (b) $\sqrt{x}-1$; (c) 1 **19.** (a) $\dfrac{2x}{1+3x}$; Domain: $\left(-\infty,-\dfrac{1}{3}\right)\cup\left(-\dfrac{1}{3},0\right)\cup(0,\infty)$; (b) $\dfrac{x+3}{2}$; Domain: $(-\infty,-3)\cup(-3,\infty)$ **20.** (a) $\dfrac{4}{4+x}$; Domain: $(-\infty,-4)\cup(-4,0)\cup(0,\infty)$; (b) $\dfrac{4(x+1)}{x}$; $(-\infty,-1)\cup(-1,0)\cup(0,\infty)$ **21.** (a) $\sqrt{x-2}$; Domain: $[2,\infty)$; (b) $\sqrt{x}-2$; Domain: $[0,\infty)$ **22.** 1, 2 **23.** (a) -7; (b) -21; (c) 27; (d) -37; (e) 11; (f) 25

3.7 Inverse Functions

In this section you will learn to:

- verify inverse functions
- find inverse functions
- determine whether a function is one-to-one
- understand the characteristics of inverse functions
- graph inverse functions

The function $\boldsymbol{g(x)}$ **is the inverse of** $\boldsymbol{f(x)}$ if $(f \circ g)(x) = x$ for every x in the domain of g **AND** $(g \circ f) = x$ for every x in the domain of f. or $f(g(x)) = x$ or $g(f(x)) = x$	
The **inverse of the function** $\boldsymbol{f}$ is denoted by $\boldsymbol{f^{-1}}$. (Read as "f-inverse".)	
domain of f = range of f^{-1}	range of f = domain of f^{-1}
$(f \circ f^{-1})(x) = x$ $f(f^{-1}(x)) = x$	$(f^{-1} \circ f)(x) = x$ $f^{-1}(f(x)) = x$
The graphs of f and f^{-1} are symmetric over the line $y = x$.	
If the point (a, b) is on the graph of f, then the point (b, a) is on the graph of f^{-1}.	

Example 1: Using the definition of inverse, determine whether the two functions below are inverses of each other.

$$f(x) = 3x \qquad\qquad g(x) = \frac{x}{3}$$

Example 2: Verify $g = f^{-1}$ if $f(x) = 3x + 5$ and $g(x) = \dfrac{x-5}{3}$.

Steps for finding the equation for the inverse of a function *f* :

1. Replace $f(x)$ with *y*.

* 2. Interchange *x* and *y*.

* 3. Solve for *y*. (If *y* is not a function of *x*, then *f* does not have an inverse.)

4. Replace *y* with $f^{-1}(x)$.

5. Check: Does $f(g) = g(f) = x$ (For a quick, "rough" check use the points (*a, b*) in and (*b, a*).)

Example 3: Given $f(x) = \dfrac{4}{x+7}$. Find f^{-1}.

Example 4: Given $h(x) = 27x^3 - 1$. Find h^{-1}.

Example 5: Find the inverse of $f(x) = \dfrac{2x+1}{3x-4}$.

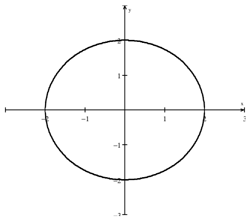
Example 6: Find the inverse of $f(x) = \dfrac{5x-3}{2x-5}$

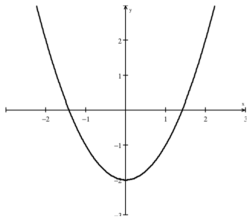
Vertical Line Test (VLT) – Use to see if *f* is a function.
Horizontal Line Test (HLT) – Use to see if f^{-1} is a function.
One-to-One Function – Passes both HLT and VLT

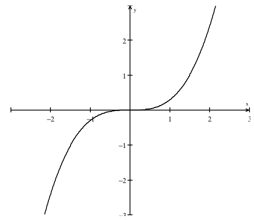
Example 7: Determine whether each function is a one-to-one function.

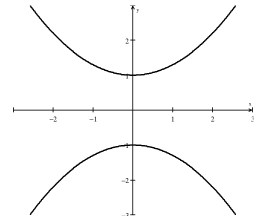
$f(x) = 5x$ ______ $f(x) = 2x^2 - 7$ ______ $h(x) = |x-3|$ ______ $g(x) = 3$ ______ $f(x) = (x-4)^2; x \geq 4$ ______

Example 8: Determine whether each graph represents a one-to-one function.



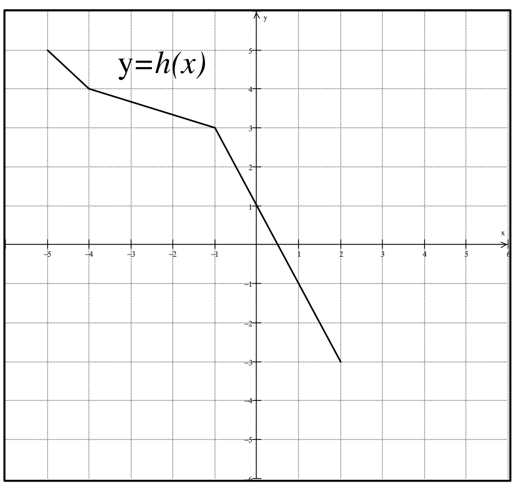


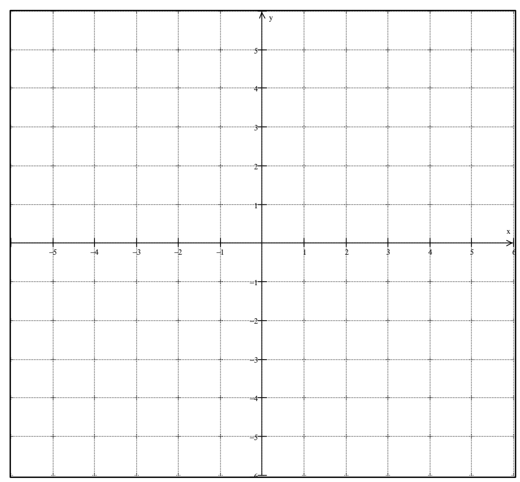




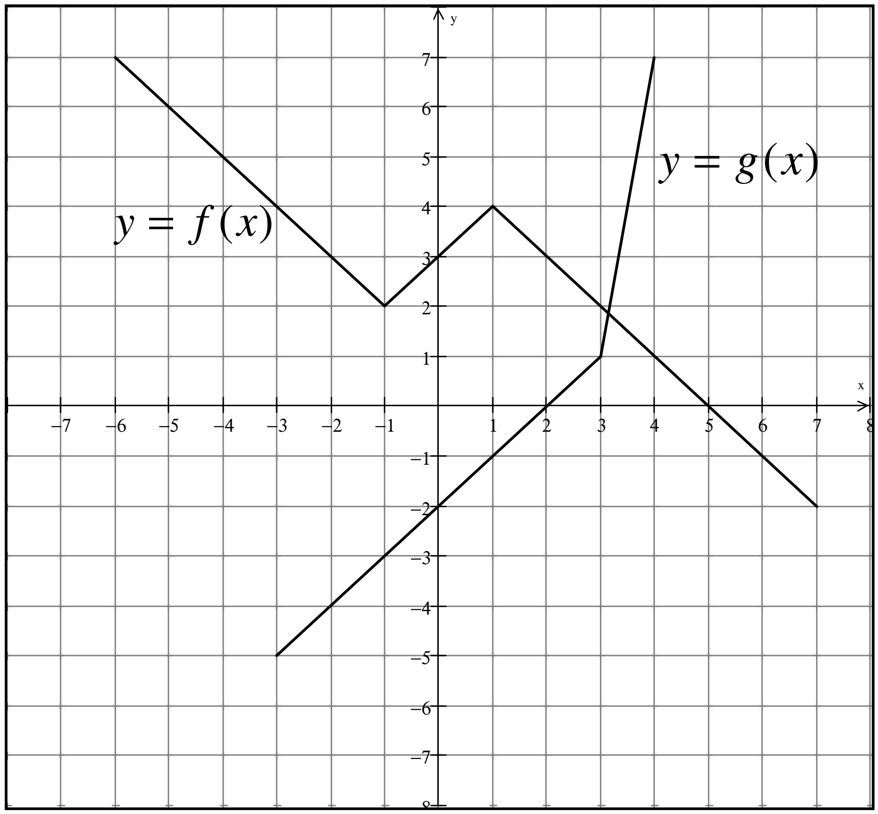
__________ __________ __________ __________

Example 9: Use the graph of *h* below to draw the graph of h^{-1}.





Example 10: Refer to the graph to complete the statements below.



(a) $(f + g)(-3) =$ ________

(b) $(f \cdot g)(2) =$ ________

(c) $\left(\dfrac{f}{g}\right)(-1) =$ ________

(d) $(f \circ g)(3) =$ ________

(e) $g^{-1}(-4) =$ ________

(f) Evaluate $(f \circ f)(2)$ _______

(g) Evaluate $g(f(g(1)))$. ________

(h) State the domain of $f + g$. _______________

(i) State the domain of $\dfrac{f}{g}$. _______________

(j) Which function is a one-to-one function? ____

(k) Evaluate $(f(3))^3 - 4g(-2)$ ________

(l) For what value(s) is $f(x) = 3$? _________

3.7 Homework Problems

1. Find $f(g(x))$ and $g(f(x))$. Then determine whether each pair of functions f and g are inverses of each other.

   (a) $f(x) = 4x$ and $g(x) = \dfrac{x}{4}$

   (b) $f(x) = 5x - 9$ and $g(x) = \dfrac{x+5}{9}$

   (c) $f(x) = \dfrac{3}{x-4}$ and $g(x) = \dfrac{3}{x} + 4$

   (d) $f(x) = \sqrt[3]{x-4}$ and $g(x) = x^3 + 4$

For Problems 2 – 10 find an equation for $f^{-1}(x)$.

2. $f(x) = x + 3$

3. $f(x) = 2x$

4. $f(x) = x^3 + 2$

5. $f(x) = \dfrac{1}{x-2}$

6. $f(x) = \dfrac{7}{x} - 3$

7. $f(x) = \dfrac{2x+1}{x-3}$

8. $f(x) = \dfrac{2x-3}{x+1}$

9. $f(x) = \dfrac{x}{2x-1}$

10. $f(x) = \dfrac{3x}{2x-5}$

For Problems 11 and 12, find the domain and range of *f*. (Find the range by finding the domain of f^{-1}).

11. $f(x) = \dfrac{x}{x-2}$

12. $f(x) = \dfrac{2x+1}{x-3}$

3.7 Homework Answers: 1. (a) $f(g(x)) = g(f(x)) = x$; f and g are inverses; (b) $f(g(x)) = \dfrac{5x-56}{9}$; $g(f(x)) = \dfrac{5x-4}{9}$; f and g are not inverses; (c) $f(g(x)) = g(f(x)) = x$; f and g are inverses; (d) $f(g(x)) = g(f(x)) = x$; f and g are inverses **2.** $f^{-1}(x) = x - 3$ **3.** $f^{-1} = \dfrac{x}{2}$ **4.** $f^{-1}(x) = \sqrt[3]{x-2}$

5. $f^{-1}(x) = \dfrac{1}{x} + 2,\ x \neq 0$ **6.** $f^{-1}(x) = \dfrac{7}{x+3},\ x \neq -3$ **7.** $f^{-1}(x) = \dfrac{3x+1}{x-2},\ x \neq 2$

8. $f^{-1}(x) = \dfrac{-x-3}{x-2},\ x \neq 2$ **9.** $f^{-1}(x) = \dfrac{x}{2x-1},\ x \neq \dfrac{1}{2}$ **10.** $f^{-1}(x) = \dfrac{5x}{2x-3}$

11. Domain: $(-\infty, 2) \cup (2, \infty)$; Range: $(-\infty, 1) \cup (1, \infty)$ **12.** Domain: $(-\infty, 3) \cup (3, \infty)$; Range: $(-\infty, 2) \cup (2, \infty)$