Activity: Square Roots and Complex Numbers

Definition of a Square Root: If a is a real number, then b is said to be a square root of a if $b^2 = a$. For example, $b = 5$ is a square root of 25.

Positive and Negative Square Roots: If b is a square root of a, then $-b$ is also a square root of a since $(-b)^2 = b^2 = a$. Every positive real number $a > 0$ has two distinct square roots, one positive (denoted $\sqrt{a}$) and one negative (denoted $-\sqrt{a}$). Zero only has one square root, namely zero.

Multiplicative Property of Square Roots: If $a, b \geq 0$, then $\sqrt{ab} = \sqrt{a}\sqrt{b}$. WARNING: This property is not true if a or b is a negative number.

1. Using the multiplicative property you can simplify radicals. For example, $\sqrt{32} = \sqrt{16(2)} = \sqrt{16}\sqrt{2} = 4\sqrt{2}$. Simplify each of the following radicals.
   (a) $\sqrt{75}$
   (b) $\sqrt{243}$
   (c) $\sqrt{4000}$
2. CAUTION: $\sqrt{a+b} \neq \sqrt{a} + \sqrt{b}$ in general. Give at least one example. Which seems to be bigger, $\sqrt{a+b}$ or $\sqrt{a} + \sqrt{b}$? Is this always true? Is it ever true that $\sqrt{a+b} = \sqrt{a} + \sqrt{b}$?

The Imaginary Number i: Let i be a number with the property that $i^2 = -1$. In other words, i is a square root of -1. Of course, $-i$ is also a square root of -1. The number i is not a real number, since the square of any real number is non-negative. Nonetheless, the real number system can be extended to the complex number system (see below).

Square Roots of Negative Numbers: If $a > 0$, then $\sqrt{-a} = \sqrt{a}i$ by definition. For example, $a = 25 > 0$, so $\sqrt{-25} = \sqrt{25}i = 5i$ by definition. This is a correct definition since $(5i)^2 = (5i)(5i) = 5^2 i^2 = 25(-1) = -25$, in other words $5i$ is a square root of -25. What is the other square root of -25?

Sound Advice: Whenever you encounter the square root of a negative number, immediately re-write the square root using i. For example, $\sqrt{-42}$ should be re-written as $\sqrt{42}i$.

What happens if you ignore this advice? Well, we might be tempted to say the following:

$$-1 = i^2 = \sqrt{-1}\sqrt{-1}) = \sqrt{(-1)(-1)} = \sqrt{1} = 1.$$

The conclusion is that $-1 = 1$. Clearly something is wrong. The mistake is that the multiplicative property of square roots only applies for non-negative numbers. So, for example, $\sqrt{5}\sqrt{3} = \sqrt{15}$, but $\sqrt{-5}\sqrt{-3}$ is not equal to $\sqrt{15}$. What is $\sqrt{-5}\sqrt{-3}$ equal to? (Use the sound advice!)

Rewrite each radical using the imaginary number i and then simplify the expression.

1. $\sqrt{-64}$
2. $\sqrt{-128}$
3. $\sqrt{-3}\sqrt{-5}\sqrt{-15}$

The Field of Complex Numbers: An expression of the form $a + bi$, where a and b are real numbers and i has the property that $i^2 = -1$ is called a *complex number*. The value a is called the real part and the value b is called the imaginary part. Two complex numbers are equal if they have equal real and imaginary parts. Thus $3 + 2i$ is not equal to $3 - 2i$ since $2 \neq -2$.

Define the sum of complex numbers as follows:
$(a + bi) + (c + di) = (a + c) + (b + d)i$.

Define the product of complex numbers as follows:
$(a + bi)(c + di) = (ac - bd) + (ac + bd)i$.

With these definitions, complex numbers behave like polynomials. For example, you expand $(2 + 3i)(4 + 5i)$ as you would $(2 + 3x)(4 + 5x)$, namely you use the distributive property.

Even better, every non-zero complex number has a reciprocal:

$$\frac{1}{a+bi} = \frac{a}{a^2+b^2} + \frac{-b}{a^2+b^2}i.$$

Modulus (also called absolute value): The modulus of $a + bi$ is the number, denoted by $|a + bi|$, equal to $\sqrt{a^2 + b^2}$. The modulus is a non-negative real number.

Complex Conjugate: The complex conjugate of $a + bi$, denoted by $\overline{a + bi}$, is equal to $a - bi$. So, the complex conjugate is another complex number. When is a complex number equal to its own conjugate?

Conjugates, Modulus, and Reciprocals: These three properties are related by the following equations:

$$(a + bi)(a - bi) = a^2 + b^2 \quad \text{and} \quad \frac{1}{a+bi} = \frac{a-bi}{a^2+b^2}.$$

1. Verify that the left equation above is correct by expanding $(a + bi)(a - bi)$.
2. Deduce that the right equation above is true by using the left equation.