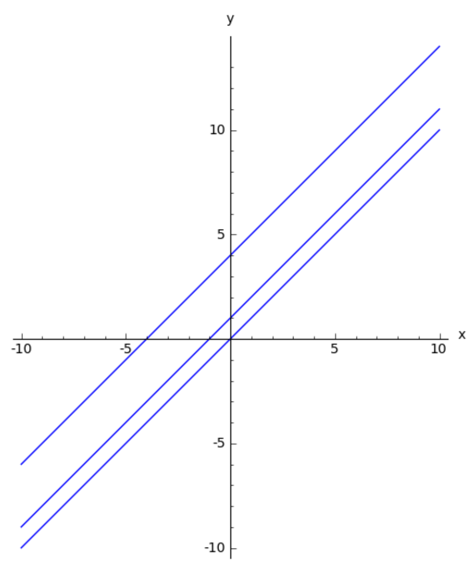
Solutions to Homework 4

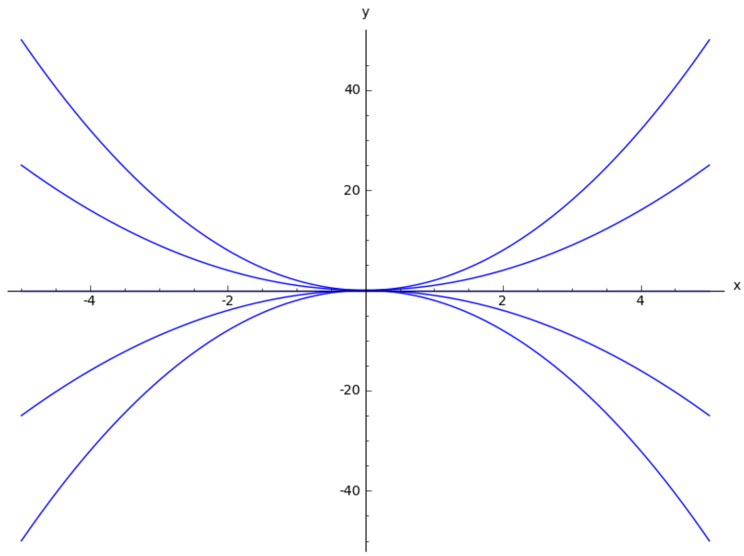
1. For each function below, identify the function's domain and range. Then sketch at least three non-empty level curves of each function. Finally, determine if the domain is open, closed, or neither.

   (a) $f(x, y) = \sqrt{y - x}$
   Solution: $y - x \geq 0$ is the domain. This consists of all points in the plane above or on the line $y = x$. This is a closed set. The range of the function is $[0, \infty)$. Here is a sketch of the level curves $f = 0$, $f = 1$, and $f = 2$:

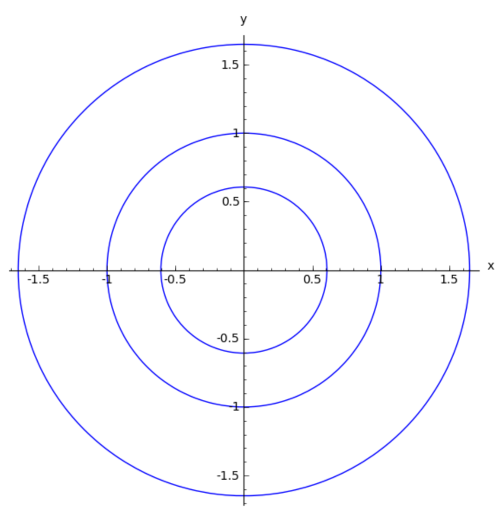
   The curves are the lines $y - x = 0$, $y - x = 1$, and $y - x = 4$, respectively.

   (b) $f(x, y) = y/x^2$
   Solution: $x^2 \neq 0$ is the domain. This consists of all points in the plane which do not lie on the line $x = 0$. This is an open set. The range of the function is $(-\infty, \infty)$. Here is a sketch of the level curves $f = -2$, $f = -1$, $f = 0$, $f = 1$ and $f = 2$:



The figure above needs to be modified: all points on the y-axis, i.e. on the line $x = 0$, need to be exluded. This can be achieved by drawing an open circle at the origin in the figure above.

(c) $f(x, y) = \ln(x^2 + y^2)$
Solution: $x^2 + y^2 > 0$ is the domain. This consists of all points in the plane except $(0, 0)$. This is an open set. The range of the function is $(-\infty, \infty)$. Here is a skecth of the level curves
$f = -1$, $f = 0$, and $f = 1$.



2. By considering different paths of approach show that the function $f(x, y) = x^4/(x^4 + y^2)$ has no limit as (x, y) approaches $(0, 0)$.

   Solution: The limit as $y = 0$ and $x \to 0^+$ is equal to 1. The limit as $x = 0$ and $y \to 0^+$ is 0. Therefore, the limit of $f(x, y)$ as $(x, y) \to (0, 0)$ does not exist.

3. Suppose that $f(x_0, y_0) = 3$. What can you conlcude about

   $$\lim_{(x,y)\to(0,0)} f(x, y)$$

   if f is continuous at (x_0, y_0)? What can you conclude if f is not continuous at (x_0, y_0)? Explain your reasoning.

   Solution: If f is continuous at (x_0, y_0), the $f(x_0, y_0) = 3$. If f is not continuous at this point, then either the limit of $f(x, y)$ as (x, y) approaches (x_0, y_0) does not exist, or this limit does exist but its value is not equal to 3.

   The reason is that the definition of f being continous at this point is that the limit at this point exists and is equal to the value of the function at this point.

4. Determine f_x, f_y, and f_z for the following functions:

   (a) $f(x, y, z) = \dfrac{1}{\sqrt{x^2 + y^2 + z^2}}$

   Solution: $f_x = -x/(x^2 + y^2 + z^2)^{3/2}$. The expression for f_y (respectively f_z) is the same, but interchange the roles of y (respectively z) and x.

   (b) $f(x, , y, z) = yz \ln(xy)$

   Solution:
   $f_x = ((yz)/(xy))(y) = yz/x$
   $f_y = z \ln(xy) + yz(1/xy)(x) = z \ln xy + z$
   $f_z = y \ln(xy)$

5. Determine the value of $\partial x/\partial z$ at the point $(1, -1, -3)$ if the equation

   $$xy + z^3x - 2yz = 0$$

   defines x as a function of the two independent variables y and z. (Hint: implicit differentiation.)

Solution:

$$\frac{\partial}{\partial z}\left(xy + z^3x - 2yz = 0\right) \implies \frac{\partial x}{\partial z}y + (3z^2x + z^3\frac{\partial x}{\partial z}) - 2y = 0,$$

where the parentheses are there to emphasize that the product rule has been applied to the term z^3x– both z^3 and x are functions of x in this problem.

Solving the above equation yields the following:

$$\frac{\partial x}{\partial z} = \frac{2y - 3z^2x}{y + z^3}.$$

Finally, plugging in the point $(1, -1, -3)$, one obtains the answer $\partial x/\partial z(1, -1, -3) = (2(-1) - 3(-3)^2(1))/(-1 + (-3)^3) = 29/28$. There is an error in the statement of the problem, however. This point does not lie on the surface! I should have given a point which makes the equation true. For, instance, $(1, 1, 1)$ would work.

There is another way to solve this problem by appealing to a formula derived in the text: if $F(x, y, z) = 0$ and x is implicitly a function of y and z, then

$$\frac{\partial x}{\partial z} = -\frac{F_z}{F_x}.$$

This formula is derived by the same process as the above: implicitly differentiate the equation $F(x, y, z) = 0$ and solve for $\partial x/\partial z$. By the chain rule, we have that

$$\frac{\partial}{\partial z}\left(F(x, y, z) = 0 = 0\right) \implies F_x\frac{\partial x}{\partial z} + F_y\frac{\partial y}{\partial x} + F_z\frac{\partial z}{\partial z} = 0.$$

Since $\partial z/\partial z = 1$ and $\partial y/\partial z = 0$ (since y and z are regarded as the independent variables in this setup), the equation above simplifies and upon solving for $\partial x/\partial z$ one obtains the formula above.

6. The (two dimensional) Laplace equation is

$$\frac{\partial^2 f}{\partial x^2} + \frac{\partial^2 f}{\partial y^2} = 0.$$

Show that the function below is a solution to the Laplace equation:

$$f(x, y) = \ln\sqrt{x^2 + y^2}.$$

Solution: Simplify first: $\ln\sqrt{x^2+y^2} = (1/2)\ln(x^2+y^2)$. Then $f_x = (1/2)(1/(x^2+y^2))(2x) = x/(x^2+y^2)$. By symmetry, $f_y = y/(x^2+y^2)$.

Differentiating again (using the quptient rule), one determines that

$$f_{xx} = \frac{(x^2+y^2) - 2x^2}{(x^2+y^2)^2} = \frac{y^2 - x^2}{(x^2+y^2)^2}.$$

By symmetry, $f_{yy} = (x^2-y^2)/(x^2+y^2)^2$. And so, $f_{xx} + f_{yy} = 0$.