NUCLEATION OF INSTABILITY OF THE MEISSNER STATE OF 3-DIMENSIONAL SUPERCONDUCTORS

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Abstract

This paper concerns a nonlinear partial differential system in a 3-dimensional domain involving the operator curl^2, which is a simplified model used to examine nucleation of instability of the Meissner state of a superconductor as the applied magnetic field reaches the superheating field. We derive a priori $C^{2+\alpha}$ estimates for a weak solution $\mathbf{H}$, the curl of the magnetic potential, and determine the location of the maximal points of $|\mathrm{curl}\,\mathbf{H}|$ which correspond to the nucleation of instability of the Meissner state. We show that, if the penetration length is small, the solution exhibits a boundary layer. If the applied magnetic field is homogeneous, $|\mathrm{curl}\,\mathbf{H}|$ is maximal around the points on the boundary where the applied field is tangential to the surface.

§1. Introduction

In this paper we study the following elliptic system:

$$-\lambda^2\mathrm{curl}^2\mathbf{A} = (1-|\mathbf{A}|^2)\mathbf{A} \quad \text{in } \Omega, \qquad \lambda\,(\mathrm{curl}\,\mathbf{A})_T = \mathcal{H}^e_T \quad \text{on } \partial\Omega, \tag{1.1}$$

where Ω is a bounded and smooth domain in $\mathbb{R}^3$, $\mathbf{A}(x) = (A_1(x), A_2(x), A_3(x))$, and $\mathcal{H}^e$ is a given vector field on $\partial\Omega$. Here, the subscript T denotes the tangential

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component on $\partial\Omega$, respectively, and $0 < \lambda \ll 1$. For reasons that will become clear below, we are interested in the solutions of (1.1) satisfying

$$\|\mathbf{A}\|_{L^\infty(\Omega)} < \frac{1}{\sqrt{3}}. \tag{1.2}$$

We shall investigate the location of the maximum points of $|\mathbf{A}(x)|$ for small λ.

Equation (1.1) is an approximation of the Ginzburg-Landau system of superconductivity with large value of Ginzburg-Landau parameter κ, which was derived by Chapman [C1] (also see [Mon]) to describe loss of stability of the Meissner state when the applied magnetic field reaches the superheating field H_{sh}. In the system $\mathcal{H}^e$ is the applied magnetic field, $\mathbf{A}$ is the magnetic potential, $\operatorname{curl}\mathbf{A}$ is the induced magnetic field, and λ is the penetration depth, which is small typically. In fact, the instability occurs when the condition (1.2) is violated [C1].

It was first discovered in [C1] (also see [Mon]) that, under the condition (1.2), equation (1.1) is equivalent to the following quasilinear system:

$$-\lambda^2 \operatorname{curl}\left[F(\lambda^2|\operatorname{curl}\mathbf{H}|^2)\operatorname{curl}\mathbf{H}\right] = \mathbf{H} \quad \text{in } \Omega, \qquad \mathbf{H}_T = \mathcal{H}^e_T \quad \text{on } \partial\Omega. \tag{1.3}$$

More precisely, if $\mathbf{A} \in C^3(\Omega,\mathbb{R}^3) \cap C^2(\bar{\Omega},\mathbb{R}^3)$ is a solution of (1.1) and satisfies (1.2), and letting $\mathbf{H} = \lambda \operatorname{curl}\mathbf{A}$, then $\mathbf{H}$ solves (1.3), and the following estimate holds:

$$\lambda\|\operatorname{curl}\mathbf{H}\|_{L^\infty(\Omega)} < \sqrt{\frac{4}{27}}. \tag{1.4}$$

Here the function F is determined by the relation

$$v = F(t^2)t \iff t = (1 - v^2)v, \tag{1.5}$$

and $F(0) = 1$. F is uniquely defined for $0 \le t \le \sqrt{\frac{4}{27}}$, i.e., for $0 \le v \le \frac{1}{\sqrt{3}}$. On the other hand, if in addition Ω is a simply-connected domain, and if $\mathbf{H} \in C^3(\Omega,\mathbb{R}^3) \cap C^2(\bar{\Omega},\mathbb{R}^3)$ is a solution of (1.3) and satisfies (1.4), and letting

$$\mathbf{A} = -\lambda F(\lambda^2|\operatorname{curl}\mathbf{H}|^2)\operatorname{curl}\mathbf{H},$$

then $\mathbf{A} \in C^2(\Omega,\mathbb{R}^3)$ is the unique solution of (1.1) and satisfies (1.2). Moreover, the maximum points of $|\mathbf{A}(x)|$ coincide with the maximum points of $|\operatorname{curl}\mathbf{H}(x)|$.[1]

Let us recall that [dG, MS, Fn, Kra, FP1, FP2, C1, BH], for a type II superconductor subjected to an applied magnetic field, the Meissner state is a global minimizer

[1]This conclusion can be proved by using the argument in [PK], where the equivalence of two equations (1.6) and (1.7) in the 2-dimensional case was proved.

of the Ginzburg-Landau energy and hence globally stable if the applied field is below the first critical field H_{C_1}, it is locally stable if the applied field is between H_{C_1} and H_{sh}, and it is not stable if the applied field is above H_{sh}. It is interesting to calculate the superheating field H_{sh} for a superconductor with general shape ([C1, p.1258]), and to explore how the Meissner state losses its stability as the applied magnetic field increases and reaches H_{sh}.

Consider a cylinder of infinite height with its axis in the x_3-direction, and placed in an applied axial magnetic field $\mathcal{H}^e = (0, 0, h)$. Then the Ginzburg-Landau system is reduced to an elliptic system on the 2-dimensional cross section D of the cylinder. Chapman [C1] derived the following system as a large κ limit of the Ginzburg-Landau system:

$$\begin{cases} -\lambda^2 \mathrm{curl}^2 \mathbf{A} = (1 - |\mathbf{A}|^2)\mathbf{A} & \text{in } D, \\ \nu \cdot \mathbf{A} = 0, \quad \lambda \,\mathrm{curl}\, \mathbf{A} = h & \text{on } \partial D, \end{cases} \tag{1.6}$$

where $\mathbf{A} = (A_1(x_1, x_2), A_2(x_1, x_2))$, and ν is the outer normal vector of ∂D. He showed that, if $\|\mathbf{A}\|_{L^\infty(D)} \le \frac{1}{\sqrt{3}}$, (1.6) can be transformed to a quasilinear equation in divergence form for the scalar function $H = \lambda \,\mathrm{curl}\, \mathbf{A} = \lambda(\partial_1 A_2 - \partial_2 A_1)$:

$$\lambda^2 \mathrm{div}\, [F(\lambda^2 |\nabla H|^2)\nabla H] = H \quad \text{in } \Omega, \qquad H = h \quad \text{on } \partial\Omega. \tag{1.7}$$

He also showed that, as the value of h increases, the solution $\mathbf{A}$ of (1.6) loses its stability when $\max |\mathbf{A}(x)|$ reaches $\frac{1}{\sqrt{3}}$, and the instability will occur first at the boundary of the sample. Berestycki, Bonnet and Chapman [BBC] rigorously proved that, if $\|\mathbf{A}\|_{L^\infty(D)} \le \frac{1}{\sqrt{3}}$, then the maximum points of $|\mathbf{A}(x)|$ must lie on the boundary ∂D. Chapman [C2] further showed by formal analysis that, as $\lambda \to 0$, $|\mathbf{A}(x)|$ reaches the maximum value at the points of largest negative curvature of boundary ∂D. This conclusion was rigorously verified by Pan and Kwek [PK]. Moreover, it was proved in [PK] that the solutions of (1.6) exhibit boundary layer behavior when λ is small, which is recognized as the Meissner effect of superconductivity (see for instance [FP1]). The approximation of Meissner solutions by the solutions of (1.6) was verified by Bonnet-Chapman-Monneau [BCM].

For a bulk superconductor occupying a bounded domain Ω in $\mathbb{R}^3$, Monneau [Mon] derived an approximation system of the Ginzburg-Landau system as $\kappa \to \infty$[2]:

$$\begin{cases} -\lambda^2 \operatorname{curl}^2 \mathbf{A} = (1 - |\mathbf{A}|^2)\mathbf{A} & \text{in } \Omega, \\ \operatorname{curl}^2 \mathbf{A} = \mathbf{0} & \text{in } \mathbb{R}^3 \setminus \bar{\Omega}, \\ [\nu \times \mathbf{A}] = \mathbf{0}, \quad [\nu \times \operatorname{curl} \mathbf{A}] = \mathbf{0} & \text{on } \partial\Omega, \\ \lambda \operatorname{curl} \mathbf{A} - \mathcal{H}^e \to \mathbf{0} \quad \text{as } |x| \to \infty, \end{cases} \tag{1.8}$$

where $[\cdot]$ denotes the jump in the enclosed quantity across $\partial\Omega$. System (1.1) can be viewed as an approximation of (1.8) if we restrict ourselves in the domain Ω and take the boundary condition $\lambda (\operatorname{curl} \mathbf{A})_T = \mathcal{H}^e_T$ on $\partial\Omega$. Note that if $\mathbf{A}$ is a solution of (1.1) in Ω, and if $\mathbf{B}$ is a solution in $\Omega^c = \mathbb{R}^3 \setminus \bar{\Omega}$ of the system

$$\lambda \operatorname{curl} \mathbf{B} = \mathcal{H}^e \quad \text{in } \Omega^c, \qquad \mathbf{B}_T = \mathbf{A}_T \quad \text{on } \partial\Omega,$$

and if we define $\tilde{\mathbf{A}} = \mathbf{A}$ in $\bar{\Omega}$ and $\tilde{\mathbf{A}} = \mathbf{B}$ in Ω^c, then $\tilde{\mathbf{A}}$ is a solution of (1.8).

Monneau [Mon] proved by using the implicit function theorem that, if $\|\mathcal{H}^e_T\|_{C^{2+\alpha}(\partial\Omega)}$ is small, then (1.3) (in the case where $\lambda = 1$) has a unique solution $\mathbf{H} \in C^{2+\alpha}(\bar{\Omega}, \mathbb{R}^3)$, and $|\operatorname{curl} \mathbf{H}|$ attains its maximum only on $\partial\Omega$. If $\|\mathcal{H}^e_T\|_{C^{2+\alpha}(\partial\Omega)}$ is large, then (1.3) has no solution. However, the smallness condition of the boundary data depends on λ, and the dependence on λ was not obtained in [Mon]. Thus no criterion has been given for the boundary data that allows the existence of solutions for all λ.

Because we wish to determine the location of the points where $|\mathbf{A}(x)|$ reaches its maximum, it is necessary to consider the limit as $\lambda \to 0$. In particular we must establish optimal bounds for boundary data and show solvability for all small λ. This requires the introduction of weak solutions. So, the analysis in [Mon], while very useful, must be significantly extended.

In this paper we shall find a bound C of the boundary data such that (1.3) has a solution $\mathbf{H}^\lambda$ for all small λ if $\|\mathcal{H}^e_T\|_{C^{2+\alpha}(\partial\Omega)} < C$, and determine the location of the maximum points of $|\operatorname{curl} \mathbf{H}^\lambda|$. We shall also establish an a prior $C^{2+\alpha}$ estimate of the solutions to (1.1) and (1.3), which is useful in our study of the asymptotic behavior of solutions. We shall show that, as $\lambda \to 0$, the points where the maximum of $|\mathbf{A}(x)|$ (and of $|\operatorname{curl} \mathbf{H}(x)|$) is attained must approach points in $(\partial\Omega)(\mathcal{H}^e_T)$, where

$$\partial\Omega(\mathcal{H}^e_T) = \{x \in \partial\Omega : \ |\mathcal{H}^e_T(x)| = \|\mathcal{H}^e_T\|_{C^0(\partial\Omega)}\}. \tag{1.9}$$

[2]The Ginzburg-Landau system has two parameters: the penetration depth λ and the coherence length ξ. The ratio $\kappa = \lambda/\xi$ is the Ginzburg-Landau parameter (see [GL, CHO]). In this paper we are interested in the behavior of Meissner solutions of the Ginzburg-Landau system with small λ and large κ. For the Meissner state, one may write the order parameter in the form $\psi = fe^{i\chi}$ with $f > 0$. We first fix λ and let κ go to infinity to formally get (1.8) as the limiting equations of the Ginzburg-Landau system (see [C1]); then we investigate the solutions of (1.8) with small λ. We would like to mention that one may rescale the Ginzburg-Landau functional in a way that λ is reduced to 1. This is the case in [Mon]. In that setting, the change in λ in our present paper corresponds the change of the domain size of the superconductor. See also [DP, Remark 1.4].

In the special case of a homogeneous applied field, namely $\mathcal{H}^e = \mathbf{h}$ a constant vector, which is of most importance in applications, $\mathcal{H}^e_T = \mathbf{h}_T = \mathbf{h} - (\mathbf{h}\cdot\nu)\nu$, and $|\mathbf{h}_T|$ is maximal at the points where $\mathbf{h}$ is tangential to $\partial\Omega$. Thus

$$\partial\Omega(\mathbf{h}_T) = (\partial\Omega)_{\mathbf{h}} \equiv \{x \in \partial\Omega : \ \mathbf{h} \text{ is tangential to } \partial\Omega \text{ at } x\}.$$

In this case the optimal bound of boundary data for solvability is $C = \sqrt{\frac{5}{18}}$, which equals the value of the superheating field H_{sh} found in [C1].[3]

We now state our main results. In this paper $H^k(\Omega, \mathbb{R}^3)$ and $C^{k+\alpha}(\bar\Omega, \mathbb{R}^3)$ denote the usual Sobolev space, resp. the usual Hölder space of vector fields, and $C^{k+\alpha}(\Omega, \mathbb{R}^3)$ denotes the space $C^{k+\alpha}_{\mathrm{loc}}(\Omega, \mathbb{R}^3)$.

Theorem 1. *Let Ω be a bounded and simply-connected domain in $\mathbb{R}^3$ with C^4 boundary, and let $\mathcal{H}^e_T$ satisfy*

$$\mathcal{H}^e_T \in C^{2+\alpha}(\partial\Omega, \mathbb{R}^3), \quad \|\mathcal{H}^e_T\|_{C^0(\partial\Omega)} < \sqrt{\frac{5}{18}}, \quad \nu\cdot\operatorname{curl}\mathcal{H}^e_T = 0 \ \text{on } \partial\Omega. \tag{1.10}$$

Then for all $\lambda > 0$ small, (1.3) has a unique solution $\mathbf{H}^\lambda \in C^3(\Omega, \mathbb{R}^3) \cap C^{2+\alpha}(\bar\Omega, \mathbb{R}^3)$ which has the following properties:

(i) *$\mathbf{H}^\lambda$ satisfies (1.4).*

(ii) *For any sequence ρ_λ such that $\rho_\lambda \le \frac{c}{\lambda}$ and $\rho_\lambda \to \infty$ as $\lambda \to 0$, we have*

$$\lim_{\lambda\to 0} \sup_{\operatorname{dist}(x,\partial\Omega)\ge\lambda\rho_\lambda} |\mathbf{H}^\lambda(x)| = 0. \tag{1.11}$$

(iii) *Let $\mu = \|\mathcal{H}^e_T\|_{C^0(\partial\Omega)}$. We have*

$$\lim_{\lambda\to 0} \lambda\|\operatorname{curl}\mathbf{H}^\lambda\|_{C^0(\partial\Omega)} = [1 - (1-2\mu^2)^{1/2}](1-2\mu^2). \tag{1.12}$$

(iv) *If P^λ is a maximum point of $|\operatorname{curl}\mathbf{H}^\lambda(x)|$and if $P^\lambda \to P_0$ for a sequence $\lambda = \lambda_n \to 0$, then $P_0 \in \partial\Omega(\mathcal{H}^e_T)$.*

(v) *In particular, if $\mathcal{H}^e = \mathbf{h}$, a constant vector, and if*

$$|\mathbf{h}| < \sqrt{\frac{5}{18}}, \tag{1.13}$$

then $P_0 \in (\partial\Omega)_{\mathbf{h}}$.

Using the equivalence between classical solutions of (1.1) and (1.3), we can rewrite Theorem 1 with respect to equation (1.1):

[3]The optimal bound of h allowing the existence of solutions for all small λ for the 2-dimensional problem (1.7) is also equal to $\sqrt{\frac{5}{18}}$, (see [PK]).

Theorem 1′. *Under the conditions of Theorem 1, for all $\lambda > 0$ small, (1.1) has a unique solution $\mathbf{A}^\lambda \in C^3(\Omega, \mathbb{R}^3) \cap C^{2+\alpha}(\bar{\Omega}, \mathbb{R}^3)$ which satisfies (1.2). Moreover, the maximum points of $|\mathbf{A}^\lambda(x)|$ satisfy the conclusions (iv) and (v) in Theorem 1.*

Remark 1.1. In order to obtain the optimal bound of boundary data for solvability of (1.3), we need to study the weak limit of solutions as the boundary data approach the bound (see Lemma 7.1), and hence we have to work in the framework of weak solutions.

Remark 1.2. It is interesting to compare the phenomena of nucleation of superconductivity and instability of Meissner state in the 3-dimensional case with that in the 2-dimensional case.

In the 2-dimensional case (cylindrical superconductors subject to an axial homogeneous magnetic field):

(i) as the applied field decreases from H_{C_3} superconductivity nucleates at the boundary where the curvature is the *maximal* ([HP]);

(ii) as the applied field increases to H_{sh} the Meissner state losses its stability at the boundary where the curvature is the *minimal* ([PK]).

In the 3-dimensional case (bulk superconductors subjected to a homogeneous magnetic field $\mathcal{H}^e = \mathbf{h}$):

(i) as the applied field decreases from H_{C_3} superconductivity nucleates in $(\partial\Omega)_{\mathbf{h}}$ ([LP, P]);[4]

(ii) as the applied field increases to H_{sh} the Meissner state losses its stability at points in the set $(\partial\Omega)_{\mathbf{h}}$.[5]

Remark 1.3. Chapman [C1, p.1250] also conjectured that the instability of the Meissner state at H_{sh} leads to formation of vortices in the sample, and leads to transitions from the Meissner state to the mixed state. It would be interesting to verify the conjecture, and to locate the points of vortex nucleation. Combining our Theorem 1 with Chapman's conjecture, one may expect that when a type II superconductor changes from the Meissner state to the mixed state, vortices will start to nucleate in $(\partial\Omega)_{\mathbf{h}}$. In particular, for a superconductor occupying an ellipsoidal domain and placed in a magnetic field directed parallel to its major axis, one may expect that vortices will start to nucleate first at the equator.[6]

Remark 1.4 From the physical background of our problem, the third condition in

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[4] Moreover, the analysis in [P] and [HP] suggests that, superconductivity nucleates at some points in $(\partial\Omega)_{\mathbf{h}}$ where the curvature function $P(x)$ is the *minimal* among all points in $(\partial\Omega)_{\mathbf{h}}$.

[5] But we do not know yet whether at the location of instability the curvature function $P(x)$ is the *maximal* among all points in $(\partial\Omega)_{\mathbf{h}}$.

[6] We should mention that, Du and Ju [DJ] computed a reduced Ginzburg-Landau equation for a superconducting hollow sphere subjected to a constant applied magnetic field and their computational results show that vortex pairs nucleate on that equator of the sphere, which is everywhere perpendicular to the field.

(1.10) is natural: For the applied magnetic field $\mathcal{H}^e$ we always assume $\mathrm{curl}\,\mathcal{H}^e \equiv \mathbf{0}$. Hence $\nu \cdot \mathrm{curl}\,(\mathcal{H}^e_T) = \nu \cdot \mathrm{curl}\,(\mathcal{H}^e) = 0$ on $\partial\Omega$. In particular, if $\mathcal{H}^e = \mathbf{h}$, a constant vector, then the condition holds. We shall see that this condition forces the solutions of (1.1) to satisfy an additional boundary condition $\mathbf{A} \cdot \nu = 0$ on $\partial\Omega$, (see Lemma 2.5).

Remark 1.5. Besides the physical motivation, we are interested in systems (1.1) and (1.3) also because of their mathematical structure. For nonlinear differential systems with operator curl^2, the existence and regularity of solutions sensitively depends on the nonlinearity. For (1.1) and its 2-dimensional version (1.6), without the smallness condition (1.2), the regularity of the solutions fails. In fact, [PQ, Theorem 4] gives a solution of (1.6) with a singularity in the unit disk; $|\mathbf{A}(x)| \equiv 1$ and $\mathrm{curl}\,\mathbf{A}$ is a constant a.e., while $\mathrm{div}\,\mathbf{A} \notin L^2(\Omega)$.

The interaction between the nonlinearity of the equations and pointwise degeneracy and the global ellipticity of the operator curl^2 is most interesting to us. Due to the pointwise degeneracy of ellipticity of curl^2, the classical Schauder estimates for elliptic systems (see [G, Mor]) do not apply to (1.1) and (1.3). On the other hand, restricted in the space of divergence free vector fields, curl^2 is coercive (globally elliptic).[7] For linear systems involving curl^2, one may apply the Hodge decomposition theory and work in the space of divergence free vector fields. One can use the standard difference-quotient-method to derive H^k estimates of the solutions for any positive integer k, and get a C^k estimate by using the Sobolev imbedding theorem, see Dautray-Lions [DL]. However, for the quasilinear system (1.3), the difference-quotient-method allows us to obtain only H^2 estimates. The $C^{2+\alpha}$ estimates require more involved arguments, and we sketch our main ideas here.

We begin with considering weak solutions of (1.3) satisfying

$$\lambda \|\mathrm{curl}\,\mathbf{H}\|_{L^\infty(\Omega)} \le M < \sqrt{\frac{4}{27}}. \tag{1.14}$$

An H^2 estimate and the Sobolev imbedding theorem yield a C^δ estimate with $0 < \delta < 1/2$. Since Ω is simply-connected, there exists $\mathcal{B} \in C^{1+\delta}(\bar{\Omega}, \mathbb{R}^3)$ such that

$$\lambda\,\mathrm{curl}\,\mathcal{B} = \mathbf{H} \quad \text{and} \quad \mathrm{div}\,\mathcal{B} = 0 \quad \text{in } \Omega, \quad \nu \cdot \mathcal{B} = 0 \quad \text{on } \partial\Omega. \tag{1.15}$$

From (1.3) and (1.15) we find $\mathrm{curl}\,[\lambda F(\lambda^2|\mathrm{curl}\,\mathbf{H}|^2)\mathrm{curl}\,\mathbf{H} + \mathcal{B}] = \mathbf{0}$. So there exists a function $\varphi \in H^1(\Omega)$ such that $\lambda F(\lambda^2|\mathrm{curl}\,\mathbf{H}|^2)\mathrm{curl}\,\mathbf{H} + \mathcal{B} = \nabla\varphi$. From the boundary condition and the last equality in (1.10) we have $\nu \cdot \mathrm{curl}\,\mathbf{H} = 0$ in the sense of trace in $H^{1/2}(\partial\Omega)$. Thus φ is a weak solution of a quasilinear equation

$$\mathrm{div}\,[(1 - |\nabla\varphi - \mathcal{B}|^2)(\nabla\varphi - \mathcal{B})] = 0 \quad \text{in } \Omega, \qquad \frac{\partial\varphi}{\partial\nu} = 0 \quad \text{on } \partial\Omega. \tag{1.16}$$

[7] This is one of the reasons why we prefer studying (1.3) instead of studying (1.1) directly.

Condition (1.14) implies that (1.16) is an elliptic equation for φ. So we can derive a $C^{1+\delta}$ estimate for $\nabla\varphi$ in terms of $\mathcal{B}$, hence in terms of Ω, $\mathcal{H}^e_T$, λ and M, which in turn produces a $C^{1+\delta}$ estimate for $\mathbf{H}$. Then the regularity theory of (1.15) implies $\mathcal{B}\in C^{2+\delta}(\bar\Omega,\mathbb{R}^3)$ and

$$\|\mathcal{B}\|_{C^{2+\delta}(\bar\Omega)}\le C(\Omega,\delta)\|\mathrm{curl}\,\mathcal{B}\|_{C^{1+\delta}(\bar\Omega)}. \tag{1.17}$$

So we can write (1.16) in a non-divergence form with $C^{1+\delta}$ coefficients and derive Schauder estimate of φ. Finally we establish a $C^{2+\alpha}$ estimate for $\mathbf{H}$.

Remark 1.6. Due to the boundary layer behavior of the solutions of (1.3), the global H^2 estimate blows-up as $\lambda\to 0$, see (4.1). On the other hand, we have a uniform L^∞ estimate (8.1), which enables us to establish the H^2_{loc} and $C^{2+\alpha}_{\mathrm{loc}}$ estimates for the rescaled vector fields in Lemma 8.2. The local estimates are sufficient for us to prove Theorem 1.

This paper is organized as follows. In section 2 we provide some properties of the curl operator, and give $C^{1+\alpha}$ estimates of gradients of vector fields in terms of the divergence, curl and tangential or normal trace of the vector fields. In particular, we shall prove inequality (1.17) for the solutions of (1.15). In section 3 we introduce the definition of weak solutions of (1.1) and (1.3). The H^2 estimate of weak solutions of (1.3) is given in section 4, and the $C^{2+\alpha}$ estimate is established in section 5. In section 6 we classify the solutions of limiting equations in $\mathbb{R}^3$ and in $\mathbb{R}^3_+$. In section 7, based on the $C^{2+\alpha}$ estimate of section 5, we use blow-up arguments to derive a criterion for the boundary data that implies solvability. In section 8 we examine the asymptotic behavior of the solutions as $\lambda\to 0$, and prove Theorem 1.

We would like to mention that (1.15) is a special case of div-curl systems. The solvability and regularity of div-curl systems and the applications in physical and mathematical problems have been studied by many authors, see for instance Kress [Kre], Dautray-Lions [DL], Wahl [W1], Schwarz [S], Bourgain-Brezis [BB] and the references therein. We should also mention that Yin [Y] studied an equation involving curl^2 and obtained $C^{1+\alpha}$ regularity.

Inequalities to control gradients of vector fields by using their divergence and curl and traces have been studied for many years. Control in the framework of Sobolev spaces has been well-established, see Dautray-Lions [DL]. Control of the C^α norm of the gradients of vector fields by using C^α norms of divergence and curl has been established by Bolik-Wahl [BW], which enables us to derive a $C^{1+\alpha}$ version of the Bolik-Wahl inequality in section 2. Inequalities of this form play an important rule in our study of (1.3).

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§2. Some Analysis Aspects of the Operator curl

In this section we discuss some questions involving the curl operator for vector fields. Our first question is to ask in which context $\nabla\mathbf{B}$ can be controlled by $\operatorname{div}\mathbf{B}$ and $\operatorname{curl}\mathbf{B}$ in Ω, and by either $\nu\cdot\mathbf{B}$ or $\mathbf{B}_T$ on $\partial\Omega$. If Ω is a bounded and simply-connected domain in $\mathbb{R}^3$ with smooth boundary, the following are known:[8]

(i) The control is true in $H^k(\Omega,\mathbb{R}^3)$ for any $k\geq 0$: It follows from the classical results of Agmon, Douglis and Nirenberg [ADN] and Dautray and Lions [DL] (Theorem 3 on p.209, Proposition 6′ on p.237) that, if Ω is a bounded and simply-connected domain with C^{k+1} boundary, then

$$\|\mathbf{B}\|_{H^{k+1}(\Omega)}\leq C(\Omega,k)\Big\{\|\operatorname{div}\mathbf{B}\|_{H^k(\Omega)}+\|\operatorname{curl}\mathbf{B}\|_{H^k(\Omega)}+\left\|\begin{matrix}\nu\cdot\mathbf{B}\\ \nu\times\mathbf{B}\end{matrix}\right\|_{H^{k+1/2}(\partial\Omega)}\Big\}. \tag{2.1}$$

Here and in the following $\left\|\begin{matrix}\nu\cdot\mathbf{B}\\ \nu\times\mathbf{B}\end{matrix}\right\|_*$ stands for "either $\|\nu\cdot\mathbf{B}\|_*$ or $\|\nu\times\mathbf{B}\|_*$". When $k=0$, $H^0(\Omega)$ is interpreted as $L^2(\Omega)$. Note that the formula given in [DL, p.209] contains the L^2 norm of $\mathbf{B}$ in the right of the inequality. However, under our assumption on the domain Ω, we have, for any $\mathbf{B}\in H^1(\Omega,\mathbb{R}^3)$,

$$\|\mathbf{B}\|_{L^2(\Omega)}\leq C(\Omega)\Big\{\|\operatorname{div}\mathbf{B}\|_{L^2(\Omega)}+\|\operatorname{curl}\mathbf{B}\|_{L^2(\Omega)}+\left\|\begin{matrix}\nu\cdot\mathbf{B}\\ \nu\times\mathbf{B}\end{matrix}\right\|_{H^{1/2}(\partial\Omega)}\Big\}, \tag{2.2}$$

(see Appendix A.2 for the proof). Hence the term $\|\mathbf{B}\|_{L^2(\Omega)}$ in the right of the inequality can be dropped.[9]

(ii) The control is true in $C^\alpha(\bar\Omega,\mathbb{R}^3)$ for any $0<\alpha<1$: It follows from the results of Bolik-Wahl [BW] that, if Ω is a bounded and simply-connected domain with C^2 boundary, then

$$\|\nabla\mathbf{B}\|_{C^\alpha(\bar\Omega)}\leq C(\Omega,\alpha)\Big\{\|\operatorname{div}\mathbf{B}\|_{C^\alpha(\bar\Omega)}+\|\operatorname{curl}\mathbf{B}\|_{C^\alpha(\bar\Omega)}+\left\|\begin{matrix}\nu\cdot\mathbf{B}\\ \nu\times\mathbf{B}\end{matrix}\right\|_{C^{1+\alpha}(\partial\Omega)}\Big\}. \tag{2.3}$$

[8]The results proved by Dautray-Lions [DL], Wahl[W2] and Bolik-Wahl [BW] cover more general cases. These results cited here are limited to the case of simply-connected domains.

[9]The control is also true in $L^p(\Omega,\mathbb{R}^3)$ for any $1<p<\infty$ if either $\nu\cdot\mathbf{B}=0$ or $\mathbf{B}_T=\mathbf{0}$ on $\partial\Omega$, (see Wahl [W2]).

For the purposes of this paper, we need control in $C^{k+\alpha}(\bar{\Omega},\mathbb{R}^3)$ with $k\geq 1$. Let $\nabla_{\partial\Omega}$ denote the tangential gradient operator on $\partial\Omega$, and let

$$\|\mathbf{B}\|_{C^{k+\alpha}(\partial\Omega)}=\|\mathbf{B}\|_{C^0(\partial\Omega)}+\sum_{j=1}^{k-1}\|\nabla^j_{\partial\Omega}\mathbf{B}\|_{C^0(\partial\Omega)}+\|\nabla^k_{\partial\Omega}\mathbf{B}\|_{C^\alpha(\partial\Omega)}.$$

Proposition 2.1. *Let $k\geq 1$ be an integer and $0\leq\alpha<1$. Assume Ω is a bounded and simply-connected domain in $\mathbb{R}^3$ with C^{k+2} boundary. There exists $C(\Omega,k,\alpha)>0$ such that*

$$\|\mathbf{B}\|_{C^{k+1+\alpha}(\bar{\Omega})}\leq C(\Omega,k,\alpha)\Big\{\|\operatorname{div}\mathbf{B}\|_{C^{k+\alpha}(\bar{\Omega})}+\|\operatorname{curl}\mathbf{B}\|_{C^{k+\alpha}(\bar{\Omega})}+\left\|\begin{matrix}\nu\cdot\mathbf{B}\\ \nu\times\mathbf{B}\end{matrix}\right\|_{C^{k+1+\alpha}(\partial\Omega)}\Big\}. \tag{2.4}$$

The proof of Proposition 2.1 will be given in Appendix A.3

Our second question is the following: Given $\mathbf{H}$ with $\operatorname{div}\mathbf{H}=0$, and let $\mathbf{B}$ be a solution of

$$\operatorname{curl}\mathbf{B}=\mathbf{H}\quad\text{and}\quad\operatorname{div}\mathbf{B}=0\quad\text{in }\Omega,\qquad\nu\cdot\mathbf{B}=0\quad\text{on }\partial\Omega, \tag{2.5}$$

what can one say about the regularity of $\mathbf{B}$? For the Sobolev regularity, it is well-known that (see [DL]), if Ω is a bounded and simply-connected domain with smooth boundary, and if $\mathbf{H}\in L^2(\Omega,\mathbb{R}^3)$, the existence of $\mathbf{B}$ satisfying (2.5) is a consequence of the Hodge decomposition; and it follows from (2.1) that, if $\partial\Omega$ is of class C^{k+2} with $k\geq 0$ and if $\mathbf{H}\in H^k(\Omega,\mathbb{R}^3)$, then $\mathbf{B}\in H^{k+1}(\Omega,\mathbb{R}^3)$, and

$$\|\mathbf{B}\|_{H^{k+1}(\Omega)}\leq C(\Omega,k)\|\mathbf{H}\|_{H^k(\Omega)}. \tag{2.6}$$

For the Hölder regularity, the result follows from Proposition 2.1 and a local version of (2.4) (see the proof of Proposition 2.1 in Appendix A.3).

Corollary 2.2. *Let k be a non-negative integer and $0\leq\alpha<1$. Assume that Ω is a bounded, simply-connected domain with C^{k+2} boundary.*

(i) *If $\mathbf{H}\in C^{k+\alpha}(\bar{\Omega},\mathbb{R}^3)$, then (2.5) has a unique solution $\mathbf{B}\in C^{k+1+\alpha}(\bar{\Omega},\mathbb{R}^3)$, and*

$$\|\mathbf{B}\|_{C^{k+1+\alpha}(\bar{\Omega})}\leq C(\Omega,k,\alpha)\|\mathbf{H}\|_{C^{k+\alpha}(\bar{\Omega})}. \tag{2.7}$$

(ii) *Furthermore, if $\mathbf{H}\in C^{k+1+\alpha}(\Omega,\mathbb{R}^3)$, then $\mathbf{B}\in C^{k+2+\alpha}(\Omega,\mathbb{R}^3)$.*

Our third question is about extensions of vector fields on $\partial\Omega$ to all of Ω. The existence of such extensions has been proved in [Mon, Lemma 3.1] in the $C^{2+\alpha}$ case. Here we give a minimality characterization of the extension, and give some estimates.

Lemma 2.3. *Assume Ω is a bounded and simply-connected domain in $\mathbb{R}^3$ with C^4 boundary, and $\mathcal{H}_T^e \in H^{3/2}(\partial\Omega, \mathbb{R}^3)$.*

(i) *There exists a vector field $\mathbf{H} \in H^2(\Omega, \mathbb{R}^3)$ such that*

$$\operatorname{div} \mathbf{H} = 0 \quad \text{in } \Omega, \quad \mathbf{H}_T = \mathcal{H}_T^e \quad \text{on } \partial\Omega, \tag{2.8}$$

and

$$\|\mathbf{H}\|_{H^2(\Omega)} \leq C(\Omega)\|\mathcal{H}_T^e\|_{H^{3/2}(\partial\Omega)}. \tag{2.9}$$

Moreover, $\mathbf{H}$ can be chosen such that the L^2 norm of $\operatorname{curl} \mathbf{H}$ is minimal among all vector fields satisfying (2.8).

(ii) *If in addition $\mathcal{H}_T^e \in C^{2+\alpha}(\partial\Omega, \mathbb{R}^3)$ with $0 \leq \alpha < 1$, then $\mathbf{H} \in C^{2+\alpha}(\bar{\Omega}, \mathbb{R}^3)$ and*

$$\|\mathbf{H}\|_{C^{2+\alpha}(\bar{\Omega})} \leq C(\Omega, \alpha)\|\mathcal{H}_T^e\|_{C^{2+\alpha}(\partial\Omega)}. \tag{2.10}$$

(iii) *Furthermore, if $\mathcal{H}_T^e$ satisfies*

$$\nu \cdot \operatorname{curl}(\mathcal{H}_T^e) = 0 \quad \text{on } \partial\Omega, \tag{2.11}$$

then $\mathbf{H} = \nabla\phi$ for some function ϕ. Moreover, $\phi \in H^3(\Omega)$ if $\mathcal{H}_T^e \in H^{3/2}(\partial\Omega, \mathbb{R}^3)$, and $\phi \in C^{3+\alpha}(\bar{\Omega})$ if $\mathcal{H}_T^e \in C^{2+\alpha}(\partial\Omega, \mathbb{R}^3)$.

The proof of Lemma 2.3 will be given in Appendix A.4. We would like to mention that, a minimization problem for the L^2 norm of $\operatorname{curl} \mathbf{u}$ with prescribed whole trace $\mathbf{u} = \mathbf{u}_0$ on $\partial\Omega$ has been studied in [PQ].

Our fourth question is to ask in which context the condition

$$\mathbf{B}_T = \mathcal{B}_T \quad \text{on } \partial\Omega, \tag{2.12}$$

allows us to conclude that

$$\nu \cdot \operatorname{curl} \mathbf{B} = \nu \cdot \operatorname{curl} \mathcal{B} \quad \text{on } \partial\Omega, \tag{2.13}$$

where ν is the unit outer normal vector to $\partial\Omega$. For C^1 vector fields on $\bar{\Omega}$, it is well-known that the value of $\nu \cdot \operatorname{curl} \mathbf{B}$ on $\partial\Omega$ depends only on the tangential component $\mathbf{B}_T$ (see [Mon]), and hence (2.12) implies (2.13). Using a density argument we see that the conclusion remains true for H^2 vector fields. We include it here for reader's convenience.

Lemma 2.4. *Assume $\partial\Omega$ is of class C^2.*

(i) *Let $\mathbf{B} \in C^1(\partial\Omega, \mathbb{R}^3)$ and let $\mathbf{B}_T$ be its tangential component on $\partial\Omega$. Then the function $x \to \nu \cdot \operatorname{curl}(\mathbf{B}_T)(x)$ is well-defined on $\partial\Omega$, and its value depends only on the tangential derivatives of $\mathbf{B}_T$. For any $\mathcal{B} \in C^1(\bar{\Omega}, \mathbb{R}^3)$ such that (2.12) holds pointwise on $\partial\Omega$, then (2.13) holds pointwise on $\partial\Omega$.*

(ii) *Let $\mathbf{B} \in H^{3/2}(\partial\Omega, \mathbb{R}^3)$. Then $\nu \cdot \operatorname{curl} \mathbf{B}_T$ is well-defined on $\partial\Omega$ as an element in $H^{1/2}(\partial\Omega)$. For any $\mathcal{B} \in H^2(\Omega, \mathbb{R}^3)$ such that (2.12) holds in the sense of trace in $H^{1/2}(\partial\Omega, \mathbb{R}^3)$, then (2.13) holds in the sense of trace in $H^{1/2}(\partial\Omega)$.*

See Appendix A.5 for a proof of (i). As a direct consequence of Lemma 2.4 we have

Lemma 2.5. *Assume $\mathcal{H}_T^e \in C^2(\partial\Omega, \mathbb{R}^3)$ and satisfies (2.11). If $\mathbf{A} \in C^2(\bar{\Omega}, \mathbb{R}^3)$ is a solution of (1.1) satisfying $\|\mathbf{A}\|_{L^\infty(\Omega)} < 1$, then $\mathbf{A}$ satisfies the additional boundary condition*

$$\mathbf{A} \cdot \nu = 0 \quad \text{on } \partial\Omega. \tag{2.14}$$

Proof. Let $\mathbf{A} \in C^2(\bar{\Omega}, \mathbb{R}^3)$ be a solution of (1.1) satisfying $\|\mathbf{A}\|_{L^\infty(\Omega)} < 1$, and let $\mathbf{H} = \lambda \operatorname{curl} \mathbf{A}$. From the equation in (1.1) we see that, (2.14) holds if and only if $\nu \cdot \operatorname{curl} \mathbf{H} = 0$ on $\partial\Omega$. From Lemma 2.4 and using the boundary condition $\mathbf{H}_T = \mathcal{H}_T^e$ we have $\nu \cdot \operatorname{curl} \mathbf{H} = \nu \cdot \operatorname{curl}(\mathbf{H}_T) = \nu \cdot \operatorname{curl}(\mathcal{H}_T^e) = 0$. Thus (2.14) holds. $\square$

§3. Weak Solutions

Let us define

$$\begin{aligned}
&\mathcal{H}(\Omega, \operatorname{curl}) = \{\mathbf{A} \in L^2(\Omega, \mathbb{R}^3) : \ \operatorname{curl} \mathbf{A} \in L^2(\Omega, \mathbb{R}^3)\}, \\
&\mathcal{H}(\Omega, \operatorname{curl} 0) = \{\mathbf{A} \in L^2(\Omega, \mathbb{R}^3) : \operatorname{curl} \mathbf{A} = \mathbf{0}\}, \\
&\|\mathbf{A}\|_{\mathcal{H}(\Omega, \operatorname{curl})} = \left[\|\mathbf{A}\|^2_{L^2(\Omega)} + \|\operatorname{curl} \mathbf{A}\|^2_{L^2(\Omega)}\right]^{1/2}.
\end{aligned}$$

Definition 3.1. *Let $\mathcal{H}_T^e \in H^{1/2}(\partial\Omega, \mathbb{R}^3)$. $\mathbf{A}$ is called a weak solution of (1.1) if $\mathbf{A} \in \mathcal{H}(\Omega, curl) \cap L^\infty(\Omega, \mathbb{R}^3)$, and for all $\mathbf{B} \in \mathcal{H}(\Omega, curl)$ the following equality holds:*

$$\int_\Omega \{\lambda^2 \operatorname{curl} \mathbf{A} \cdot \operatorname{curl} \mathbf{B} + (1 - |\mathbf{A}|^2)\mathbf{A} \cdot \mathbf{B}\} dx + \int_{\partial\Omega} \lambda(\mathcal{H}_T^e \times \mathbf{B}_T) \cdot \nu dS = 0. \tag{3.1}$$

To see that the definition is meaningful, if $\mathbf{B} \in \mathcal{H}(\Omega, \operatorname{curl})$, from the trace theorem for $H(\Omega, \operatorname{curl})$ (see [DL, p.204]), the tangential component $\mathbf{B}_T \in H^{-1/2}(\partial\Omega, \mathbb{R}^3)$, hence the surface integral in (3.1) is well defined if $\mathcal{H}_T^e \in H^{1/2}(\partial\Omega, \mathbb{R}^3)$. Now we give an H^1 estimate.

Lemma 3.2. *Assume $\mathcal{H}_T^e \in H^{1/2}(\partial\Omega, \mathbb{R}^3)$. Let $\mathbf{A} \in \mathcal{H}(\Omega, curl)$ be a weak solution of (1.1) which satisfies the equation for a.e. $x \in \Omega$ and suppose that $\|\mathbf{A}\|_{L^\infty(\Omega)} \le 1$. Then $\mathbf{H} = \lambda \operatorname{curl} \mathbf{A} \in H^1(\Omega, \mathbb{R}^3)$, and*

$$\begin{aligned}
&\|\mathbf{H}\|^2_{L^2(\Omega)} \le \lambda \|\mathcal{H}_T^e\|_{L^1(\partial\Omega)}, \\
&\|\mathbf{H}\|^2_{H^1(\Omega)} \le C(\Omega)\lambda^{-1} \|\mathcal{H}_T^e\|_{L^1(\partial\Omega)} + C(\Omega)\|\mathcal{H}_T^e\|^2_{H^{1/2}(\partial\Omega)},
\end{aligned} \tag{3.2}$$

where $C(\Omega)$ depends only on Ω. Moreover, the equality $\mathbf{H}_T = \mathcal{H}_T^e$ holds in the sense of trace in $H^{1/2}(\partial\Omega, \mathbb{R}^3)$.

Proof. Taking $\mathbf{B} = \mathbf{A}$ in (3.1), we get

$$\int_\Omega \{|\mathbf{H}|^2 + (1 - |\mathbf{A}|^2)|\mathbf{A}|^2\} dx = \lambda \int_{\partial\Omega} (\mathbf{A}_T \times \mathcal{H}_T^e) \cdot \nu dS. \tag{3.3}$$

From (1.1), $\lambda^2|\mathrm{curl}\,\mathbf{H}(x)|^2 = (1-|\mathbf{A}(x)|^2)^2|\mathbf{A}(x)|^2$. Substituting this into (3.3) yields

$$\int_\Omega \{|\mathbf{H}|^2 + \lambda^2|\mathrm{curl}\,\mathbf{H}|^2 + |\mathbf{A}|^4(1-|\mathbf{A}|^2)\}dx = \lambda \int_{\partial\Omega} (\mathbf{A}_T \times \mathcal{H}_T^e)\cdot \nu dS.$$

Since $|\mathbf{A}(x)| \le 1$ on $\bar\Omega$, we have

$$\|\mathbf{H}\|^2_{L^2(\Omega)} + \lambda^2\|\mathrm{curl}\,\mathbf{H}\|^2_{L^2(\Omega)} \le \lambda\|\mathcal{H}_T^e\|_{L^1(\partial\Omega)}.$$

Since $\mathrm{div}\,\mathbf{H} = 0$, using (2.1) with $k = 0$ we get (3.2).

Since $\mathbf{A}$ satisfies (1.1) for a.e. $x \in \Omega$, we can multiply (1.1) by any smooth vector field $\mathbf{B}$ and integrate over Ω. Since $\mathbf{H} \in H^1(\Omega,\mathbb{R}^3)$, we have $\mathbf{H}_T \in \mathbf{H}^{1/2}(\partial\Omega,\mathbb{R}^3)$, and we can integrate by parts to get

$$\int_\Omega \{\lambda^2 \mathrm{curl}\,\mathbf{A}\cdot \mathrm{curl}\,\mathbf{B} + (1-|\mathbf{A}|^2)\mathbf{A}\cdot\mathbf{B}\}dx + \int_{\partial\Omega} \lambda(\mathbf{H}_T\times\mathbf{B}_T)\cdot\nu dS = 0.$$

From this and (3.1) we see that the surface integrals are equal for any smooth vector field $\mathbf{B}$. Thus $\mathbf{H}_T = \mathcal{H}_T^e$ in the sense of trace. $\square$

Under the conditions of Lemma 3.2, it is not always the case that $\mathbf{A} \in H^1(\Omega,\mathbb{R}^3)$ even in 2-dimensions, (see Remark 1.5).

Definition 3.3. *Let $\mathcal{H}_T^e \in H^{1/2}(\partial\Omega,\mathbb{R}^3)$. $\mathbf{H}$ is called a weak solution of (1.3) if the following conditions are satisfied:*

(i) $\mathbf{H} \in H^1(\Omega,\mathbb{R}^3)$;
(ii) $\mathrm{curl}\,\mathbf{H} \in L^\infty(\Omega,\mathbb{R}^3)$ *and (1.4) holds;*
(iii) $\mathbf{H}_T = \mathcal{H}_T^e$ *holds on $\partial\Omega$ in the sense of trace in $H^{1/2}(\partial\Omega,\mathbb{R}^3)$;*
(iv) $(\mathrm{curl}\,\mathbf{H})_T \in H^{-1/2}(\partial\Omega,\mathbb{R}^3)$;
(v) *for all $\mathbf{B} \in H^1(\Omega,\mathbb{R}^3)$, the following equality holds:*

$$\begin{aligned}&\int_\Omega \{\lambda^2 F(\lambda^2|\mathrm{curl}\,\mathbf{H}|^2)\mathrm{curl}\,\mathbf{H}\cdot\mathrm{curl}\,\mathbf{B} + \mathbf{H}\cdot\mathbf{B}\}dx \\ &+ \int_{\partial\Omega} \lambda^2 F(\lambda^2|\mathrm{curl}\,\mathbf{H}|^2)((\mathrm{curl}\,\mathbf{H})_T\times\mathbf{B}_T)\cdot\nu dS = 0.\end{aligned} \tag{3.4}$$

If $\mathbf{B} \in H^1(\Omega,\mathbb{R}^3)$, then $\mathbf{B}_T \in H^{1/2}(\partial\Omega,\mathbb{R}^3)$, and hence the surface integral in (3.4) is well-defined if $\mathbf{H}$ satisfies condition (iv). To see that condition (iv) is reasonable for a weak solution of (1.3), note that we expect the weak solution to satisfy the condition $\mathrm{curl}\,[F(\lambda^2|\mathrm{curl}\,\mathbf{H}|^2)\mathrm{curl}\,\mathbf{H}] \in L^2(\Omega,\mathbb{R}^3)$. From this and condition (1.4) we have $F(\lambda^2|\mathrm{curl}\,\mathbf{H}|^2)\mathrm{curl}\,\mathbf{H} \in \mathcal{H}(\Omega,\mathrm{curl})$. Then it follows from the trace theorem for $\mathcal{H}(\Omega,\mathrm{curl})$ that $F(\lambda^2|\mathrm{curl}\,\mathbf{H}|^2)(\mathrm{curl}\,\mathbf{H})_T \in H^{-1/2}(\partial\Omega,\mathbb{R}^3)$. Since $F(\lambda^2|\mathrm{curl}\,\mathbf{H}|^2) \ge 1$, $\mathbf{H}$ must satisfy (iv).

Lemma 3.4. *Assume $\mathcal{H}_T^e \in H^{1/2}(\partial\Omega, \mathbb{R}^3)$. Let $\mathbf{H} \in H^1(\Omega, \mathbb{R}^3)$ be a weak solution of (1.3) satisfying (1.14). Then*

$$\begin{aligned}
&\|\mathbf{H}\|^2_{L^2(\Omega)} \le \lambda F(M^2) M \|\mathcal{H}_T^e\|_{L^1(\partial\Omega)}, \\
&\textit{and} \\
&\|\mathbf{H}\|^2_{H^1(\Omega)} \le C(\Omega)\lambda^{-1} F(M^2) M \|\mathcal{H}_T^e\|_{L^1(\partial\Omega)} + C(\Omega)\|\mathcal{H}_T^e\|^2_{H^{1/2}(\partial\Omega)},
\end{aligned} \tag{3.5}$$

where $C(\Omega)$ depends only on Ω.

Proof. In (3.4) take $\mathbf{B} = \mathbf{H}$. Using (1.14) and since $1 \le F(\lambda^2|\mathrm{curl}\,\mathbf{H}|^2) \le F(M^2)$ and $\mathbf{H}_T = \mathcal{H}_T^e$, we have

$$\lambda^2\|\mathrm{curl}\,\mathbf{H}\|^2_{L^2(\Omega)} + \|\mathbf{H}\|^2_{L^2(\Omega)} \le \lambda F(M^2) M \|\mathcal{H}_T^e\|_{L^1(\partial\Omega)}.$$

Using this and (2.1) we get (3.5). $\square$

Lemma 3.5. *Given $\mathcal{H}_T^e \in H^{1/2}(\partial\Omega, \mathbb{R}^3)$, problem (1.3) has at most one weak solution* $\mathbf{H}$ *satisfying (1.4). Therefore (1.1) has at most one weak solution* $\mathbf{A}$ *such that* $\mathbf{A}$ *satisfies (1.2) and that $\mathbf{H} = \lambda\,\mathrm{curl}\,\mathbf{A}$ is a weak solution of (1.3).*

The proof is omitted here, as we shall prove in section 6 a uniqueness result in unbounded domains, and the argument there applies to bounded domains as well.

§4. H^2 Estimates of Weak Solutions

Theorem 4.1. *Let Ω be a bounded and simply-connected domain in $\mathbb{R}^3$ with C^3 boundary, and $\mathcal{H}_T^e \in H^{3/2}(\partial\Omega, \mathbb{R}^3)$. Let $\mathbf{H} \in H^1(\Omega, \mathbb{R}^3)$ be a weak solution of (1.3) satisfying (1.14). Then $\mathbf{H} \in H^2(\Omega, \mathbb{R}^3)$, and*

$$\|\mathbf{H}\|^2_{H^2(\Omega)} \le C(\Omega, M)\{(1+\lambda^{-2})\|\mathcal{H}_T^e\|^2_{H^{3/2}(\partial\Omega)} + (\lambda + \lambda^{-3})\|\mathcal{H}_T^e\|_{L^1(\partial\Omega)}\}, \tag{4.1}$$

where $C(\Omega, M)$ depends only on Ω and M.

Proof. We may assume that $\mathcal{H}_T^e$ has been extended to $\bar\Omega$ in the way as stated in Lemma 2.3 (i). In particular,

$$\mathrm{div}\,\mathcal{H}^e = 0 \quad \text{in } \Omega, \qquad \|\mathcal{H}^e\|_{H^2(\Omega)} \le C(\Omega)\|\mathcal{H}_T^e\|_{H^{3/2}(\partial\Omega)}. \tag{4.2}$$

Step 1. Interior estimate.

Let $\mathbf{e}$ denote one of the coordinate vectors $\mathbf{e}_j$, $j = 1, 2, 3$. For $\sigma > 0$ small we define

$$\begin{aligned}
&\mathbf{h}_\sigma(x) = \frac{1}{\sigma}[\mathbf{H}(x + \sigma\mathbf{e}) - \mathbf{H}(x)], \\
&\mathbf{H}_{t,\sigma}(x) = \mathbf{H}(x) + t[\mathbf{H}(x+\sigma\mathbf{e}) - \mathbf{H}(x)] = \mathbf{H}(x) + t\sigma\mathbf{h}_\sigma(x), \\
&u_{t,\sigma} = |\mathrm{curl}\,\mathbf{H}_{t,\sigma}|^2.
\end{aligned} \tag{4.3}$$

For any subdomain $\Omega' \Subset \Omega$, and for all σ sufficiently small, we have ([G, Proposition 2.1]):

$$\|\mathbf{h}_\sigma\|_{L^2(\Omega')} \le C(\Omega, \Omega')\|D\mathbf{H}\|_{L^2(\Omega)}.$$

From (3.4), for any C^1 vector field $\mathbf{B}$ supported in the interior of Ω,

$$\int_\Omega \{\lambda^2 F(\lambda^2|\mathrm{curl}\,\mathbf{H}(x+\sigma\mathbf{e})|^2)\mathrm{curl}\,\mathbf{H}(x+\sigma\mathbf{e})\cdot\mathrm{curl}\,\mathbf{B} + \mathbf{H}(x+\sigma\mathbf{e})\cdot\mathbf{B}\}dx = 0.$$

We subtract (3.4) from this to get

$$\begin{aligned}\int_\Omega \Big\{&\lambda^2\big[F(\lambda^2|\mathrm{curl}\,\mathbf{H}(x+\sigma\mathbf{e})|^2)\mathrm{curl}\,\mathbf{H}(x+\sigma\mathbf{e}) - F(\lambda^2|\mathrm{curl}\,\mathbf{H}(x)|^2)\mathrm{curl}\,\mathbf{H}(x)\big]\cdot\mathrm{curl}\,\mathbf{B}\\ &+ [\mathbf{H}(x+\sigma\mathbf{e}) - \mathbf{H}(x)]\cdot\mathbf{B}\Big\}dx = 0.\end{aligned} \tag{4.4}$$

Let us write

$$\begin{aligned}&F(\lambda^2|\mathrm{curl}\,\mathbf{H}(x+\sigma\mathbf{e})|^2)\mathrm{curl}\,\mathbf{H}(x+\sigma\mathbf{e}) - F(\lambda^2|\mathrm{curl}\,\mathbf{H}(x)|^2)\mathrm{curl}\,\mathbf{H}(x)\\ &= \int_0^1 \frac{d}{dt}[F(\lambda^2 u_{t,\sigma})\mathrm{curl}\,\mathbf{H}_{t,\sigma}]dt\\ &= \sigma\int_0^1 \big\{F(\lambda^2 u_{t,\sigma})\mathrm{curl}\,\mathbf{h}_\sigma + 2\lambda^2 F'(\lambda^2 u_{t,\sigma})(\mathrm{curl}\,\mathbf{H}_{t,\sigma}\cdot\mathrm{curl}\,\mathbf{h}_\sigma)\mathrm{curl}\,\mathbf{H}_{t,\sigma}\big\}dt\\ &= \sigma a_\sigma(x)\mathrm{curl}\,\mathbf{h}_\sigma + 2\sigma Q_\sigma(x)\mathrm{curl}\,\mathbf{h}_\sigma,\end{aligned}$$

where $a_\sigma(x)$ is a scalar function defined by

$$a_\sigma(x) = \int_0^1 F(\lambda^2 u_{t,\sigma})dt,$$

and $Q_\sigma(x) = (q_{\sigma,ij}(x))$ is a 3×3 matrix-valued function with entries

$$q_{\sigma,ij}(x) = \lambda^2\int_0^1 F'(\lambda^2 u_{t,\sigma})(\mathrm{curl}\,\mathbf{H}_{t,\sigma})_i(\mathrm{curl}\,\mathbf{H}_{t,\sigma})_j dt,$$

where $(\mathrm{curl}\,\mathbf{H}_{t,\sigma})_i$ is the i-th component of $\mathrm{curl}\,\mathbf{H}_{t,\sigma}$. Since $\lambda^2 u_{t,\sigma} \le M^2 < 4/27$, there exists a constant $C(M)$ independent of λ such that

$$\begin{aligned}&F(\lambda^2 u_{t,\sigma}) + F'(\lambda^2 u_{t,\sigma}) + a_\sigma(x) + |Q_\sigma(x)| \le C(M), \qquad a_\sigma(x) \ge 1,\\ &Q_\sigma(x) \text{ is non-negative definite for all } x\in\Omega \text{ and } \lambda > 0.\end{aligned} \tag{4.5}$$

Returning to (4.4) we have, for all $\mathbf{B} \in H^1(\Omega, \mathbb{R}^3)$ with compact support in Ω,

$$\int_\Omega \big\{\lambda^2 a_\sigma(x)\mathrm{curl}\,\mathbf{h}_\sigma \cdot \mathrm{curl}\,\mathbf{B} + 2\lambda^2\big\langle Q_\sigma(x)\mathrm{curl}\,\mathbf{h}_\sigma, \mathrm{curl}\,\mathbf{B}\big\rangle + \mathbf{h}_\sigma \cdot \mathbf{B}\big\}\,dx = 0. \tag{4.6}$$

Let η be a smooth function with support in Ω, and let $\mathbf{B} = \eta^2\mathbf{h}_\sigma$. From (4.6) we get

$$\begin{aligned}&\int_\Omega \big\{\lambda^2 a_\sigma(x)|\mathrm{curl}\,(\eta\mathbf{h}_\sigma)|^2 + 2\lambda^2\langle Q_\sigma(x)\mathrm{curl}\,(\eta\mathbf{h}_\sigma), \mathrm{curl}\,(\eta\mathbf{h}_\sigma)\rangle + \eta^2|\mathbf{h}_\sigma|^2\big\}dx\\ =&\int_\Omega \big\{\lambda^2 a_\sigma(x)|\nabla\eta \times \mathbf{h}_\sigma|^2 + 2\lambda^2\langle Q_\sigma(x)\nabla\eta\times\mathbf{h}_\sigma, \nabla\eta\times\mathbf{h}_\sigma\rangle\big\}dx.\end{aligned} \tag{4.7}$$

Using this and (4.5) we have, for all $\sigma > 0$ small,

$$\begin{aligned}&\lambda^2\|\mathrm{curl}\,(\eta\mathbf{h}_\sigma)\|^2_{L^2(\Omega)} + \|\eta\mathbf{h}_\sigma\|^2_{L^2(\Omega)}\\ \le&\int_\Omega \big\{\lambda^2 a_\sigma(x)|\nabla\eta\times\mathbf{h}_\sigma|^2 + 2\lambda^2\langle Q_\sigma(x)\nabla\eta\times\mathbf{h}_\sigma, \nabla\eta\times\mathbf{h}_\sigma\rangle\big\}dx\\ \le& C(M)\lambda^2\|\nabla\eta\times\mathbf{h}_\sigma\|^2_{L^2(\Omega)}.\end{aligned}$$

Note that $\mathrm{div}\,(\eta\mathbf{h}_\sigma) = \nabla\eta\cdot\mathbf{h}_\sigma$. Since $\eta = 0$ on $\partial\Omega$, we use (2.1) to get

$$\begin{aligned}\|\eta\mathbf{h}_\sigma\|^2_{H^1(\Omega)} &\le C(\Omega)\{\|\mathrm{curl}\,(\eta\mathbf{h}_\sigma)\|^2_{L^2(\Omega)} + \|\mathrm{div}\,(\eta\mathbf{h}_\sigma)\|^2_{L^2(\Omega)}\}\\ &\le C(\Omega, M)\|\nabla\eta\times\mathbf{h}_\sigma\|^2_{L^2(\Omega)} + C(\Omega)\|\nabla\eta\cdot\mathbf{h}_\sigma\|^2_{L^2(\Omega)}.\end{aligned}$$

Hence for any subdomain $\Omega' \Subset \Omega$ and σ sufficiently small,

$$\|\mathbf{h}_\sigma\|_{H^1(\Omega')} \le C(\Omega, \Omega', M)\|\mathbf{h}_\sigma\|_{L^2(\Omega)} \le C(\Omega, \Omega', M)\|D\mathbf{H}\|_{L^2(\Omega)}.$$

It follows from this and [G, Proposition 2.1] that $\partial_j\mathbf{H} \in H^1(\Omega', \mathbb{R}^3)$, and $\mathbf{h}_\sigma \to \partial_j\mathbf{H}$ in $H^1(\Omega', \mathbb{R}^3)$ as $\sigma \to 0$.

The above conclusion is true for $j = 1, 2, 3$. So $\mathbf{H} \in H^2(\Omega', \mathbb{R}^3)$, and

$$\|\mathbf{H}\|_{H^2(\Omega')} \le C(\Omega, \Omega', M)\|\mathbf{H}\|_{H^1(\Omega)}. \tag{4.8}$$

In particular, $D^2\mathbf{H}$ exists for a.e. $x \in \Omega$. Using (3.4) we can show that $\mathbf{H}$ satisfies (1.3) for a.e. $x \in \Omega$. So $\mathrm{curl}\,[\lambda^2 F(\lambda^2|\mathrm{curl}\,\mathbf{H}|^2)\mathrm{curl}\,\mathbf{H}] = -\mathbf{H} \in L^2(\Omega, \mathbb{R}^3)$. From (1.14), $\lambda^2 F(\lambda^2|\mathrm{curl}\,\mathbf{H}|^2)\mathrm{curl}\,\mathbf{H} \in L^\infty(\Omega, \mathbb{R}^3) \subset L^2(\Omega, \mathbb{R}^3)$. Thus

$$\lambda^2 F(\lambda^2|\mathrm{curl}\,\mathbf{H}|^2)\mathrm{curl}\,\mathbf{H} \in \mathcal{H}(\Omega, \mathrm{curl}\,). \tag{4.9}$$

By the trace theorem for $\mathcal{H}(\Omega, \mathrm{curl}\,)$, $\lambda^2 F(\lambda^2|\mathrm{curl}\,\mathbf{H}|^2)\mathrm{curl}\,\mathbf{H} \in H^{-1/2}(\partial\Omega, \mathbb{R}^3)$.

Step 2. Boundary estimate: tangential derivatives.

Let $\mathbf{b} = \mathbf{H} - \mathcal{H}^e$. Then $\mathbf{b}_T = \mathbf{0}$ on $\partial\Omega$. From (4.9), $\lambda^2 F(\lambda^2|\mathrm{curl}\,\mathbf{H}|^2)\mathrm{curl}\,\mathbf{b} \in H^{-1/2}(\partial\Omega, \mathbb{R}^3)$. From (3.4) we have,

$$\begin{aligned}
&\int_\Omega \big\{\lambda^2 F(\lambda^2|\mathrm{curl}\,\mathbf{H}|^2)\mathrm{curl}\,\mathbf{b}\cdot\mathrm{curl}\,\mathbf{B} + \mathbf{b}\cdot\mathbf{B}\\
&\qquad + \lambda^2 F(\lambda^2|\mathrm{curl}\,\mathbf{H}|^2)\mathrm{curl}\,\mathcal{H}^e\cdot\mathrm{curl}\,\mathbf{B} + \mathcal{H}^e\cdot\mathbf{B}\big\}dx\\
&+\int_{\partial\Omega} \lambda^2\big\{F(\lambda^2|\mathrm{curl}\,\mathbf{H}|^2)((\mathrm{curl}\,\mathbf{b})_T\times\mathbf{B}_T)\cdot\nu\\
&\qquad + \lambda^2 F(\lambda^2|\mathrm{curl}\,\mathbf{H}|^2)((\mathrm{curl}\,\mathcal{H}^e)_T\times\mathbf{B}_T)\cdot\nu\big\}dS = 0.
\end{aligned}\tag{4.10}$$

We consider a subset Ω' of Ω that is located near $\partial\Omega$. To avoid technical complexity, we assume that $\Gamma = \partial\Omega\cap\bar{\Omega}'$ is flat. Therefore we may write

$$\Omega' = B_{2R}^+ = \{x\in B_{2R}:\ x_3>0\},\qquad \Gamma_{2R} = \{x:\ |x|<2R,\ x_3=0\}.$$

Let $\mathbf{e} = \mathbf{e}_1$ or $\mathbf{e}_2$, and set

$$\mathbf{b}_\sigma(x) = \frac{1}{\sigma}[\mathbf{b}(x+\sigma\mathbf{e}) - \mathbf{b}(x)],\qquad \mathcal{H}^e_\sigma(x) = \frac{1}{\sigma}[\mathcal{H}^e(x+\sigma\mathbf{e}) - \mathcal{H}^e(x)].$$

Note that $\mathbf{b}_{\sigma T} = \mathbf{0}$ on Γ_{2R}. Let $\mathbf{B}$ be supported in $B^+_{2R}\cup\Gamma_{2R}$ and satisfy $\mathbf{B}_T = \mathbf{0}$ on $\partial\Omega$. From (4.10) we have,

$$\begin{aligned}
&\int_\Omega\big\{\lambda^2F(\lambda^2|\mathrm{curl}\,\mathbf{H}(x+\sigma\mathbf{e})|^2)\mathrm{curl}\,\mathbf{b}(x+\sigma\mathbf{e})\cdot\mathrm{curl}\,\mathbf{B} + \mathbf{b}(x+\sigma\mathbf{e})\cdot\mathbf{B}\\
&\quad + \lambda^2F(\lambda^2|\mathrm{curl}\,\mathbf{H}(x+\sigma\mathbf{e})|^2)\mathrm{curl}\,\mathcal{H}^e(x+\sigma\mathbf{e})\cdot\mathrm{curl}\,\mathbf{B} + \mathcal{H}^e(x+\sigma\mathbf{e})\cdot\mathbf{B}\big\}dx = 0.
\end{aligned}$$

As in Step 1, we subtract (4.10) from this equality and derive

$$\begin{aligned}
&\int_\Omega\big\{\lambda^2a_\sigma(x)\mathrm{curl}\,\mathbf{b}_\sigma\cdot\mathrm{curl}\,\mathbf{B} + 2\lambda^2\langle Q_\sigma(x)\mathrm{curl}\,\mathbf{b}_\sigma,\mathrm{curl}\,\mathbf{B}\rangle + \mathbf{b}_\sigma\cdot\mathbf{B}\big\}dx\\
&= -\int_\Omega\big\{\lambda^2a_\sigma(x)\mathrm{curl}\,\mathcal{H}^e_\sigma\cdot\mathrm{curl}\,\mathbf{B} + 2\lambda^2\langle Q_\sigma(x)\mathrm{curl}\,\mathcal{H}^e_\sigma,\mathrm{curl}\,\mathbf{B}\rangle + \mathcal{H}^e_\sigma\cdot\mathbf{B}\big\}dx,
\end{aligned}\tag{4.11}$$

where a_σ and Q_σ are the same as in Step 1.

Now we choose $\mathbf{B} = \eta^2\mathbf{b}_\sigma$, where η is a smooth cut-off function supported in B_{2R}. Then $(\eta^2\mathbf{b}_\sigma)_T = \mathbf{0}$ on $\partial\Omega$. Using this in (4.11) we get (comparing with (4.7))

$$\begin{aligned}
&\int_\Omega\big\{\lambda^2a_\sigma(x)|\mathrm{curl}\,(\eta\mathbf{b}_\sigma)|^2 + 2\lambda^2\langle Q_\sigma(x)\mathrm{curl}\,(\eta\mathbf{b}_\sigma),\mathrm{curl}\,(\eta\mathbf{b}_\sigma)\rangle + \eta^2|\mathbf{b}_\sigma|^2\big\}dx\\
&= \int_\Omega\big\{\lambda^2a_\sigma(x)|\nabla\eta\times\mathbf{b}_\sigma|^2 + 2\lambda^2\langle Q_\sigma(x)\nabla\eta\times\mathbf{b}_\sigma,\nabla\eta\times\mathbf{b}_\sigma\rangle\big\}dx\\
&-\int_\Omega\{\lambda^2a_\sigma(x)(\eta\,\mathrm{curl}\,\mathcal{H}^e_\sigma)\cdot\mathrm{curl}\,(\eta\mathbf{b}_\sigma) + \lambda^2a_\sigma(x)(\eta\,\mathrm{curl}\,\mathcal{H}^e_\sigma)\cdot(\nabla\eta\times\mathbf{b}_\sigma) + \eta^2\mathbf{b}_\sigma\cdot\mathcal{H}^e_\sigma\}dx\\
&-\int_\Omega 2\lambda^2\{\langle Q_\sigma(x)\eta\,\mathrm{curl}\,\mathcal{H}^e_\sigma,\mathrm{curl}\,(\eta\mathbf{b}_\sigma)\rangle + \langle Q_\sigma(x)\eta\,\mathrm{curl}\,\mathcal{H}^e_\sigma,\nabla\eta\times\mathbf{b}_\sigma\rangle\}dx.
\end{aligned}$$

Using this, (4.5), and applying the Holder inequality, we find

$$\begin{aligned}&\int_\Omega \{\lambda^2|\mathrm{curl}\,(\eta\mathbf{b}_\sigma)|^2+\eta^2|\mathbf{b}_\sigma|^2\}dx\\ \le& C\big\{\lambda^2\big[\|\mathbf{b}_\sigma\|^2_{L^2(B_R^+)}+\|\mathrm{curl}\,\mathcal{H}^e_\sigma\|^2_{L^2(B_R^+)}\big]+\|\mathcal{H}^e_\sigma\|^2_{L^2(B_R^+)}\big\},\end{aligned}\tag{4.12}$$

where C depends on Ω, M and R. Note that

$$\begin{aligned}&\|\mathbf{b}_\sigma\|_{L^2(B_R^+)}\le C\|D\mathbf{b}\|_{L^2(\Omega)}\le C\{\|\mathbf{H}\|_{H^1(\Omega)}+\|\mathcal{H}^e\|_{H^1(\Omega)}\},\\ &\|\mathrm{curl}\,\mathcal{H}^e_\sigma\|_{L^2(B_R^+)}\le C\|D\mathrm{curl}\,\mathcal{H}^e\|_{L^2(\Omega)}\le C\|\mathcal{H}^e\|_{H^2(\Omega)}.\end{aligned}$$

Also recall that $\mathrm{div}\,\mathbf{b}_\sigma=0$ in Ω and $(\eta\mathbf{b}_\sigma)_T=\mathbf{0}$ on $\partial\Omega$. Now we use (2.1) and (4.12) to get

$$\begin{aligned}\|\eta\mathbf{b}_\sigma\|^2_{H^1(\Omega)}\le& C\big\{\|\mathrm{curl}\,(\eta\mathbf{b}_\sigma)\|^2_{L^2(\Omega)}+\|\mathrm{div}\,(\eta\mathbf{b}_\sigma)\|^2_{L^2(\Omega)}\big\}\\ \le& C\big\{\|\mathrm{curl}\,(\eta\mathbf{b}_\sigma)\|^2_{L^2(\Omega)}+\|\nabla\eta\cdot\mathbf{b}_\sigma\|^2_{L^2(\Omega)}\big\}\\ \le& C\big\{\|\mathbf{b}_\sigma\|^2_{L^2(\Omega)}+\|\mathrm{curl}\,\mathcal{H}^e_\sigma\|^2_{L^2(\Omega)}\big\}+C\lambda^{-2}\|\mathcal{H}^e_\sigma\|^2_{L^2(\Omega)}\\ \le& C\|\mathbf{H}\|^2_{H^1(\Omega)}+C(1+\lambda^{-2})\|\mathcal{H}^e\|^2_{H^2(\Omega)},\end{aligned}\tag{4.13}$$

where C depends on Ω, M and η only.

Letting $\sigma\to 0$ in (4.13), we see that, for $j=1,2$, $\partial_j\mathbf{b}\in H^1(B_R^+,\mathbb{R}^3)$, and

$$\|\partial_j\mathbf{b}\|^2_{H^1(B_R^+)}\le C\|\mathbf{H}\|^2_{H^1(\Omega)}+C(1+\lambda^{-2})\|\mathcal{H}^e\|^2_{H^2(\Omega)}.$$

Hence $\partial_j\mathbf{H}\in H^1(B_R^+,\mathbb{R}^3)$, and

$$\|\partial_j\mathbf{H}\|_{H^1(B_R^+)}\le C\|\mathbf{H}\|_{H^1(\Omega)}+C(1+\lambda^{-1})\|\mathcal{H}^e\|_{H^2(\Omega)},\tag{4.14}$$

where C depends only on Ω, M and R.

Step 3. Boundary estimates: normal derivatives.

As mentioned in the last part of step 1, $D^2\mathbf{H}$ exists a.e. in Ω, and $\mathbf{H}$ satisfies equation (1.3) for a.e. $x\in\Omega$. Therefore,

$$\lambda^2F(\lambda^2u)\Delta\mathbf{H}-\lambda^4F'(\lambda^2u)\nabla u\times\mathrm{curl}\,\mathbf{H}=\mathbf{H},\quad\text{for a.e. } x\in\Omega,\tag{4.15}$$

where $u(x)=|\mathrm{curl}\,\mathbf{H}(x)|^2$. Write $\mathbf{J}(x)=\mathrm{curl}\,\mathbf{H}(x)=(J_1,J_2,J_3)$. We compute

$$\nabla u\times\mathrm{curl}\,\mathbf{H}=\begin{bmatrix}2J_1J_2\partial_{33}H_2-2J_2^2\partial_{33}H_1+f_1\\ -2J_1^2\partial_{33}H_2+2J_1J_2\partial_{33}H_1+f_2\\ f_3\end{bmatrix},$$

where

$$\begin{aligned}
f_1 &= -2J_2[J_1\partial_{23}H_3 - J_2\partial_{13}H_3 + J_2(\partial_{13}H_2 - \partial_{23}H_1)] + J_3\partial_2 u,\\
f_2 &= 2J_1[J_1\partial_{23}H_3 - J_2\partial_{13}H_3 + J_2(\partial_{13}H_2 - \partial_{23}H_1)] - J_3\partial_1 u,\\
f_3 &= J_2\partial_1 u - J_1\partial_2 u.
\end{aligned}$$

Since

$$|\mathbf{J}|^2 = u \le M^2\lambda^{-2}, \text{ and } |\partial_j u| = 2|\mathbf{J}\cdot\partial_j\mathbf{J}| \le 2M\lambda^{-1}\sum_{(i,j)\neq(3,3)}|\partial_{ij}\mathbf{H}|,$$

we have

$$|f_k| \le C(M)\lambda^{-2}\sum_{(i,j)\neq(3,3)}|\partial_{ij}\mathbf{H}|. \tag{4.16}$$

Now we write (4.15) as follows:

$$a\partial_{33}H_1 + b\partial_{33}H_2 = g_1, \quad b\partial_{33}H_1 + c\partial_{33}H_2 = g_2, \quad \lambda^2F(\lambda^2u)\partial_{33}H_3 = g_3, \tag{4.17}$$

where

$$\begin{aligned}
a &= \lambda^2F(\lambda^2u) + 2\lambda^4F'(\lambda^2u)J_2^2,\\
b &= -2\lambda^4F'(\lambda^2u)J_1J_2,\\
c &= \lambda^2F(\lambda^2u) + 2\lambda^4F'(\lambda^2u)J_1^2,\\
g_1 &= H_1 - \lambda^2F(\lambda^2u)(\partial_{11}H_1 + \partial_{22}H_1) + \lambda^4F'(\lambda^2u)f_1,\\
g_2 &= H_2 - \lambda^2F(\lambda^2u)(\partial_{11}H_2 + \partial_{22}H_2) + \lambda^4F'(\lambda^2u)f_2,\\
g_3 &= H_3 - \lambda^2F(\lambda^2u)(\partial_{11}H_3 + \partial_{22}H_3) + \lambda^4F'(\lambda^2u)f_3.
\end{aligned}$$

We have $|a| + |b| + |c| \le C(M)\lambda^2$, and

$$ac - b^2 = \lambda^4F(\lambda^2u)^2 + 2\lambda^6F'(\lambda^2u)(J_1^2 + J_2^2) \ge \lambda^4.$$

We solve $\partial_{33}H_i$ from (4.17):

$$\partial_{33}H_1 = \frac{cg_1 - bg_2}{ac - b^2}, \quad \partial_{33}H_2 = \frac{-bg_1 + ag_2}{ac - b^2}, \quad \partial_{33}H_3 = \frac{g_3}{\lambda^2F(\lambda^2u)}. \tag{4.18}$$

Now we consider a subdomain Ω' located near $\partial\Omega$. As in step 2 we may assume that $\Omega' = B_{2R}^+$, and $\Gamma_{2R} = \{x:\ |x| < 2R,\ x_3 = 0\}$. From (4.14) and (4.16),

$$\begin{aligned}
\|g_k\|_{L^2(B_R^+)} \le& \|\mathbf{H}\|_{L^2(B_R^+)} + C(M)\lambda^2\sum_{(i,j)\neq(3,3)}\|\partial_{ij}\mathbf{H}\|_{L^2(B_R^+)}\\
\le& \|\mathbf{H}\|_{L^2(\Omega)} + C(\Omega,M)\lambda^2\{\|\mathbf{H}\|_{H^1(\Omega)} + (1+\lambda^{-1})\|\mathcal{H}^e\|_{H^2(\Omega)}\}.
\end{aligned}$$

From this and (4.18) we obtain

$$\|\partial_{33}\mathbf{H}\|_{L^2(B_R^+)} \le C(\Omega, M, R)\big\{\lambda^{-2}\|\mathbf{H}\|_{L^2(\Omega)} + \|\mathbf{H}\|_{H^1(\Omega)} + (1+\lambda^{-1})\|\mathcal{H}^e\|_{H^2(\Omega)}\big\}. \tag{4.19}$$

Step 4. Covering a neighborhood of $\partial\Omega$ by a finite number of subdomains which are diffeomorphic to a half ball, and applying (4.14) and (4.19) on each of such half balls, and then using (4.8), we find

$$\|\mathbf{H}\|^2_{H^2(\Omega)} \le C(\Omega, M)\big\{\|\mathbf{H}\|^2_{H^1(\Omega)} + (1+\lambda^{-4})\|\mathbf{H}\|^2_{L^2(\Omega)} + (1+\lambda^{-2})\|\mathcal{H}^e\|^2_{H^2(\Omega)}\big\}.$$

Using this together with (3.5) and (4.2) we get (4.1). □

From Theorem 4.1 and the Sobolev imbedding theorem we have:

Corollary 4.2. *Under the conditions of Theorem 4.1, the solution $\mathbf{H} \in C^\alpha(\bar\Omega, \mathbb{R}^3)$ for any $\alpha \in (0, \frac{1}{2})$, and*

$$\|\mathbf{H}\|^2_{C^\alpha(\bar\Omega)} \le C(\Omega, M, \alpha)\Big\{(1+\lambda^{-2})\|\mathcal{H}^e_T\|^2_{H^{3/2}(\partial\Omega)} + (\lambda+\lambda^{-3})\|\mathcal{H}^e_T\|_{L^1(\partial\Omega)}\Big\}, \tag{4.20}$$

where $C(\Omega, M, \alpha)$ depends only on Ω, M and α.

§5. $C^{2+\alpha}$ Regularity of the Weak Solutions

Theorem 5.1. *Assume that Ω is a bounded and simply-connected domain in $\mathbb{R}^3$ with C^4 boundary, and*

$$\mathcal{H}^e_T \in C^{2+\alpha}(\partial\Omega), \qquad \nu \cdot \operatorname{curl}\mathcal{H}^e_T = 0 \ \text{on } \partial\Omega, \tag{5.1}$$

where $0 < \alpha < 1$. Let $\mathbf{H} \in H^1(\Omega, \mathbb{R}^3)$ be a weak solution of (1.3) satisfying (1.14). Then $\mathbf{H} \in C^3(\Omega, \mathbb{R}^3) \cap C^{2+\alpha}(\bar\Omega, \mathbb{R}^3)$, and

$$\|\mathbf{H}\|_{C^{2+\alpha}(\bar\Omega)} \le C(\Omega, \lambda, M, \alpha, \|\mathcal{H}^e_T\|_{C^{2+\alpha}(\partial\Omega)}). \tag{5.2}$$

Proof. We may assume $\mathcal{H}^e_T$ has been extended over $\bar\Omega$ in the way as stated in Lemma 2.3, (ii). In particular

$$\operatorname{div}\mathcal{H}^e = 0 \quad \text{in } \Omega, \qquad \|\mathcal{H}^e\|_{C^{2+\alpha}(\bar\Omega)} \le C(\Omega, \alpha)\|\mathcal{H}^e_T\|_{C^{2+\alpha}(\partial\Omega)}. \tag{5.3}$$

In the following, for simplicity of presentation, we only give the proof in the case where $\lambda = 1$. Let $\mathbf{H}$ be the weak solution of (1.3).

Step 1. Find a vector field $\mathcal{B}$ satisfying (1.15) (with $\lambda = 1$).

From Corollary 4.2, $\mathbf{H} \in C^\delta(\bar{\Omega}, \mathbb{R}^3)$ for any $0 < \delta < 1/2$. For the moment let us fix $\delta < \min\{\alpha, \frac{1}{2}\}$. Since $\operatorname{div} \mathbf{H} = 0$, from Corollary 2.2 (i) we see that there exists a vector field $\mathcal{B} \in C^{1+\delta}(\Omega, \mathbb{R}^3)$, $\mathcal{B} = (\mathcal{B}_1, \mathcal{B}_2, \mathcal{B}_3)$, such that (1.15) (with $\lambda = 1$) holds, and

$$\|\mathcal{B}\|_{C^{1+\delta}(\bar{\Omega})} \leq C(\Omega, \delta)\|\mathbf{H}\|_{C^\delta(\bar{\Omega})} \leq C(\Omega, \lambda, M, \delta, \|\mathcal{H}^e\|_{H^2(\Omega)}). \tag{5.4}$$

Step 2. Find the scalar function φ satisfying (1.16).

Since $\mathbf{H}$ satisfies (1.3) for a.e. $x \in \Omega$, we have $\operatorname{curl}[F(|\mathbf{J}|^2)\mathbf{J} + \mathcal{B}] = 0$ for a.e. $x \in \Omega$. Thus $F(|\mathbf{J}|^2)\mathbf{J} + \mathcal{B} \in \mathcal{H}(\Omega, \operatorname{curl} 0)$. Since Ω is simply-connected, there exists $\varphi \in H^1(\Omega)$ such that $F(|\mathbf{J}|^2)\mathbf{J} + \mathcal{B} = \nabla\varphi$ (see [DL, p.219, Proposition 2]). From condition (5.1) and using Lemma 2.4 (ii) we have

$$\nu \cdot \mathbf{J} = \nu \cdot \operatorname{curl} \mathbf{H} = \nu \cdot \operatorname{curl} \mathbf{H}_T = \nu \cdot \operatorname{curl} \mathcal{H}^e_T = \nu \cdot \operatorname{curl} \mathcal{H}^e = 0$$

on $\partial\Omega$. Since we also have $\nu \cdot \mathcal{B} = 0$ on $\partial\Omega$, it holds that $\frac{\partial\varphi}{\partial\nu} = 0$ on $\partial\Omega$.

From (1.14) and (1.5), there exists $b(M) < 1/\sqrt{3}$ such that

$$\begin{aligned} &|\nabla\varphi - \mathcal{B}| = F(|\mathbf{J}|^2)|\mathbf{J}| \leq b(M) < \frac{1}{\sqrt{3}}, \\ &|\mathbf{J}| = (1 - |\nabla\varphi - \mathcal{B}|^2)|\nabla\varphi - \mathcal{B}|, \qquad F(|\mathbf{J}|^2) = \frac{|\nabla\varphi - \mathcal{B}|}{|\mathbf{J}|} = \frac{1}{1 - |\nabla\varphi|^2}, \\ &\mathbf{J} = \frac{\nabla\varphi - \mathcal{B}}{F(|\mathbf{J}|^2)} = (1 - |\nabla\varphi - \mathcal{B}|^2)(\nabla\varphi - \mathcal{B}). \end{aligned} \tag{5.5}$$

Since $\operatorname{div} \mathbf{J} = 0$ in Ω and $\frac{\partial\varphi}{\partial\nu} = 0$ on $\partial\Omega$, φ satisfies (1.16).

Step 3. $C^{2+\delta}$ regularity for the quasilinear Neumann problem (1.16).

Write

$$\begin{aligned} &\mathbf{p} = (p_1, p_2, p_3), \qquad \mathcal{Q}_b = \{(x, \mathbf{p}) : \ x \in \bar{\Omega}, \ \mathbf{p} \in \mathbb{R}^3, \ |\mathbf{p} - \mathcal{B}(x)| \leq b\}, \\ &A(x, \mathbf{p}) = (1 - |\mathbf{p} - \mathcal{B}(x)|^2)(\mathbf{p} - \mathcal{B}(x)). \end{aligned}$$

Then (1.16) can be written as

$$\operatorname{div} A(x, \nabla\varphi) = 0 \quad \text{in } \Omega, \qquad \frac{\partial\varphi}{\partial\nu} = 0 \quad \text{on } \partial\Omega. \tag{5.6}$$

We claim that if $0 < b < \frac{1}{\sqrt{3}}$, then $A(x, \mathbf{p})$ satisfies the following conditions on $\mathcal{Q}_b$:

(i) $A_i \in C^{1+\delta}(\mathcal{Q}_b)$;

(ii) $\displaystyle\sum_{i,j=1}^{3} \frac{\partial A_i}{\partial p_j}(x, \mathbf{p})\xi_i\xi_j \geq \lambda(b)|\xi|^2 \quad \text{for all } \xi \in \mathbb{R}^3;$

(iii) $|A(x, \mathbf{0})| \leq m$;

(iv) $$\left|\frac{\partial A_i}{\partial x_j}(x, \mathbf{p})| + (1 + |\mathbf{p}|)|A_i(x, \mathbf{p})| + (1 + |\mathbf{p}|^2)|\frac{\partial A_i}{\partial p_j}(x, \mathbf{p})\right| \leq \Lambda(b)(1 + |\mathbf{p}|^2), \tag{5.7}$$

where $\lambda(b)$ and $\Lambda(b)$ depend only on b, and m depends on $\mathcal{B}$.

(i) is true because $\mathcal{B} \in C^{1+\delta}(\bar\Omega, \mathbb{R}^3)$. (ii) is true for $\lambda(b) = 1 - 3b^2 > 0$. In fact,

$$\begin{aligned}
&\frac{\partial A_i}{\partial p_j}(x, \mathbf{p}) = (1 - |\mathbf{p} - \mathcal{B}(x)|^2)\delta_{ij} - 2(p_i - \mathcal{B}_i(x))(p_j - \mathcal{B}_j(x)),\\
&\sum_{i,j=1}^3 \frac{\partial A_i}{\partial p_j}(x, \mathbf{p})\xi_i\xi_j = (1 - |\mathbf{p} - \mathcal{B}(x)|^2)|\xi|^2 - 2((\mathbf{p} - \mathcal{B}(x)) \cdot \xi)^2\\
\geq &(1 - 3|\mathbf{p} - \mathcal{B}(x)|^2)|\xi|^2 \geq (1 - 3b^2)|\xi|^2.
\end{aligned}$$

Because $|A(x, \mathbf{0})| = |1 - |\mathcal{B}(x)|^2||\mathcal{B}(x)|$, (iii) is true for $m = \max_{x\in\bar\Omega} |1 - |\mathcal{B}(x)|^2||\mathcal{B}(x)|$. To verify (iv), we compute on $\mathcal{Q}_b$:

$$\begin{aligned}
|A_i(x, \mathbf{p})| &= (1 - |\mathbf{p} - \mathcal{B}(x)|^2)|p_i - \mathcal{B}_i| \leq |\mathbf{p} - \mathcal{B}(x)|,\\
\left|\frac{\partial A_i}{\partial p_j}(x, \mathbf{p})\right| &\leq 1 + |\mathbf{p} - \mathcal{B}(x)|^2,\\
\left|\frac{\partial A_i}{\partial x_j}(x, \mathbf{p})\right| &= \left|(1 - |\mathbf{p} - \mathcal{B}(x)|^2)\frac{\partial \mathcal{B}_i}{\partial x_j} + 2(p_i - \mathcal{B}_i(x))\sum_{k=1}^3 (p_k - \mathcal{B}_k(x))\frac{\partial \mathcal{B}_k}{\partial x_j}\right|\\
&\leq (1 + 3|\mathbf{p} - \mathcal{B}(x)|^2)\|\nabla\mathcal{B}\|_{C^0(\bar\Omega)}.
\end{aligned}$$

For the number $b(M)$ given in the first line in (5.5), we choose b such that $b(M) < b < \frac{1}{\sqrt{3}}$. Then we extend $A(x, \mathbf{p})$ to the whole region $\bar\Omega \times \mathbb{R}^3$ such that conditions (i)-(iv) are satisfied for all $x \in \bar\Omega$ and all $\mathbf{p} \in \mathbb{R}^3$. Then we can apply the classical regularity theory for quasilinear elliptic equations to (5.6) to conclude that $\varphi \in C^{2+\delta}(\bar\Omega)$, and $\|\varphi\|_{C^{2+\delta}(\bar\Omega)}$ can be estimated in terms of M, b and $\|\mathcal{B}\|_{C^{1+\delta}(\bar\Omega)}$, (see [CW, chapter 5]). Regarding the regularity at the boundary, we would like to mention that, since $\nu \cdot \mathcal{B} = 0$ and $\frac{\partial\varphi}{\partial\nu} = 0$ on $\partial\Omega$, after straightening a portion of boundary, we can extend $\mathcal{B}$ and φ by reflection: even extension for φ and for the tangential component of $\mathcal{B}$, and odd extension of the normal component of $\mathcal{B}$. Then the regularity at the boundary can be established by applying the interior regularity theory to the extended functions.

Since b can be chosen to depend only on M, and using (5.4), we have

$$\|\varphi\|_{C^{2+\delta}(\bar\Omega)} \leq C(\Omega, \lambda, M, \delta, \|\mathcal{H}^e\|_{H^2(\Omega)}). \tag{5.8}$$

Step 4. $C^{2+\delta}$ estimates for $\mathbf{H}$.

Since $\mathcal{B}$, $\nabla\varphi \in C^{1+\delta}(\bar\Omega, \mathbb{R}^3)$, we have $\mathbf{J} = (1 - |\nabla\varphi - \mathcal{B}|^2)(\nabla\varphi - \mathcal{B}) \in C^{1+\delta}(\bar\Omega, \mathbb{R}^3)$. $\mathbf{H}$ satisfies

$$\operatorname{curl}\mathbf{H} = \mathbf{J} \quad \text{and} \quad \operatorname{div}\mathbf{H} = 0 \quad \text{in } \Omega, \quad \mathbf{H}_T = \mathcal{H}^e_T \quad \text{on } \partial\Omega. \tag{5.9}$$

Applying (2.4) to $\mathbf{H}$ we find $\mathbf{H} \in C^{2+\delta}(\bar{\Omega}, \mathbb{R}^3)$, and

$$\|\mathbf{H}\|_{C^{2+\delta}(\bar{\Omega})} \le C(\Omega, \delta)\{\|\mathbf{J}\|_{C^{1+\delta}(\bar{\Omega})} + \|\mathcal{H}_T^e\|_{C^{2+\delta}(\partial\Omega)}\} \le C, \tag{5.10}$$

where $C = C(\Omega, \lambda, M, \delta, \|\mathcal{H}^e\|_{H^2(\Omega)}, \|\mathcal{H}_T^e\|_{C^{2+\delta}(\partial\Omega)})$.

Step 5. Interior $C^{3+\delta}$ regularity and global $C^{2+\alpha}$ estimates of $\mathbf{H}$.

Using (5.10) we can improve the regularity of $\mathcal{B}$ obtained in step 1: Applying Corollary 2.2 (ii) to equation (1.15) (with $\lambda = 1$) for $\mathcal{B}$ we have $\mathcal{B} \in C^{2+\delta}(\bar{\Omega}, \mathbb{R}^3)$. Then the functions $A_i(x, \mathbf{p})$ defined in Step 3 are in $C^{2+\delta}$ in x. Write equation (5.6) in the form

$$\sum_{i,j=1}^{3} a_{ij}(x) \frac{\partial^2 \varphi}{\partial x_i \partial x_j} + f(x) = 0 \quad \text{in } \Omega, \qquad \frac{\partial \varphi}{\partial \nu} = 0 \quad \text{on } \partial\Omega, \tag{5.11}$$

where

$$a_{ij}(x) = (1 - |\nabla\varphi(x) - \mathcal{B}(x)|^2)\delta_{ij} - 2(\frac{\partial \varphi}{\partial x_i}(x) - \mathcal{B}_i(x))(\frac{\partial \varphi}{\partial x_j}(x) - \mathcal{B}_j(x)),$$

and

$$f(x) = 2\sum_{i,j=1}^{3} (\frac{\partial \varphi}{\partial x_i}(x) - \mathcal{B}_i(x))(\frac{\partial \varphi}{\partial x_j}(x) - \mathcal{B}_j(x))\frac{\partial \mathcal{B}_j}{\partial x_i}(x).$$

Note that a_{ij}, $f \in C^{1+\delta}(\bar{\Omega})$, and $\sum_{i,j=1}^{3} a_{ij}(x)\xi_i\xi_j \ge \lambda(b)|\xi|^2$. Applying the Schauder estimates to (5.11) (see [GT, CW]) we conclude that $\varphi \in C^{3+\delta}(\bar{\Omega})$ and $\|\varphi\|_{C^{3+\delta}(\bar{\Omega})}$ is bounded by a constant depending only on Ω, a_{ij}, f and δ. Hence $\mathbf{J} \in C^{2+\delta}(\bar{\Omega}, \mathbb{R}^3)$. Using (5.9) and checking the interior estimate in the proof of Proposition 2.1 we see that $\mathbf{H} \in C^{3+\delta}(\Omega, \mathbb{R}^3)$.

Now we derive the global $C^{2+\alpha}$ estimates. If $\alpha < \frac{1}{2}$, then we can choose $\delta = \alpha$ in the above discussions, and we are done. If $\frac{1}{2} \le \alpha < 1$, from the discussion in the last paragraph we have $\mathbf{J} \in C^{2+\delta}(\bar{\Omega}, \mathbb{R}^3) \subset C^{1+\alpha}(\bar{\Omega}, \mathbb{R}^3)$. Using (5.9) and (2.4) we find $\mathbf{H} \in C^{2+\alpha}(\bar{\Omega}, \mathbb{R}^3)$, and (5.10) holds with δ replaced by α. □

From Lemma 4.1 and Theorem 5.1 we have the following conclusion:

Theorem 5.2. *Under the condition of Theorem 5.1, if (1.3) has a weak solution* $\mathbf{H}$ *satisfying (1.4), then (1.1) has a solution* $\mathbf{A} \in C^{2+\alpha}(\bar{\Omega}, \mathbb{R}^3)$ *satisfying (1.2). Furthermore if* $\partial\Omega$ *is of class* C^5 *then* $\mathbf{A} \in C^{3+\alpha}(\bar{\Omega}, \mathbb{R}^3)$.

Proof. We keep the notations used in the proof of Theorem 5.1, and we shall prove Theorem 5.2 in the case where $\lambda = 1$. From Theorem 5.1 we have $\mathbf{H} \in C^{2+\alpha}(\bar{\Omega}, \mathbb{R}^3)$. Using Corollary 2.2, $\mathcal{B} \in C^{3+\alpha}(\bar{\Omega}, \mathbb{R}^3)$. From the proof of Theorem 5.1 we have

$\varphi \in C^{3+\alpha}(\bar\Omega)$. Hence a_{ij} and f in (5.11) are in $C^{2+\alpha}(\bar\Omega)$. Applying the Schauder estimates to (5.11) we see that $\varphi \in C^{4+\alpha}(\bar\Omega)$ if $\partial\Omega$ is C^5, and $\varphi \in C^{3+\alpha}(\bar\Omega)$ if $\partial\Omega$ is C^4. Let $\mathbf{A} = \mathcal{B} - \nabla\varphi$. Then $\mathbf{A} \in C^{3+\alpha}(\bar\Omega, \mathbb{R}^3)$ if $\partial\Omega$ is C^5 and $\mathbf{A} \in C^{2+\alpha}(\bar\Omega, \mathbb{R}^3)$ if $\partial\Omega$ is C^4. Using (5.5) we have

$$\mathrm{curl}^2\mathbf{A} = \mathrm{curl}^2\mathcal{B} = \mathrm{curl}\,\mathbf{H} = \mathbf{J} = -(1-|\mathcal{B}-\nabla\varphi|^2)(\mathcal{B}-\nabla\varphi) = -(1-|\mathbf{A}|^2)\mathbf{A},$$

So $\mathbf{A}$ is a solution of (1.1). From the first line in (5.5), $\mathbf{A}$ satisfies (1.2). $\square$

Remark 5.3. In Theorem 5.1, if we further assume that $\mathcal{H}_T^e \in C^{k+\alpha}(\partial\Omega)$ with $k \geq 2$, then $\mathbf{H} \in C^{k+\alpha}(\bar\Omega, \mathbb{R}^3)$, and a global $C^{k+\alpha}$ estimate holds.

Note that (1.16) is of hyperbolic type in the region where $\frac{1}{\sqrt{3}} < |\nabla\varphi - \mathcal{B}(x)| < 1$. Loss of regularity of φ when $|\nabla\varphi - \mathcal{B}(x)|$ increases to $\frac{1}{\sqrt{3}}$ may be relevant to loss of regularity of the solutions of (1.1) when condition (1.2) is violated.

§6. Classification of Solutions in $\mathbb{R}^3$ and in $\mathbb{R}^3_+$

The uniqueness results established in this section will be essential for exploring the asymptotic behavior of solutions of (1.3) in section 8. For the 2-dimensional case, the uniqueness for C^2 solutions in the entire plane and in the half plane that satisfy the condition $|\mathbf{A}(x)| < 1/\sqrt{3}$ was proved in [PK, Lemma 2.3], and the argument in [PK] could be used in the 3-dimensional case for C^2 solutions in this section. However, our proof below gives the uniqueness result under a weaker condition $|\mathbf{A}(x)| \leq 1 - \varepsilon_0$, and it works also for weak solutions (which can be defined as in section 3). We first consider the problems in the entire space:

$$-\mathrm{curl}^2\mathbf{A} = (1-|\mathbf{A}|^2)\mathbf{A} \quad \text{in } \mathbb{R}^3. \tag{6.1}$$

and

$$-\mathrm{curl}\left[F(|\mathrm{curl}\,\mathbf{H}|^2)\mathrm{curl}\,\mathbf{H}\right] = \mathbf{H} \quad \text{in } \mathbb{R}^3. \tag{6.2}$$

Proposition 6.1. (i) *If $\mathbf{A}$ is a weak solution of (6.1) satisfying $\|\mathbf{A}\|_{L^\infty(\mathbb{R}^3)} < 1$, then $\mathbf{A} \equiv \mathbf{0}$.*

(ii) *If $\mathbf{H}$ is a weak solution of (6.2) satisfying*

$$\|\mathrm{curl}\,\mathbf{H}\|_{L^\infty(\mathbb{R}^3)} \leq \sqrt{\frac{4}{27}}, \tag{6.3}$$

then $\mathbf{H} \equiv \mathbf{0}$.

Proof. (i) Let $\mathbf{A}$ be a solution as stated in the proposition. Choose $\varepsilon_0 > 0$ such that $\|\mathbf{A}\|_{L^\infty(\mathbb{R}^3)} < 1 - \varepsilon_0$. For any smooth vector field $\mathbf{B}$ with compact support we have

$$\int_{\mathbb{R}^3} \{\mathrm{curl}\,\mathbf{A} \cdot \mathrm{curl}\,\mathbf{B} + (1-|\mathbf{A}|^2)\mathbf{A}\cdot\mathbf{B}\}dx = 0.$$

By a density argument, this must also hold for $\mathbf{B} \in \mathcal{H}(\mathbb{R}^3, \mathrm{curl})$. Taking $\mathbf{B} = \eta^2 \mathbf{A}$, where η is a smooth function with compact support, we get

$$\int_{\mathbb{R}^3} \{|\mathrm{curl}\,(\eta \mathbf{A})|^2 + (1 - |\mathbf{A}|^2)|\eta \mathbf{A}|^2\} dx = \int_{\mathbb{R}^3} |\nabla \eta \times \mathbf{A}|^2 dx. \tag{6.4}$$

Choose $\eta = e^{-\delta r}\xi(r)$ in (6.4), where $r = |x|$, δ is a number satisfying $0 < \delta < \frac{\sqrt{\varepsilon_0}}{2}$, and $\xi(r)$ is a smooth nonincreasing cut-off function such that $\xi(r) = 1$ for $r < R$, $\xi(r) = 0$ for $r > R + 1$, and $|\xi'(r)| \le 2$. Identity (6.4) gives, for some $C = C(\delta, \varepsilon_0)$,

$$\varepsilon_0 \int_{B_R} e^{-2\delta r} |\mathbf{A}|^2 dx \le 2\delta^2 \int_{B_{R+1}} e^{-2\delta r} |\mathbf{A}|^2 dx + 8 \int_{B_{R+1} \setminus B_R} e^{-2\delta r} dx.$$

Letting $R \to \infty$ we find $\mathbf{A} \equiv \mathbf{0}$.

(ii) Let $\mathbf{H}$ be a solution as stated in the proposition. For any C^1 vector field $\mathbf{B}$ with bounded support,

$$\int_{\mathbb{R}^3} \{F(|\mathrm{curl}\,\mathbf{H}|^2) \mathrm{curl}\,\mathbf{H} \cdot \mathrm{curl}\,\mathbf{B} + \mathbf{H} \cdot \mathbf{B}\} dx = 0.$$

We choose $\mathbf{B} = \eta^2 \mathbf{H}$, where η is a smooth function with bounded support, and get

$$\int_{\mathbb{R}^3} \{F(|\mathrm{curl}\,\mathbf{H}|^2)|\mathrm{curl}\,(\eta \mathbf{H})|^2 + |\eta \mathbf{H}|^2\} dx = \int_{\mathbb{R}^3} |\nabla \eta \times \mathbf{H}|^2 dx.$$

Take η as in the proof of part (i), and find

$$\int_{B_R} e^{-2\delta r} |\mathbf{H}|^2 dx \le 2\delta^2 \int_{B_{R+1}} e^{-2\delta r} |\mathbf{H}|^2 dx + 8e^{-2\delta R} \int_{R_{R+1} \setminus B_R} |\mathbf{H}|^2 dx.$$

Letting $R \to \infty$ we find $\mathbf{H} \equiv \mathbf{0}$. $\square$

Next we discuss the problem in the half-space $\mathbb{R}^3_+ = \{(x_1, x_2, x_3) : \ x_3 > 0\}$:

$$-\mathrm{curl}^{\,2} \mathbf{A} = (1 - |\mathbf{A}|^2)\mathbf{A} \quad \text{in } \mathbb{R}^3_+, \qquad (\mathrm{curl}\,\mathbf{A})_T = \mathbf{h} \quad \text{on } \partial \mathbb{R}^3_+, \tag{6.5}$$

where $\mathbf{h} = (h_1, h_2, 0)$ is a constant vector which is tangential to $\partial \mathbb{R}^3_+$. We look for a solution of (6.5) satisfying

$$\|\mathbf{A}\|_{L^\infty(\mathbb{R}^3_+)} < \frac{1}{\sqrt{3}}. \tag{6.6}$$

If $\mathbf{A}$ is such a solution, then $\mathbf{H} = \mathrm{curl}\,\mathbf{A}$ solves the following equation

$$-\mathrm{curl}\,[F(|\mathrm{curl}\,\mathbf{H}|^2)\mathrm{curl}\,\mathbf{H}] = \mathbf{H} \quad \text{in } \mathbb{R}^3_+, \qquad \mathbf{H}_T = \mathbf{h} \quad \text{on } \partial \mathbb{R}^3_+, \tag{6.7}$$

and satisfies

$$\|\mathrm{curl}\,\mathbf{H}\|_{L^\infty(\mathbb{R}^3_+)} < \sqrt{\frac{4}{27}}. \tag{6.8}$$

Recall the function $M(\rho)$ used in [PK]:

$$M(\rho) = [1 - (1 - 2\rho^2)^{1/2}](1 - 2\rho^2). \tag{6.9}$$

Proposition 6.2. *Assume* $\mathbf{h} = (h_1, h_2, 0)$ *satisfies (1.13).*

(i) *Equation (6.5) has a smooth solution* $\mathbf{A}$ *which satisfies*

$$\begin{aligned}
&\|\mathbf{A}\|_{L^\infty(\mathbb{R}^3_+)} = \|\mathbf{A}\|_{L^\infty(\partial\mathbb{R}^3_+)} < \frac{1}{\sqrt{3}},\\
&\|\operatorname{curl}\mathbf{A}\|_{L^\infty(\mathbb{R}^3_+)} = \|\operatorname{curl}\mathbf{A}\|_{L^\infty(\partial\mathbb{R}^3_+)} = |\mathbf{h}|,\\
&\|\operatorname{curl}^2\mathbf{A}\|_{L^\infty(\mathbb{R}^3_+)} = \|\operatorname{curl}^2\mathbf{A}\|_{L^\infty(\partial\mathbb{R}_+)} = M(|\mathbf{h}|) < \sqrt{\frac{4}{27}},
\end{aligned} \tag{6.10}$$

(ii) *Equation (6.7) has a unique weak solution satisfying (6.8), and it is a smooth solution.*

Proof. Step 1. Since $\operatorname{curl}\mathbf{h} = \mathbf{0}$, as in the proof of Lemma 2.5 we can show that, if $\mathbf{A}$ is a C^2 solution of (6.5) satisfying (6.6) in $\mathbb{R}^3_+$, then $\mathbf{A}$ must satisfy an additional boundary condition $A_3 = \nu \cdot \mathbf{A} = 0$ on $\partial\mathbb{R}^3_+$. So we look for a solution such that $A_3 \equiv 0$. Write $\mathbf{h} = \rho(\cos\theta, \sin\theta, 0)$, where the constants $\rho \geq 0$ and $0 \leq \theta < 2\pi$. Let $\mathbf{A} = f(x_3)(-\sin\theta, \cos\theta, 0)$. By computation,

$$\mathbf{H} = \operatorname{curl}\mathbf{A} = -f'(x_3)(\cos\theta, \sin\theta, 0), \quad \operatorname{curl}\mathbf{H} = -f''(x_3)(-\sin\theta, \cos\theta, 0).$$

$\mathbf{A}$ is a solution of (6.5) if and only if $f(t)$ satisfies

$$f'' = (1 - f^2)f \quad \text{for } t > 0, \qquad f'(0) = -\rho. \tag{6.12}$$

It has been shown in [PK, Lemma 2.4] that, if $0 \leq \rho < 1/\sqrt{2}$, then (6.12) has a unique solution $f(t)$ satisfying $|f(t)| < 1$. Moreover,

$$\begin{aligned}
&\|f\|_{L^\infty(\mathbb{R}_+)} = f(0) = [1 - (1 - 2\rho^2)^{1/2}]^{1/2},\\
&\|f'\|_{L^\infty(\mathbb{R}_+)} = |f'(0)| = \rho.
\end{aligned}$$

Also,

$$\|f''\|^2_{L^\infty(\mathbb{R}_+)} = |f''(0)|^2 = M(\rho)$$

if $\rho = |\mathbf{h}|$ satisfies condition (1.13). With this condition we have

$$\|\mathbf{A}\|_{L^\infty(\mathbb{R}^3_+)} = \|f\|_{L^\infty(\mathbb{R}^3_+)} < \frac{1}{\sqrt{3}}, \qquad \|\operatorname{curl}\mathbf{H}\|_{L^\infty(\mathbb{R}^3_+)} = \|f''\|_{L^\infty(\mathbb{R}_+)} < \sqrt{\frac{4}{27}}.$$

Thus $\mathbf{A}$ satisfies condition (6.10). Let $\mathbf{H} = \operatorname{curl}\mathbf{A}$. Then $\mathbf{H}$ is a smooth solution of (6.7) and satisfies (6.8).

Step 2. We show that, if $\mathbf{H}_1$, $\mathbf{H}_2$ are weak solutions of (6.7) satisfying (6.8), then for any $R > 1$,

$$\int_{B_R^+} |\mathbf{H}_2 - \mathbf{H}_1|^2 dx \le CR^2, \tag{6.13}$$

where $B_R^+ = B_R \cap \mathbb{R}^3_+$, and C is a constant independent of R and the solutions.

To prove this, let $\mathbf{A}_j$ be the solution of (6.5) corresponding to $\mathbf{H}_j$, namely, $\mathbf{H}_j = \operatorname{curl} \mathbf{A}_j$. Then for any smooth vector field $\mathbf{B}$ with compact support and for each j, we have an integral equality similar to (3.1). Subtracting one from the other we get

$$\int_{\mathbb{R}^3_+} \{\operatorname{curl}(\mathbf{A}_2 - \mathbf{A}_1) \cdot \operatorname{curl} \mathbf{B} + [(1 - |\mathbf{A}_2|^2)\mathbf{A}_2 - (1 - |\mathbf{A}_1|^2)\mathbf{A}_1] \cdot \mathbf{B}\} dx = 0.$$

Write $\mathbf{A}_t(x) = \mathbf{A}_1(x) + t(\mathbf{A}_2(x) - \mathbf{A}_1(x))$. We have

$$\begin{aligned} &[(1 - |\mathbf{A}_2(x)|^2)\mathbf{A}_2(x) - (1 - |\mathbf{A}_1(x)|^2)\mathbf{A}_1(x)] \cdot \mathbf{B}(x) \\ =& \int_0^1 \frac{d}{dt}(1 - |\mathbf{A}_t(x)|^2)\mathbf{A}_t(x) \cdot \mathbf{B} dt \\ =& \int_0^1 [(1 - |\mathbf{A}_t|^2)(\mathbf{A}_2 - \mathbf{A}_1) \cdot \mathbf{B} - 2(\mathbf{A}_t \cdot (\mathbf{A}_2 - \mathbf{A}_1))(\mathbf{A}_t \cdot \mathbf{B})] dt. \end{aligned}$$

Let η be a smooth cut-off function with bounded support, and let $\mathbf{B} = \eta^2(\mathbf{A}_2 - \mathbf{A}_1)$. We have

$$\begin{aligned} &\int_{\mathbb{R}^3_+} \int_0^1 \{|\operatorname{curl}(\eta(\mathbf{A}_2 - \mathbf{A}_1))|^2 + (1 - |\mathbf{A}_t|^2)|\eta(\mathbf{A}_2 - \mathbf{A}_1)|^2 \\ &\qquad - 2[\mathbf{A}_t \cdot (\eta(\mathbf{A}_2 - \mathbf{A}_1))]^2\} dt dx \\ =& \int_{\mathbb{R}^3_+} |\nabla \eta \times (\mathbf{A}_2 - \mathbf{A}_1)|^2 dx. \end{aligned}$$

Note that for $0 \le t \le 1$, $|\mathbf{A}_t(x)| < \frac{1}{\sqrt{3}}$. Choose η such that $\eta(x) = 1$ for $|x| < R$, $\eta(x) = 0$ for $|x| > R + 1$, and $|\nabla \eta| \le 2$. Then the above identity gives

$$\int_{B_R^+} |\operatorname{curl}(\mathbf{A}_2 - \mathbf{A}_1)|^2 dx \le \int_{B_{R+1}^+ \setminus B_R^+} |\nabla \eta \times (\mathbf{A}_2 - \mathbf{A}_1)|^2 dx \le 8\pi[(R+1)^3 - R^3] \le CR^2.$$

Thus (6.13) is true.

Step 3. We show that (6.7) has a unique weak solution satisfying condition (6.8).

Let $\mathbf{H}_1$ and $\mathbf{H}_2$ be two weak solutions of (6.7) satisfying (6.8). There exists a positive number $M < \sqrt{\frac{4}{27}}$ such that

$$\|\mathrm{curl}\,\mathbf{H}_j\|_{L^\infty(\mathbb{R}^3_+)} \le M. \tag{6.14}$$

For any C^1 vector field $\mathbf{B}$ with bounded support, and for each j we have an integral equality similar to (3.4). We subtract one from the other, and choose $\mathbf{B} = \eta^2(\mathbf{H}_2 - \mathbf{H}_1)$, where η is a smooth function with bounded support. This choice of $\mathbf{B}$ is allowable by density. Noting that $\mathbf{B}_T = \mathbf{0}$ on $\partial\mathbb{R}^3_+$, we get

$$\begin{aligned}\int_{\mathbb{R}^3_+} &\{[F(|\mathrm{curl}\,\mathbf{H}_2|^2)\mathrm{curl}\,\mathbf{H}_2 - F(|\mathrm{curl}\,\mathbf{H}_1|^2)\mathrm{curl}\,\mathbf{H}_1]\cdot \mathrm{curl}\,[\eta^2(\mathbf{H}_2-\mathbf{H}_1)] \\ &+ \eta^2|\mathbf{H}_2-\mathbf{H}_1|^2\}dx = 0.\end{aligned} \tag{6.15}$$

We compute as in the proof of Theorem 4.1 (see (4.7)) and find

$$\begin{aligned}\int_{\mathbb{R}^3_+} &\{a(x)|\mathrm{curl}\,\big(\eta(\mathbf{H}_2-\mathbf{H}_1)\big)|^2 + 2\big\langle Q(x)\mathrm{curl}\,\big(\eta(\mathbf{H}_2-\mathbf{H}_1)\big), \mathrm{curl}\,\big(\eta(\mathbf{H}_2-\mathbf{H}_1)\big)\big\rangle \\ &+ \eta^2|\mathbf{H}_2-\mathbf{H}_1|^2\}dx \\ =\int_{\mathbb{R}^3_+} &\{a(x)|\nabla\eta\times(\mathbf{H}_2-\mathbf{H}_1)|^2 + 2\big\langle Q(x)\nabla\eta\times(\mathbf{H}_2-\mathbf{H}_1), \nabla\eta\times(\mathbf{H}_2-\mathbf{H}_1)\big\rangle\}dx,\end{aligned} \tag{6.16}$$

where

$$\begin{aligned}&\mathbf{H}_t(x) = \mathbf{H}_1(x) + t\big(\mathbf{H}_2(x)-\mathbf{H}_1(x)\big), \qquad u_t(x) = |\mathrm{curl}\,\mathbf{H}_t(x)|^2, \\ &a(x) = \int_0^1 F(u_t(x))dt, \qquad Q(x) = (q_{ij}(x)), \\ &q_{ij}(x) = \int_0^1 F'(u_t(x))(\mathrm{curl}\,\mathbf{H}_t(x))_i(\mathrm{curl}\,\mathbf{H}_t(x))_j dt.\end{aligned}$$

From (6.14), $u_t(x) \le M^2 < \frac{4}{27}$. So $F(u_t(x))$ and $F'(u_t(x))$, hence $a(x)$ and $q_{ij}(x)$, are uniformly bounded on $\mathbb{R}^3_+$, $a(x)\ge 1$, and $Q(x)$ is non-negative definite for all x. From (6.16) we have

$$\int_{\mathbb{R}^3_+}\eta^2|\mathbf{H}_2-\mathbf{H}_1|^2dx \le C\int_{\mathbb{R}^3_+}|\nabla\eta|^2|\mathbf{H}_2-\mathbf{H}_1|^2dx.$$

Taking $\eta = e^{-\delta r}\xi(r)$, where $\xi(r)$ is a cut-off function defined in the proof of Proposition 6.1, and $\delta^2 = 1/(4C)$, we have

$$\begin{aligned}&\int_{B^+_R} e^{-2\delta r}|\mathbf{H}_2-\mathbf{H}_1|^2dx \\ \le&2C\delta^2\int_{B^+_{R+1}} e^{-2\delta r}|\mathbf{H}_2-\mathbf{H}_1|^2dx + 8Ce^{-2\delta R}\int_{R^+_{R+1}\setminus B^+_R}|\mathbf{H}_2-\mathbf{H}_1|^2dx.\end{aligned}$$

Letting $R \to \infty$ and using (6.13) we find $\mathbf{H}_2 - \mathbf{H}_1 \equiv \mathbf{0}$. Thus the only weak solution is actually a smooth solution which has been obtained in step 1. $\square$

§7. Existence of Solutions in a Bounded Domain

In this section we introduce a parameter $\mu > 0$ into the boundary condition and consider the following boundary value problem

$$-\lambda^2 \mathrm{curl}\,[F(\lambda^2|\mathrm{curl}\,\mathbf{H}|^2)\mathrm{curl}\,\mathbf{H}] = \mathbf{H} \quad \text{in } \Omega, \qquad \mathbf{H}_T = \mu\mathcal{H}_T^e \quad \text{on } \partial\Omega, \tag{7.1}$$

where $\mathcal{H}_T^e$ is given and satisfies (5.1). We shall prove that there exists a constant μ^* such that (7.1) has a solution for $0 < \mu < \mu^*$, and give a lower bound estimate of μ^* when λ is small. This yields the existence part of Theorem 1. In the following we assume that $\mathcal{H}_T^e$ has been extended to $\mathcal{H}^e \in C^{2+\alpha}(\bar{\Omega}, \mathbb{R}^3)$ in the way as stated in Lemma 2.3 (ii), and satisfies (5.3). From Theorem 5.1, a weak solution of (7.1) satisfying (1.4) must be a classical solution.

Lemma 7.1. *Let Ω be a bounded and simply-connected domain with C^4 boundary, $\lambda > 0$, and let $\mathcal{H}_T^e$ satisfy (5.1). Then there is a constant $\mu^* = \mu^*(\mathcal{H}_T^e, \lambda) > 0$ such that the following conclusions hold:*

(i) *For all $0 \le \mu < \mu^*$, equation (7.1) has a unique solution $\mathbf{H}_\mu$ satisfying (1.4), and $|\mathrm{curl}\,\mathbf{H}_\mu(x)|$ attains its maximum only on $\partial\Omega$.*

(ii) *$\mu \mapsto \|\mathrm{curl}\,\mathbf{H}_\mu\|_{C^0(\bar{\Omega})}$ is continuous for $\mu \in (0, \mu^*)$.*

(iii) *We have*

$$\lim_{\mu \to \mu^*} \lambda \|\mathrm{curl}\,\mathbf{H}_\mu\|_{C^0(\bar{\Omega})} = \sqrt{\frac{4}{27}}. \tag{7.2}$$

Proof. While (i) and (ii) follow from Monneau [Mon], the proof of (iii) relies on the a priori estimate established in our Theorem 5.1.

Step 1. Define

$$\mu^*(\mathcal{H}_T^e, \lambda) = \sup\{b > 0 : \text{ (7.1) has a solution satisfying (1.4) for each } \mu \in (0, b)\}. \tag{7.3}$$

From [Mon, Theorem 1.2], $0 < \mu^*(\mathcal{H}_T^e, \lambda) < \infty$. From Lemma 3.5, for any $0 < \mu < \mu^*(\mathcal{H}_T^e, \lambda)$, the solution $\mathbf{H}_\mu$ of (7.1) is unique. From [Mon, Theorem 1.3], $|\mathrm{curl}\,\mathbf{H}(x)|$ attains its maximum value only on $\partial\Omega$. Hence (i) is true for $\mu^* = \mu^*(\mathcal{H}_T^e, \lambda)$.

Step 2. Now we prove (ii). For simplicity we assume $\lambda = 1$. We use the implicit function theorem as in [Mon], with a slight modification. Let $\mu_0 \in (0, \mu^*)$ and let $\mathbf{H}_{\mu_0}$ be the solution of (7.1) associated with μ_0. Write $\mathbf{H}_{\mu_0} = \mu\mathcal{H}^e + \mathbf{u}_0$. Then we can choose $\delta_0 > 0$ such that $\|\mathrm{curl}\,\mathbf{H}_{\mu_0}\|_{C^0(\bar{\Omega})} \le M \le \sqrt{\frac{4}{27}} - 2\delta_0$. For any $0 < \delta < \delta_0$, we define a smooth function $F_\delta(t)$ which is equal to $F(t)$ for $t \le \frac{4}{27} - \delta$. Then $\mathbf{H}_{\mu_0}$ is a solution of

$$-\mathrm{curl}\,[F_\delta(|\mathrm{curl}\,\mathbf{H}|^2)\mathrm{curl}\,\mathbf{H}] = \mathbf{H} \quad \text{in } \Omega, \qquad \mathbf{H}_T = \mu\mathcal{H}^e_T \quad \text{on } \partial\Omega. \tag{7.4}$$

Write $\mathbf{H} = \mu\mathcal{H}^e + \mathbf{u}$ and $\mathbf{H}_{\mu_0} = \mu_0\mathcal{H}^e + \mathbf{u}_{\mu_0}$, where $\mathcal{H}^e$ is the extension of $\mathcal{H}^e_T$. Then (7.4) can be written as an equation in $\mathbf{u}$

$$\begin{cases} -\mathrm{curl}\,[F_\delta(|\mathrm{curl}\,(\mu\mathcal{H}^e + \mathbf{u})|^2)\mathrm{curl}\,(\mu\mathcal{H}^e + \mathbf{u})] = \mu\mathcal{H}^e + \mathbf{u} & \text{in } \Omega, \\ \mathbf{u}_T = \mathbf{0} & \text{on } \partial\Omega. \end{cases} \tag{7.5}$$

Define

$$\begin{aligned} &C^{2+\alpha}_{t0}(\bar\Omega, \mathrm{div}\,0) = \{\mathbf{u} \in C^{2+\alpha}(\bar\Omega, \mathbb{R}^3) \ : \ \mathrm{div}\,\mathbf{u} = 0 \text{ in } \Omega, \ \mathbf{u}_T = \mathbf{0} \text{ on } \partial\Omega\}, \\ &C^{\alpha}(\bar\Omega, \mathrm{div}\,0) = \{\mathbf{Y} \in C^{\alpha}(\bar\Omega, \mathbb{R}^3) \ : \ \mathrm{div}\,\mathbf{Y} = 0 \text{ in } \Omega\}, \\ &\mathcal{F}_\delta : \ \mathbb{R} \times C^{2+\alpha}_{t0}(\bar\Omega, \mathrm{div}\,0) \to C^{\alpha}(\bar\Omega, \mathrm{div}\,0), \ \text{ and} \\ &\mathcal{F}_\delta(\mu, \mathbf{u}) = \mathrm{curl}\,[F_\delta(|\mathrm{curl}\,(\mu\mathcal{H}^e + \mathbf{u})|^2)\mathrm{curl}\,(\mu\mathcal{H}^e + \mathbf{u})] + \mu\mathcal{H}^e + \mathbf{u}. \end{aligned}$$

From [Mon], $\mathcal{F}_\delta$ is continuously differentiable in μ and $\mathbf{u}$ in the sense of Frechet, and $\frac{\partial}{\partial u}\mathcal{F}_\delta(\mu_0, \mathbf{u}_{\mu_0})$ is a homeomorphism from $C^{2+\alpha}_{t0}(\bar\Omega, \mathrm{div}\,0)$ to $C^{\alpha}(\bar\Omega, \mathrm{div}\,0)$. Applying the implicit function theorem to the operator equation

$$\mathcal{F}_\delta(\mu, \mathbf{u}) = \mathbf{0}, \tag{7.6}$$

we conclude that there exists $a > 0$ such that for all $\mu \in (\mu_0 - a, \mu_0 + a)$, (7.6) has a solution $\mathbf{u}_\mu$ in a neighborhood of $\mathbf{u}_0$ in $C^{2+\alpha}_{t0}(\bar\Omega, \mathrm{div}\,0)$. Choosing a small, we have $\|\mathbf{u}_\mu - \mathbf{u}_{\mu_0}\|_{C^{2+\alpha}(\bar\Omega)} \le \delta$. Then $\mu\mathcal{H}^e + \mathbf{u}_\mu$ is a solution of (7.1). From conclusion (i) it coincides with $\mathbf{H}_\mu$, and $\|\mathrm{curl}\,\mathbf{H}_\mu - \mathrm{curl}\,\mathcal{H}_{\mu_0}\|_{C^0(\bar\Omega)} \le \delta$. Hence (ii) is true.

Step 3. Now we prove (iii). If (7.2) were false, there exists $\varepsilon > 0$ and a sequence $\mu_j \to \mu^*$ such that $\|\mathrm{curl}\,\mathbf{H}_{\mu_j}\|_{C^0(\bar\Omega)} \le M \le \sqrt{\frac{4}{27}} - \varepsilon$ for each j. From Theorem 4.1 we have a uniform H^2 estimate for $\{\mathbf{H}_\mu\}$, and hence from Theorem 5.1 we have a uniform $C^{2+\alpha}$ estimate $\sup_j \|\mathbf{H}_{\mu_j}\|_{C^{2+\alpha}(\bar\Omega)} \le C$, where $C = C(\Omega, \lambda, M, \alpha, \|\mathcal{H}^e_T\|_{C^{2+\alpha}(\partial\Omega)})$. Therefore we may pass to a subsequence and assume that, for some $0 < \delta < \alpha$, $\mathbf{H}_{\mu_j} \to \mathbf{H}_*$ in $C^{2+\delta}(\bar\Omega, \mathbb{R}^3)$ as $j \to \infty$, where $\mathbf{H}_*$ is the solution of (7.1) for $\mu = \mu^*$, and $\|\mathrm{curl}\,\mathbf{H}_*\|_{C^0(\bar\Omega)} \le M$. Now we apply the implicit function theorem in a neighborhood of $\mathbf{H}_*$ as in [Mon, Proposition 3.2], with an obvious modification, and find $\rho > 0$ such that, for $\mu \in (\mu^* - \rho, \mu^* + \rho)$, equation (7.1) has a solution in $C^{2+\alpha}(\bar\Omega, \mathbb{R}^3)$ that satisfies (1.4). This contradicts the definition of μ^*. Hence (iii) is true. $\square$

Remark 7.2. We conjecture that, when $\mu = \mu^*(\mathcal{H}_T^e, \lambda)$, equation (7.1) has a unique solution $\mathbf{H}$ such that $\lambda\|\mathrm{curl}\,\mathbf{H}\|_{L^\infty(\Omega)} = \sqrt{\frac{4}{27}}$, and the maximum value of $|\mathrm{curl}\,\mathbf{H}(x)|$ is obtained only on $\partial\Omega$. Thus, $\lambda|\mathrm{curl}\,\mathbf{H}(x)| < \sqrt{\frac{4}{27}}$ for all $x \in \Omega$. If this is true, then we can use the comparison in subdomains of Ω to prove the uniqueness of solutions for $\mu = \mu^*$. For a discussion on this point in the 2-dimensional case see [BBC].

Lemma 7.3. *Let Ω be a bounded and simply-connected domain with C^4 boundary, and let $\mathcal{H}_T^e \in C^{2+\alpha}(\partial\Omega, \mathbb{R}^3)$ satisfy the conditions of Theorem 5.1. Then*

$$\liminf_{\lambda\to 0}\ \mu^*(\mathcal{H}_T^e, \lambda) \geq \sqrt{\frac{5}{18}}\,(\|\mathcal{H}_T^e\|_{C^0(\partial\Omega)})^{-1}. \tag{7.7}$$

In particular, if $\mathcal{H}^e = \mathbf{h}$ is a unit vector, then

$$\liminf_{\lambda\to 0}\ \mu^*(\mathbf{h}_T, \lambda) \geq \sqrt{\frac{5}{18}}. \tag{7.8}$$

Proof. We use a blow-up argument that was used in [PK] for the two dimensional case, with a slight modification. Let us fix an arbitrary positive number $M_0 < \sqrt{\frac{4}{27}}$. Let $\{\lambda_n\}$ be any sequence of positive numbers such that $\lambda_n \to 0$ as $n \to \infty$. From Lemma 7.1 (i), (ii), for each n there exists μ_n, $0 < \mu_n < \mu^*(\lambda_n, \mathcal{H}_T^e)$, such that (1.3) has a unique solution $\mathbf{H}_{\mu_n}$ which satisfies

$$\lambda_n\|\mathrm{curl}\,\mathbf{H}_{\mu_n}\|_{C^0(\bar\Omega)} = M_0. \tag{7.9}$$

There exists $x_n \in \partial\Omega$ such that $|\mathrm{curl}\,\mathbf{H}_{\mu_n}(x_n)| = \|\mathrm{curl}\,\mathbf{H}_{\mu_n}\|_{C^0(\bar\Omega)}$. Passing to a subsequence we may assume that $x_n \to x_0$ and $\mu_n \to \tilde\mu$ as $n \to \infty$. Let us define rescaled fields

$$\tilde{\mathbf{H}}_{\mu_n}(y) = \mathbf{H}_{\mu_n}(x_n + \lambda_n y), \quad y \in \Omega_n = \frac{\Omega - x_n}{\lambda_n}.$$

Using (7.9) we can show that $\sup_n \|\mathbf{H}_{\mu_n}\|_{L^\infty(\Omega)} < \infty$. Then we can use the arguments in the proofs of Theorems 4.1 and 5.1 to show that, for any fixed $R > 0$, $\|\tilde{\mathbf{H}}_{\mu_n}\|_{C^{2+\alpha}(B_{2R}(0)\cap\Omega_n)} \leq C(M_0, R)$. The proof is omitted here, as it is similar to the proof of Lemma 8.1.

Therefore we can pass to a subsequence, still denoted by $\tilde{\mathbf{H}}_{\mu_n}$, such that $\tilde{\mathbf{H}}_{\mu_n} \to \tilde{\mathbf{H}}$ in $C^{2+\alpha}_{\mathrm{loc}}(\mathbb{R}^3_+, \mathbb{R}^3)$ as $n \to \infty$, and $\tilde{\mathbf{H}}$ is a solution of (6.7) on the half-space, with boundary condition $\tilde{\mathbf{H}}_T = \tilde\mu\mathbf{h}_0$, where $\mathbf{h}_0 = \mathcal{H}_T^e(x_0)$ is tangential to $\partial\mathbb{R}^3_+$, and $\|\mathrm{curl}\,\tilde H\|_{L^\infty(\mathbb{R}^3_+)} = |\mathrm{curl}\,\tilde{\mathbf{H}}(0)| = M_0$. Since $M_0 > 0$, we see that $\mathbf{h}_0 \neq \mathbf{0}$. Write $\tilde\mu\mathbf{h}_0 = \rho(\cos\theta, \sin\theta, 0)$, where $\rho > 0$. Then we use Proposition 6.2 and its proof to find $M_0 = \|\mathrm{curl}\,\tilde{\mathbf{H}}\|_{L^\infty(\mathbb{R}^3_+)} = M(\rho)$, where $M(\rho)$ was defined in (6.9). $M(\rho)$ is strictly increasing on $[0, \sqrt{\frac{5}{18}}]$, and hence has an inverse function $\rho(M)$ defined for $0 \le M \le \sqrt{\frac{4}{27}}$, and $\rho(M)$ is strictly increasing. Thus there exists a unique $\tilde{\rho} < \sqrt{\frac{5}{18}}$ such that $\tilde{\rho} = \rho(M_0)$. Since the solution of the limiting equation (6.7) is unique, the full sequence must converge. So $\lim_{n\to\infty} \mu_n |\mathcal{H}_T^e(x_n)| = \tilde{\mu}|\mathbf{h}_0| = \rho = \rho(M_0)$. Therefore

$$\liminf_{\lambda\to 0} \mu^*(\mathcal{H}_T^e, \lambda)\|\mathcal{H}_T^e\|_{C^0(\partial\Omega)} \ge \lim_{n\to\infty} \mu_n |\mathcal{H}_T^e(x_n)| \ge \rho(M_0).$$

Now we let M_0 approach $\sqrt{\frac{4}{27}}$. Noting that $\rho(\frac{4}{27}) = \sqrt{\frac{5}{18}}$, (7.7) is proved. $\square$

From Theorems 4.1, 5.1, 5.2 and Lemma 7.3 we get the following

Theorem 7.4. *Let Ω be a bounded and simply-connected domain with C^4 boundary and $\mathcal{H}_T^e$ satisfy (1.10). Then for all $\lambda > 0$ small we have:*

(i) Equation (1.3) has a unique solution $\mathbf{H} \in C^3(\Omega, \mathbb{R}^3) \cap C^{2+\alpha}(\bar{\Omega}, \mathbb{R}^3)$ satisfying (1.4), and

$$\|\mathbf{H}\|_{C^{2+\alpha}(\bar{\Omega})} \le C_1(\Omega, \lambda, \|\mathcal{H}_T^e\|_{C^{2+\alpha}(\partial\Omega)}). \tag{7.10}$$

(ii) Equation (1.1) has a unique solution $\mathbf{A} \in C^3(\Omega, \mathbb{R}^3) \cap C^{2+\alpha}(\bar{\Omega}, \mathbb{R}^3)$ satisfying (1.2), and

$$\|\mathbf{A}\|_{C^{2+\alpha}(\bar{\Omega})} \le C_2(\Omega, \lambda, \|\mathcal{H}_T^e\|_{C^{2+\alpha}(\partial\Omega)}). \tag{7.11}$$

If $\partial\Omega$ is of class C^5, then $\mathbf{A} \in C^4(\Omega, \mathbb{R}^3) \cap C^{3+\alpha}(\bar{\Omega}, \mathbb{R}^3)$ and

$$\|\mathbf{A}\|_{C^{3+\alpha}(\bar{\Omega})} \le C_3(\Omega, \lambda, \|\mathcal{H}_T^e\|_{C^{2+\alpha}(\partial\Omega)}).$$

§8. Asymptotic Behavior of Solutions with Small λ

We use the notation $\Omega(\delta) = \{x \in \Omega : \ \mathrm{dist}(x, \partial\Omega) \ge \delta\}$.

Lemma 8.1. *Let Ω and $\mathbf{H}_T^e$ satisfy the conditions of Theorem 4.1. For each λ small, let $\mathbf{H}^\lambda$ be a weak solution of (1.3) satisfying (1.14), where $M < \sqrt{\frac{4}{27}}$ is independent of λ. Then we have:*

(i) For any $\lambda_0 > 0$, there exists $C(\lambda_0) > 0$ such that

$$\|\mathbf{H}^\lambda\|_{L^\infty(\Omega)} \le C(\lambda_0), \quad \textit{for all } 0 < \lambda \le \lambda_0. \tag{8.1}$$

(ii) For any sequence $\{\rho_\lambda\}$ satisfying $\rho_\lambda \le \frac{c}{\lambda}$, $\rho_\lambda \to \infty$ as $\lambda \to 0$,

$$\lim_{\lambda\to 0} \sup_{x\in\Omega(\lambda\rho_\lambda)} |\mathbf{H}^\lambda(x)| = 0. \tag{8.2}$$

Proof. Step 1. We first assume (8.1) is true and prove (8.2).

Let $\mathcal{H}^e \in H^2(\bar{\Omega}, \mathbb{R}^3)$ be the extension of $\mathcal{H}^e_T$ as stated in Lemma 2.3 (i). From (1.14) and Corollary 4.2, $\mathbf{H}^\lambda \in C^\alpha(\bar{\Omega}, \mathbb{R}^3)$ for any $\alpha \in (0, 1/2)$. Let $x^\lambda \in \Omega(\lambda\rho_\lambda)$ and define

$$\mathbf{H}_\lambda(x) = \mathbf{H}^\lambda(x^\lambda + \lambda x), \quad \mathcal{H}^e_\lambda(x) = \mathcal{H}^e(x^\lambda + \lambda x), \quad x \in \Omega_\lambda = \frac{\Omega - x^\lambda}{\lambda}. \tag{8.3}$$

Then $\mathbf{H}_\lambda$ is a weak solution of the equation

$$-\mathrm{curl}\left[F(|\mathrm{curl}\,\mathbf{H}_\lambda|^2)\mathrm{curl}\,\mathbf{H}_\lambda\right] = \mathbf{H}_\lambda \quad \text{in } \Omega_\lambda, \qquad \mathbf{H}_{\lambda T} = \mathcal{H}^e_{\lambda T} \quad \text{on } \partial\Omega_\lambda, \tag{8.4}$$

and satisfies

$$\|\mathrm{curl}\,\mathbf{H}_\lambda\|_{L^\infty(\Omega_\lambda)} \le M < \sqrt{\frac{4}{27}}. \tag{8.5}$$

Since $\rho_\lambda \to \infty$ as $\lambda \to 0$, and since $\mathrm{dist}(x^\lambda, \partial\Omega) \ge \lambda\rho_\lambda$, for any $R > 0$, there exists $\lambda(R) > 0$ such that $B_{10R} \subset \Omega_\lambda$ for all $0 < \lambda < \lambda(R)$. Let η_R be a cut-off function supported in B_{2R} such that $\eta_R = 1$ on B_R and $|\nabla\eta_R(x)| \le C/R$. Multiplying (8.4) by $\eta_{4R}^2\mathbf{H}_\lambda$ and integrating by parts, we get

$$\begin{aligned}&\int_{B_{8R}} \{F(|\mathrm{curl}\,\mathbf{H}_\lambda|^2)|\mathrm{curl}\,(\eta_{4R}\mathbf{H}_\lambda)|^2 + \eta_{4R}^2|\mathbf{H}_\lambda|^2\}dx\\ =&\int_{B_{8R}} F(|\mathrm{curl}\,\mathbf{H}_\lambda|^2)|\nabla\eta_{4R} \times \mathbf{H}_\lambda|^2 dx.\end{aligned}$$

From this and using (8.1) and (2.1),

$$\|\mathbf{H}_\lambda\|^2_{H^1(B_{2R})} \le \|\eta_{2R}\mathbf{H}_\lambda\|^2_{H^1(B_{4R})} \le C_1(R)\int_{B_{4R}} \{|\mathrm{curl}\,\mathbf{H}_\lambda|^2 + |\mathbf{H}_\lambda|^2\}dx \le C_2(R).$$

Then we can apply the interior H^2 estimate (see step 1 of the proof of Theorem 4.1, with $\lambda = 1$) to $\mathbf{H}_\lambda$ on B_{2R}, and find (see (4.8))

$$\|\mathbf{H}_\lambda\|^2_{H^2(B_R)} \le C_3(R, M)\|\mathbf{H}_\lambda\|^2_{H^1(B_{2R})} \le C_4(R, M). \tag{8.6}$$

From (8.6), $\{\mathbf{H}_\lambda\}_{0<\lambda\le 1}$ is bounded in $H^2_{\mathrm{loc}}(\mathbb{R}^3, \mathbb{R}^3)$. For any sequence $\lambda_j \to 0$, we can find a subsequence, still denoted by λ_j, such that $\mathbf{H}_{\lambda_j} \to \mathbf{H}_0$ weakly in $H^2_{\mathrm{loc}}(\mathbb{R}^3, \mathbb{R}^3)$ and strongly in $H^1_{\mathrm{loc}}(\mathbb{R}^3, \mathbb{R}^3)$. Thus, $\mathrm{curl}\,\mathbf{H}_{\lambda_j} \to \mathrm{curl}\,\mathbf{H}_0$ weakly in $H^1_{\mathrm{loc}}(\mathbb{R}^3, \mathbb{R}^3)$ and strongly in $L^2_{\mathrm{loc}}(\mathbb{R}^3)$. So $|\mathrm{curl}\,\mathbf{H}_0(x)| \le M$ for a.e. $x \in \mathbb{R}^3$. Since $F(t)$ is continuous and bounded for $t \in [0, M]$, from (8.5) and the Lebesgue dominated convergence theorem, $F(|\mathrm{curl}\,\mathbf{H}_{\lambda_j}|^2)\mathrm{curl}\,\mathbf{H}_{\lambda_j} \to F(|\mathrm{curl}\,\mathbf{H}_0|^2)\mathrm{curl}\,\mathbf{H}_0$ strongly in $L^p_{\rm loc}(\mathbb{R}^3,\mathbb{R}^3)$ for any $1<p<\infty$. Therefore $\mathbf{H}_0$ is a weak solution of (6.2). From Proposition 6.1 (ii) we see that $\mathbf{H}_0\equiv\mathbf{0}$. Thus $\mathbf{H}_{\lambda_j}\to\mathbf{0}$ weakly in $H^2_{\rm loc}(\mathbb{R}^3,\mathbb{R}^3)$ and strongly in $H^1_{\rm loc}(\mathbb{R}^3,\mathbb{R}^3)$ as $j\to\infty$. Since this is true for any sequences, we must have, for any $R>0$ fixed, $\|\mathbf{H}_\lambda\|_{H^1(B_R)}\to 0$ as $\lambda\to 0$. Now we use the interior H^2 estimate (8.6) and the Sobolev imbedding theorem to find that, as $\lambda\to 0$,

$$\|\mathbf{H}_\lambda\|_{C^\alpha(B_R)}\le C_5(R)\|\mathbf{H}_\lambda\|_{H^2(B_R)}\le C_6(M,R)\|\mathbf{H}_\lambda\|_{H^1(B_{2R})}\to 0.$$

In particular, $|\mathbf{H}^\lambda(x^\lambda)|=|\mathbf{H}_\lambda(0)|\le\|\mathbf{H}_\lambda\|_{L^\infty(B_R)}\to 0$. Thus (8.2) is true.

Step 2. Now we prove (8.1). Suppose (8.1) were not true. We can find $\lambda_j\to 0$, $x^{\lambda_j}\in\bar\Omega$, such that $m_j=|\mathbf{H}^{\lambda_j}(x^{\lambda_j})|=\|\mathbf{H}^{\lambda_j}\|_{L^\infty(\Omega)}\to\infty$. For simplicity we denote λ_j by λ. For this choice of x^λ we define $\mathbf{H}_\lambda$ as in (8.3) and set $\hat{\mathbf{H}}_\lambda=\varepsilon_\lambda\mathbf{H}_\lambda$, where $\varepsilon_\lambda=1/m_j$. Then $\hat{\mathbf{H}}_\lambda$ satisfies

$$-\mathrm{curl}\left[F(|\mathrm{curl}\,\mathbf{H}|^2)\mathrm{curl}\,\hat{\mathbf{H}}_\lambda\right]=\hat{\mathbf{H}}_\lambda\quad\text{in }\Omega_\lambda,\qquad \hat{\mathbf{H}}_{\lambda T}=\varepsilon_\lambda\mathcal{H}^e_{\lambda T}\quad\text{on }\partial\Omega_\lambda,\tag{8.7}$$

and $|\hat{\mathbf{H}}_\lambda(0)|=\|\hat{\mathbf{H}}_\lambda\|_{L^\infty(\Omega_\lambda)}=1$. As in step 1 above we can show that, for all λ small,

$$\|\hat{\mathbf{H}}_\lambda\|_{H^1(B_R\cap\Omega_\lambda)}\le C_7(R,M,\|\mathcal{H}^e_{\lambda T}\|_{H^{1/2}(\partial\Omega_\lambda\cap B_{2R})}).\tag{8.8}$$

In the following we shall show that $\{\hat{\mathbf{H}}_\lambda\}$ is bounded in $H^2_{\rm loc}$:

$$\|\hat{\mathbf{H}}_\lambda\|_{H^2(B_R\cap\Omega_\lambda)}\le C_8(R,M,\|\mathcal{H}^e_{\lambda T}\|_{H^{3/2}(\partial\Omega_\lambda\cap B_{2R})}).\tag{8.9}$$

The interior $H^2_{\rm loc}$ estimate can be established as in step 1 of the proof of Theorem 4.1. Write

$$\begin{aligned}
&\mathbf{h}_{\lambda,\sigma}(x)=\frac{1}{\sigma}[\mathbf{H}_\lambda(x+\sigma\mathbf{e})-\mathbf{H}_\lambda(x)], &&\mathbf{H}_{\lambda,t,\sigma}(x)=\mathbf{H}_\lambda(x)+t\sigma\mathbf{h}_{\lambda,\sigma}(x),\\
&u_{\lambda,t,\sigma}=|\mathrm{curl}\,\mathbf{H}_{\lambda,t,\sigma}|^2, &&u_\lambda=|\mathrm{curl}\,\mathbf{H}_\lambda|^2,\\
&\hat{\mathbf{h}}_{\lambda,\sigma}(x)=\frac{1}{\sigma}[\hat{\mathbf{H}}_\lambda(x+\sigma\mathbf{e})-\hat{\mathbf{H}}_\lambda(x)], &&\hat{\mathbf{H}}_{\lambda,t,\sigma}(x)=\hat{\mathbf{H}}_\lambda(x)+t\sigma\hat{\mathbf{h}}_{\lambda,\sigma}(x).
\end{aligned}$$

Observe that

$$\begin{aligned}
&F(|\mathrm{curl}\,\mathbf{H}_\lambda(x+\sigma\mathbf{e})|^2)\mathrm{curl}\,\hat{\mathbf{H}}_\lambda(x+\sigma\mathbf{e})-F(|\mathrm{curl}\,\mathbf{H}_\lambda(x)|^2)\mathrm{curl}\,\hat{\mathbf{H}}_\lambda(x)\\
&=\sigma\int_0^1\left\{F(u_{\lambda,t,\sigma})\mathrm{curl}\,\hat{\mathbf{h}}_\sigma+2F'(u_{\lambda,t,\sigma})(\mathrm{curl}\,\mathbf{H}_{\lambda,t,\sigma}\cdot\mathrm{curl}\,\mathbf{h}_{\lambda,\sigma})\mathrm{curl}\,\hat{\mathbf{H}}_{\lambda,t,\sigma}\right\}dt\\
&=\sigma\int_0^1\left\{F(u_{\lambda,t,\sigma})\mathrm{curl}\,\hat{\mathbf{h}}_\sigma+2F'(u_{\lambda,t,\sigma})(\mathrm{curl}\,\mathbf{H}_{\lambda,t,\sigma}\cdot\mathrm{curl}\,\hat{\mathbf{h}}_{\lambda,\sigma})\mathrm{curl}\,\mathbf{H}_{\lambda,t,\sigma}\right\}dt\\
&=\sigma a_\sigma(x)\mathrm{curl}\,\hat{\mathbf{h}}_\sigma+2\sigma Q_\sigma(x)\mathrm{curl}\,\hat{\mathbf{h}}_\sigma,
\end{aligned}$$

where

$$a_\sigma(x) = \int_0^1 F(u_{\lambda,t,\sigma})dt, \qquad Q_\sigma(x) = (q_{\sigma,ij}(x)),$$

$$q_{\sigma,ij}(x) = \int_0^1 F'(u_{\lambda,t,\sigma})(\mathrm{curl}\,\mathbf{H}_{\lambda,t,\sigma})_i(\mathrm{curl}\,\mathbf{H}_{\lambda,t,\sigma})_j dt.$$

Hence for all $\mathbf{B} \in H^1(\Omega_\lambda, \mathbb{R}^3)$ with compact support in Ω_λ,

$$\int_{\Omega_\lambda} \big\{a_\sigma(x)\mathrm{curl}\,\hat{\mathbf{h}}_\sigma \cdot \mathrm{curl}\,\mathbf{B} + 2\big\langle Q_\sigma(x)\mathrm{curl}\,\hat{\mathbf{h}}_\sigma, \mathrm{curl}\,\mathbf{B}\big\rangle + \hat{\mathbf{h}}_\sigma \cdot \mathbf{B}\big\}dx = 0.$$

Then we can proceed as in step 1 of the proof of Theorem 4.1 to get the interior H^2_{loc} estimate.

To get the boundary H^2_{loc} estimate, we assume for simplicity that the $\Gamma = \partial\Omega_\lambda \cap \bar{\Omega}'$ is flat. Using the above observation, we proceed as in the step 2 of the proof of Theorem 4.1 to get the boundary H^2_{loc} estimate for the tangential derivatives. To estimate the normal derivatives, we compute as in the third step of the proof of Theorem 4.1 and write (8.7) in the form

$$F(u_\lambda)\Delta\hat{\mathbf{H}}_\lambda - F'(u_\lambda)\big(\nabla|\mathrm{curl}\,\hat{\mathbf{H}}_\lambda \cdot \mathrm{curl}\,\mathbf{H}_\lambda|^2\big) \times \mathrm{curl}\,\mathbf{H}_\lambda = \hat{\mathbf{H}}_\lambda,$$

from which we can solve $\partial_{33}\hat{H}_\lambda$ in terms of $\mathrm{curl}\,\mathbf{H}_\lambda$ and $\partial_{ij}\hat{H}_\lambda$ with $(i,j) \neq (3,3)$, see (4.15) and (4.18). Using (8.5) we get

$$\|\partial_{33}\hat{H}_\lambda\|_{L^2(B_R\cap\Omega_\lambda)} \le C(\Omega, M, R)\{\|\hat{\mathbf{H}}_\lambda\|_{L^2(B_{2R}\cap\Omega_\lambda)} + \sum_{(i,j)\neq(3,3)} \|\partial_{ij}\hat{H}_\lambda\|_{L^2(B_{2R}\cap\Omega_\lambda)}\}.$$

From this and the H^2_{loc} estimate for the tangential derivatives we derive (8.9).

From (8.9), we can find a subsequence, still denoted by $\hat{\mathbf{H}}_\lambda$, such that $\hat{\mathbf{H}}_\lambda \to \hat{\mathbf{H}}$ weakly in H^2_{loc} and strongly in H^1_{loc} as $\lambda \to 0$, and $\hat{\mathbf{H}}$ is a weak solution of the limiting equations of (8.7). Moreover, $\hat{\mathbf{H}}_\lambda \to \hat{\mathbf{H}}$ in C^α_{loc} for some $\alpha \in (0, 1/2)$. In particular $|\hat{\mathbf{H}}(0)| = 1$. We only need to consider the following two cases.

Case 1. $\lim_{\lambda\to 0} \lambda^{-1}\mathrm{dist}(x^\lambda, \partial\Omega) = \infty$. Then the limiting equation of (8.7) is (6.2). From Proposition 6.1 (ii) we have $\hat{\mathbf{H}} \equiv \mathbf{0}$, which is a contradiction.

Case 2. $\mathrm{dist}(x^\lambda, \partial\Omega) \le C$. Then after a translation in the x_3 direction, the limiting equation of (8.7) is (6.7) with $\mathbf{h} = \mathbf{0}$. From Proposition 6.2 (ii) we have $\hat{\mathbf{H}} = \mathbf{0}$, a contradiction again. $\square$

Let $\partial\Omega(\mathcal{H}^e_T)$ and $\partial\Omega(\mathbf{h}_T)$ be the sets defined in the introduction.

Lemma 8.2. *Let Ω and $\mathcal{H}_T^e$ satisfy the conditions of Theorem 5.1. For each λ small, let $\mathbf{H}^\lambda$ be a weak solution of (1.3) satisfying (1.14), where $M < \sqrt{\frac{4}{27}}$ is independent of λ.*

(i) *Let x^λ be any point on $\partial\Omega$ and assume $x^\lambda \to x^0$ as $\lambda \to 0$. Then in local coordinates near x^0, the rescaled vector field $\tilde{\mathbf{H}}_\lambda(y)$ converges in $C^{2+\alpha}_{\mathrm{loc}}(\mathbb{R}^3_+, \mathbb{R}^3)$ to the unique solution of the equation*

$$-\mathrm{curl}\left[F(|\mathrm{curl}\,\mathbf{H}|^2)\mathrm{curl}\,\mathbf{H}\right] = \mathbf{H} \quad \text{in } \mathbb{R}^3_+, \qquad \mathbf{H}_T = \tilde{\mathbf{h}} \quad \text{on } \partial\mathbb{R}^3_+, \tag{8.10}$$

where $\tilde{\mathbf{h}} = \mathcal{H}_T^e(x^0)$.

(ii) *Let $M(\rho)$ be the function defined in (7.14). We have*

$$\lim_{\lambda\to 0} \max_{x\in\bar\Omega} \lambda|\mathrm{curl}\,\mathbf{H}^\lambda(x)| = M(\|\mathcal{H}_T^e\|_{C^0(\partial\Omega)}). \tag{8.11}$$

(iii) *Let P^λ be a maximum point of $|\mathrm{curl}\,\mathbf{H}^\lambda(x)|$ and assume $P^\lambda \to P$ for a sequence $\lambda_n \to 0$. Then $P \in \partial\Omega(\mathcal{H}_T^e)$.*

Proof. Step 1. Proof of (i). We consider the rescaled vector field $\mathbf{H}_\lambda(x)$ defined in (8.3). Then $\mathbf{H}_\lambda$ satisfies (8.1) and (8.5). Hence we can apply the arguments in the proofs of Theorems 4.1 and 5.1 (see step 2 in the proof of Lemma 8.1), and conclude that, for any $R > 0$, there exists a constant C depending on Ω, $\|\mathcal{H}_T^e\|_{C^{2+\alpha}(\partial\Omega)}$, M, R and α, but independent of λ, such that

$$\|\mathbf{H}_\lambda\|_{C^{2+\alpha}(B_R^+\cap\Omega_\lambda)} \le C. \tag{8.12}$$

Now for each λ, we adopt the local coordinates y introduced by straightening the boundary in a neighborhood around x^λ such that $y = \mathbf{0}$ corresponds to x^λ, and the inner normal vector of $\partial\Omega$ at x^λ points in the positive y_3 direction (see Appendix A.1). Let $\tilde{\mathbf{H}}^\lambda(y) = \mathbf{H}^\lambda(x)$. Then we define rescaled vector fields $\tilde{\mathbf{H}}_\lambda(y) = \tilde{\mathbf{H}}^\lambda(\lambda y)$, which are well-defined for $y \in B^+_{r_\lambda}$, where $r_\lambda \to \infty$ as $\lambda \to 0$. From (8.5),

$$|\mathrm{curl}\,\tilde{\mathbf{H}}_\lambda(y)| \le M \quad \text{for all } y \in B^+_{r_\lambda}. \tag{8.13}$$

From (8.12) we have

$$\|\tilde{\mathbf{H}}_\lambda\|_{C^{2+\alpha}(B_R^+)} \le C,$$

where C depends on Ω, $\|\mathcal{H}_T^e\|_{C^{2+\alpha}(\partial\Omega)}$, M, R and α, but is independent of λ. Thus for any sequence $\lambda_j \to 0$, we can choose a subsequence, still denoted by λ_j, such that, for any $0 < \delta < \alpha$, $\tilde{\mathbf{H}}_\lambda \to \tilde{\mathbf{H}}_0$ in $C^{2+\delta}_{\mathrm{loc}}(\mathbb{R}^3_+, \mathbb{R}^3)$.

To find the limit $\tilde{\mathbf{H}}_0$, we write equation (1.3) in the new coordinates to get a quasilinear elliptic system for $\tilde{\mathbf{H}}_\lambda(y)$. Because $\tilde{\mathbf{H}}_\lambda \in C^{2+\alpha}_{\mathrm{loc}}(\mathbb{R}^3_+, \mathbb{R}^3)$, and because of condition (8.13), we can write this system in the form of an elliptic linear system with $C^{1+\alpha}$ coefficients. Then, as in the proof of Lemma 8.1, we can take the limit $\lambda_j \to 0$ and find that $\tilde{\mathbf{H}}_0$ is a classical solution of (8.10). From the choice of y coordinates, $\tilde{\mathbf{h}}$ is orthogonal to the y_3 direction. Moreover, for a.e. $y \in \mathbb{R}^3_+$,

$$|\mathrm{curl}\, \tilde{\mathbf{H}}_0(y)| \le M, \quad \text{for all } y \in \mathbb{R}^3_+. \tag{8.14}$$

From the proof of Proposition 6.2 we see that, if $\tilde{\mathbf{h}} = \mathbf{0}$ then $\tilde{\mathbf{H}}_0 \equiv \mathbf{0}$. If $\tilde{\mathbf{h}} \ne \mathbf{0}$, we write $\tilde{\mathbf{h}} = |\tilde{\mathbf{h}}|(\cos\theta, \sin\theta, 0)$ to get

$$\tilde{\mathbf{H}}_0(y) = -f'(y_3)(\cos\theta, \sin\theta, 0),$$

where f is the solution of (6.12) with $f'(0) = -\rho = -\mu|\tilde{\mathbf{h}}|$. Since the solution to the limiting equation is unique, we conclude that the full sequence $\tilde{\mathbf{H}}_\lambda$ must converge to $\tilde{\mathbf{H}}_0$ in $C^{2+\delta}_{\mathrm{loc}}(\mathbb{R}^3_+, \mathbb{R}^3)$ as $\lambda \to 0$.

Now we show that the convergence is in $C^{2+\alpha}_{\mathrm{loc}}(\mathbb{R}^3_+, \mathbb{R}^3)$. From (8.13) and (8.14) we see that

$$\begin{aligned}
&0 < F'(|\mathrm{curl}\,(t\tilde{\mathbf{H}}_0(y) + (1-t)\tilde{\mathbf{H}}_\lambda(y))|^2) \le C,\\
&0 < F''(|\mathrm{curl}\,(t\tilde{\mathbf{H}}_0(y) + (1-t)\tilde{\mathbf{H}}_\lambda(y))|^2) \le C \quad \text{for all } y \in \mathbb{R}^3_+,\ 0 \le t \le 1.
\end{aligned}$$

From the equation for $\tilde{\mathbf{H}}_\lambda$ and the equation (8.10) for $\tilde{\mathbf{H}}_0$ we derive an equation for $\tilde{\mathbf{H}}_\lambda - \tilde{\mathbf{H}}_0$, which can be written as a linear equation for $\tilde{\mathbf{H}}_\lambda - \tilde{\mathbf{H}}_0$ with $C^{1+\alpha}$ coefficients. Then we can apply the Schauder estimates to derive an estimate of $\|\tilde{\mathbf{H}}_\lambda - \tilde{\mathbf{H}}_0\|_{C^{2+\alpha}(B^+_R)}$, and finally we find that

$$\tilde{\mathbf{H}}_\lambda \to \tilde{\mathbf{H}}_0 \quad \text{in } C^{2+\alpha}_{\mathrm{loc}}(\mathbb{R}^3_+, \mathbb{R}^3) \text{ as } \lambda \to 0. \tag{8.15}$$

Thus, conclusion (i) is proved.

Step 2. Proof of (ii) and (iii). Let $P^\lambda \in \partial\Omega$ be the maximum point of $|\mathrm{curl}\, \mathbf{H}^\lambda(x)|$. Choose $\lambda_n \to 0$ such that $P^{\lambda_n} \to P$. Then conclusion (i) is valid. In particular (8.15) holds. Let $\tilde{\mathbf{h}}(q) = \mathcal{H}^e_T(q)$, and let $\mathbf{H}_q$ be the solution of (8.10) with boundary data $\tilde{\mathbf{H}}_{qT} = \tilde{\mathbf{h}}(q)$. From the proof of Proposition 6.2, $\|\mathrm{curl}\, \mathbf{H}_q\|^2_{L^\infty(\mathbb{R}^3_+)} = M(|\tilde{\mathbf{h}}(q)|)$. The function $M(\rho)$ is strictly increasing when $0 < \rho < \sqrt{\frac{5}{18}}$. Thus

$$\sup_{q\in\partial\Omega} \|\mathrm{curl}\, \mathbf{H}_q\|^2_{L^\infty(\mathbb{R}^3_+)} = \max_{q\in\partial\Omega} M(|\tilde{\mathbf{h}}(q)|) = \max_{x\in\partial\Omega} M(\|\mathcal{H}^e_T\|_{C^0(\partial\Omega)}),$$

which is achieved on $\partial\Omega(\mathcal{H}^e_T)$. From (8.15) we find

$$\begin{aligned}
&\lim_{n\to\infty} \lambda_n |\mathrm{curl}\, \mathbf{H}^{\lambda_n}(P^{\lambda_n})| = \lim_{n\to\infty} \lambda_n \|\mathrm{curl}\, \mathbf{H}^{\lambda_n}\|_{C^0(\partial\Omega)} = \lim_{n\to\infty} \|\mathrm{curl}\, \mathbf{H}_{\lambda_n}\|_{C^0(\partial\Omega)}\\
&= \lim_{n\to\infty} \|\mathrm{curl}\, \tilde{\mathbf{H}}_{\lambda_n}\|_{C^0(\partial\Omega_{\lambda_n})} = \|\mathrm{curl}\, \mathbf{H}_P\|^2_{L^\infty(\mathbb{R}^3_+)} = M(|\tilde{\mathbf{h}}(P)|).
\end{aligned}$$

So we have

$$\lim_{n\to\infty} \lambda_n |\mathrm{curl}\,\mathbf{H}^{\lambda_n}(P^{\lambda_n})| = \max_{P\in\partial\Omega} M(|\tilde{\mathbf{h}}(P)|) = M(\|\mathcal{H}_T^e\|_{C^0(\partial\Omega)}).$$

This verifies (8.11) and proves conclusion (ii).

Conclusion (iii) follows from (8.11) and (8.15) immediately. □

Proof of Theorem 1.

Under condition (1.10), the existence and uniqueness of the solutions $\mathbf{H}^\lambda$ satisfying (1.4) and their regularity have been proved in Theorem 7.4. So conclusion (i) is true.

From the second inequality in (1.10) and Lemma 7.3, we can choose $c_0 > 1$ and small $\lambda^* > 0$ such that

$$\mu^*(\mathcal{H}_T^e, \lambda) > c_0 > 1 \quad \text{for all } 0 < \lambda \le \lambda^*. \tag{8.16}$$

Now we claim that

$$\sup_{0<\lambda\le\lambda^*} \lambda \|\mathrm{curl}\,\mathbf{H}^\lambda\|_{L^\infty(\Omega)} < \sqrt{\frac{4}{27}}. \tag{8.17}$$

Suppose (8.17) were false. Then there exists a sequence λ_n such that

$$\lambda_n \|\mathrm{curl}\,\mathbf{H}^{\lambda_n}\|_{L^\infty(\Omega)} \to \sqrt{\frac{4}{27}}. \tag{8.18}$$

We may assume $\lambda_n \to \lambda_0$. If $\lambda_0 > 0$, from Lemma 7.1 (iii) we have $\mu^*(\mathcal{H}_T^e, \lambda_0) = 1$, which contradicts (8.16). Hence $\lambda_0 = 0$, and thus $\lambda_n \to 0$ as $n \to \infty$. Fix a positive number M slightly less than $\sqrt{\frac{4}{27}}$. From (8.16), (8.18) and the definition of $\mu^*(\mathcal{H}_T^e, \lambda)$, there exists $N(M) > 0$ such that, if $n > N(M)$ then $M < \lambda_n \|\mathrm{curl}\,\mathbf{H}^{\lambda_n}\|_{L^\infty(\Omega)} \le \sqrt{\frac{4}{27}}$. Then from Lemma 7.1 (iii), we can choose a positive number $\mu_n < 1$ such that for $\lambda = \lambda_n$ and $\mu = \mu_n$, (7.1) has a solution $\hat{\mathbf{H}}_n$, and $\lambda_n \|\mathrm{curl}\,\hat{\mathbf{H}}_n\|_{L^\infty(\Omega)} = M$. Now we argue as in the proof of Lemma 7.3 and show that, after passing to a subsequence, $\liminf_{n\to\infty} \mu_n \|\mathcal{H}_T^e\|_{C^0(\partial\Omega)} \ge \rho(M)$, where $\rho(M)$ is the inverse function of $M(\rho)$. Since $\mu_n < 1$, we find $\|\mathcal{H}_T^e\|_{C^0(\partial\Omega)} \ge \rho(M)$. Now we let M approach $\sqrt{\frac{4}{27}}$ and conclude that $\|\mathcal{H}_T^e\|_{C^0(\partial\Omega)} \ge \rho(\sqrt{\frac{4}{27}}) = \sqrt{\frac{5}{18}}$, contradicting (1.10). So (8.17) is true.

Using (8.17) and Lemma 8.1, we get Conclusion (ii). Using (8.17) and Lemma 8.2 (ii) and (iii), we get conclusions (iii) and (iv). Hence, Theorem 1 is proved. □

Proof of Theorem $1'$.

In the proof of Theorem 5.2 (also see Theorem 7.4) we have shown that, if $\mathbf{H} \in C^3(\Omega, \mathbb{R}^3) \cap C^{2+\alpha}(\bar{\Omega}, \mathbb{R}^3)$ is a solution of (1.3) satisfying (1.4), then (1.1) has a solution $\mathbf{A} = \mathcal{B} - \nabla\varphi \in C^3(\Omega, \mathbb{R}^3) \cap C^{2+\alpha}(\bar{\Omega}, \mathbb{R}^3)$. So Theorem $1'$ follows from Theorem 1. □

Appendix A. Some Technical Details

§A.1. Computations in local coordinates near boundary.

Let us briefly recall the local coordinates near boundary $\partial\Omega$ determined by a diffeomorphism that straightens a piece of surface, see [P, section 3]. Let us fix a point $x_0 \in \partial\Omega$, and introduce new variables y_1, y_2 such that $\partial\Omega$ can be represented (at least near x_0) by $\mathbf{r} = \mathbf{r}(y_1, y_2)$, and $\mathbf{r}(0,0) = x_0$. Here and henceforth we let $y = (y_1, y_2)$ and use the notation $\mathbf{r}_j(y) = \partial_{y_j}\mathbf{r}(y)$, $\mathbf{r}_{ij} = \partial_{y_i y_j}\mathbf{r}(y)$, etc. Let

$$\mathbf{n}(y) = \frac{\mathbf{r}_1(y) \times \mathbf{r}_2(y)}{|\mathbf{r}_1(y) \times \mathbf{r}_2(y)|}.$$

We choose (y_1, y_2) in such a way that $\mathbf{n}(y)$ is the inward normal of $\partial\Omega$, and that the y_1- and y_2-curves on $\partial\Omega$ are the lines of principal curvature; thus, $\mathbf{r}_1(y)$ and $\mathbf{r}_2(y)$ are orthogonal to each other. Let $g_{ij}(y) = \mathbf{r}_i(y) \cdot \mathbf{r}_j(y)$, $g(y) = \det(g_{ij}(y)) = g_{11}(y)g_{22}(y)$. For scalar functions f, let $f_{,j}$ denote the partial derivative with respect to y_j. Let us define a map $\mathcal{F}$ by $x = \mathcal{F}(y,z) = \mathbf{r}(y_1, y_2) + z\mathbf{n}(y_1, y_2)$. $\mathcal{F}$ is a diffeomorphism from a ball $B_R(0)$ onto a neighborhood $\mathcal{U}$ of the point x_0, and it maps the half ball $B_R^+(0)$ onto a subdomain $\mathcal{U} \cap \Omega$, and maps the disc $\{(y_1, y_2, 0) : y_1^2 + y_2^2 < R^2\}$ onto a subset of $\partial\Omega$. Let $G_{ij}(y,z) = \partial_i\mathcal{F} \cdot \partial_j\mathcal{F}$ and let $G^{ij}(y,z)$ denote the elements of the inverse of the matrix $(G_{ij}(y,z))$. Then

$$\begin{aligned}
&G_{jj}(y,z) = g_{jj}(y)[1 - \kappa_j(y)z]^2, \quad G^{jj} = \frac{1}{G_{jj}}, \quad j = 1,2,\\
&G_{12} = G_{13} = G_{23} = G^{12} = G^{13} = G^{23} = 0, \quad G_{33} = G^{33} = 1,\\
&G(y,z) \equiv \det(G_{ij}(y,z)) = G_{11}(y,z)G_{22}(y,z).
\end{aligned}$$

On $\mathcal{U}$ we have an orthogonal coordinate framework $\{\mathbf{E}_1, \mathbf{E}_2, \mathbf{E}_3\}$, where

$$\mathbf{E}_j(y,z) = \frac{\partial_j\mathcal{F}}{|\partial_j\mathcal{F}|} = \frac{\mathbf{r}_j(y)}{\sqrt{g_{jj}}}, \quad j = 1,2, \quad \mathbf{E}_3(y,z) = \frac{\partial_3\mathcal{F}}{|\partial_3\mathcal{F}|} = \mathbf{n}(y).$$

Given a vector field $\mathbf{B}$ defined on $\bar{\Omega}$, for any $x_0 \in \partial\Omega$, we can write $\mathbf{B}$ in a neighborhood of x_0 in the new variables $(y,z) \in B_R^+ = \mathcal{F}^{-1}(\mathcal{U} \cap \Omega)$ as follows:

$$\begin{aligned}
&\tilde{\mathbf{B}}(y,z) = \mathbf{B}(\mathcal{F}(y,z)) = \sum_{j=1}^{3} G^{jj}(y,z)b_j(y,z)\partial_j\mathcal{F}(y,z) = \sum_{j=1}^{3} \tilde{B}_j(y,z)\mathbf{E}_j(y,z),\\
&b_j(y,z) = \mathbf{B}(\mathcal{F}(y,z)) \cdot \partial_j\mathcal{F}(y,z), \qquad \tilde{B}_j(y,z) = \frac{b_j(y,z)}{\sqrt{G_{jj}(y,z)}}.
\end{aligned} \tag{A.1}$$

We compute, at the point $x=\mathcal{F}(y,z)$,

$$\operatorname{curl}\mathbf{B}(x)=\sum_{j=1}^{3}\tilde{R}_j(y,z)\mathbf{E}_j(y,z), \tag{A.2}$$

where

$$\begin{aligned}
\tilde{R}_1(y,z)&=\frac{1}{\sqrt{G_{22}G_{33}}}[\partial_2(\tilde{B}_3\sqrt{G_{33}})-\partial_3(\tilde{B}_2\sqrt{G_{22}})]=\frac{1}{\sqrt{G_{22}}}[\partial_2 b_3-\partial_3 b_2],\\
\tilde{R}_2(y,z)&=\frac{1}{\sqrt{G_{33}G_{11}}}[\partial_3(\tilde{B}_1\sqrt{G_{11}})-\partial_1(\tilde{B}_3\sqrt{G_{33}})]=\frac{1}{\sqrt{G_{11}}}[\partial_3 b_1-\partial_1 b_3],\\
\tilde{R}_3(y,z)&=\frac{1}{\sqrt{G_{11}G_{22}}}[\partial_1(\tilde{B}_2\sqrt{G_{22}})-\partial_2(\tilde{B}_1\sqrt{G_{11}})]=\frac{1}{\sqrt{G_{11}G_{22}}}[\partial_1 b_2-\partial_2 b_1].
\end{aligned}$$

In the following, to save the notation, we shall write $\tilde{\mathbf{B}}$ also by $\mathbf{B}$.

§A.2. Proof of (2.2).

Suppose (2.2) were false. Then there exists a sequence $\{\mathbf{B}_j\}$ with $\|\mathbf{B}_j\|_{L^2(\Omega)}=1$, such that $\|\operatorname{div}\mathbf{B}_j\|_{L^2(\Omega)}\to 0$, $\|\operatorname{curl}\mathbf{B}_j\|_{L^2(\Omega)}\to 0$, and either $\|\nu\cdot\mathbf{B}_j\|_{H^{1/2}(\partial\Omega)}\to 0$ or $\|\nu\times\mathbf{B}_j\|_{H^{1/2}(\partial\Omega)}\to 0$. From (2.1), $\{\mathbf{B}_j\}$ is bounded in $H^1(\Omega,\mathbb{R}^3)$. After passing to a subsequence we may assume that $\mathbf{B}_j\to\mathbf{B}$ weakly in $H^1(\Omega,\mathbb{R}^3)$ and strongly in $L^2(\Omega,\mathbb{R}^3)$. Then $\|\mathbf{B}\|_{L^2(\Omega)}=1$, $\|\operatorname{div}\mathbf{B}\|_{L^2(\Omega)}=0$, $\|\operatorname{curl}\mathbf{B}\|_{L^2(\Omega)}=0$, and either $\|\nu\cdot\mathbf{B}\|_{H^{1/2}(\partial\Omega)}=0$ or $\|\nu\times\mathbf{B}\|_{H^{1/2}(\partial\Omega)}=0$. Since Ω is simply connected and $\operatorname{curl}\mathbf{B}=\mathbf{0}$, there exists $\phi\in H^2(\Omega)$ such that $\mathbf{B}=\nabla\phi$. Then $\Delta\phi=\operatorname{div}\mathbf{B}=0$. Moreover, either $\frac{\partial\phi}{\partial\nu}=0$ or $(\nabla\phi)_T=\mathbf{0}$ on $\partial\Omega$ in the sense of trace in $H^{1/2}(\partial\Omega)$. Then ϕ must be a constant on Ω, and so $\mathbf{B}=\nabla\phi=\mathbf{0}$, which is impossible as we have $\|\mathbf{B}\|_{L^2(\Omega)}=1$. $\square$

§A.3. Proof of Proposition 2.1.

We only prove (2.4) for $k=1$. The general case can be proved by induction.

Step 1. Let us assume $\operatorname{div}\mathbf{B}\in C^{1+\alpha}(\bar{\Omega})$, $\operatorname{curl}\mathbf{B}\in C^{1+\alpha}(\bar{\Omega},\mathbb{R}^3)$, and either $\nu\cdot\mathbf{B}$ or $\nu\times\mathbf{B}\in C^{2+\alpha}(\partial\Omega,\mathbb{R}^3)$. We shall use (2.3) to derive (2.4). Note that, for bounded and simply-connected domains, (2.3) can be written as

$$\|\mathbf{B}\|_{C^{1+\alpha}(\bar{\Omega})}\le C(\Omega,\alpha)\Big\{\|\operatorname{div}\mathbf{B}\|_{C^{\alpha}(\bar{\Omega})}+\|\operatorname{curl}\mathbf{B}\|_{C^{\alpha}(\bar{\Omega})}+\left\|\begin{matrix}\nu\cdot\mathbf{B}\\ \nu\times\mathbf{B}\end{matrix}\right\|_{C^{1+\alpha}(\partial\Omega)}\Big\}. \tag{A.3}$$

This is because

$$\|\mathbf{B}\|_{C^{0}(\bar{\Omega})}\le C(\Omega,\alpha)\Big\{\|\nabla\mathbf{B}\|_{C^{\alpha}(\bar{\Omega})}+\left\|\begin{matrix}\nu\cdot\mathbf{B}\\ \nu\times\mathbf{B}\end{matrix}\right\|_{C^{\alpha}(\partial\Omega)}\Big\}. \tag{A.4}$$

Step 2. We first derive an interior estimate of $D^2\mathbf{B}$. Let $\mathbf{u} = \partial_j \mathbf{B}$, $j = 1, 2, 3$. Let $\Omega' \Subset \Omega$ be a subdomain of Ω, and let η be a smooth cut-off function with compact support in Ω such that $\eta = 1$ on Ω'. Applying (2.3) to $\eta\mathbf{u}$ we have

$$\begin{aligned}
&\|\nabla(\eta\mathbf{u})\|_{C^\alpha(\bar\Omega)} \le C(\Omega,\alpha)\{\|\mathrm{div}\,(\eta\mathbf{u})\|_{C^\alpha(\bar\Omega)} + \|\mathrm{curl}\,(\eta\mathbf{u})\|_{C^\alpha(\bar\Omega)}\} \\
\le& C(\Omega,\alpha)\{\|\eta\,\mathrm{div}\,\mathbf{u}\|_{C^\alpha(\bar\Omega)} + \|\nabla\eta\cdot\mathbf{u}\|_{C^\alpha(\bar\Omega)} + \|\eta\,\mathrm{curl}\,\mathbf{u}\|_{C^\alpha(\bar\Omega)} + \|\nabla\eta\times\mathbf{u}\|_{C^\alpha(\bar\Omega)}\} \\
\le& C(\Omega,\Omega',\alpha)\{\|\mathrm{div}\,\mathbf{u}\|_{C^\alpha(\bar\Omega)} + \|\mathrm{curl}\,\mathbf{u}\|_{C^\alpha(\bar\Omega)} + \|\mathbf{u}\|_{C^\alpha(\bar\Omega)}\} \\
\le& C(\Omega,\Omega',\alpha)\{\|\mathrm{div}\,\mathbf{B}\|_{C^{1+\alpha}(\bar\Omega)} + \|\mathrm{curl}\,\mathbf{B}\|_{C^{1+\alpha}(\bar\Omega)} + \|\nabla\mathbf{B}\|_{C^\alpha(\bar\Omega)}\}.
\end{aligned}$$

Now we use (2.3) to control the term $\|\nabla\mathbf{B}\|_{C^a(\bar\Omega)}$ in the right hand side, and get

$$\|\nabla\mathbf{B}\|_{C^{1+\alpha}(\bar\Omega')} \le C(\Omega,\Omega',\alpha)\Big\{\|\mathrm{div}\,\mathbf{B}\|_{C^{1+\alpha}(\bar\Omega)} + \|\mathrm{curl}\,\mathbf{B}\|_{C^{1+\alpha}(\bar\Omega)} + \left\|\begin{matrix}\nu\cdot\mathbf{B}\\ \nu\times\mathbf{B}\end{matrix}\right\|_{C^{1+\alpha}(\partial\Omega)}\Big\}. \tag{A.5}$$

Step 3. Next we consider a subset $D = \Omega\cap B_R(x_0)$ with $x_0 \in \partial\Omega$. As in §A.1, we choose $R > 0$ small but independent of x_0 such that on $B_R(x_0)$ there exists a new coordinate system (y_1, y_2, z) which corresponds to an orthonormal framework $\{\mathbf{E}_1, \mathbf{E}_2, \mathbf{E}_3\}$ such that, restricted to $\partial\Omega\cap B_R(x_0)$, $\mathbf{E}_1$, $\mathbf{E}_2$ are tangential to $\partial\Omega$, and $\mathbf{E}_3 = -\nu$.

We first consider "tangential" derivative $\mathbf{u} = \partial_{y_i}\mathbf{B}$, $i = 1, 2$. Let η be a smooth cut-off function supported in $B_R(x_0)$ such that $\eta = 1$ on $B_{R/2}(x_0)$. Write $\Omega_R = B_R(x_0)\cap\Omega$ and $\Gamma_R = B_R(x_0)\cap\partial\Omega$. Applying (2.3) to $\eta\mathbf{u}$ we have

$$\begin{aligned}
\|\nabla(\eta\mathbf{u})\|_{C^\alpha(\bar\Omega)} \le& C(\Omega,\alpha)\Big\{\|\mathrm{div}\,(\eta\mathbf{u})\|_{C^\alpha(\bar\Omega)} + \|\mathrm{curl}\,(\eta\mathbf{u})\|_{C^\alpha(\bar\Omega)} + \left\|\begin{matrix}\eta\nu\cdot\mathbf{u}\\ \eta\nu\times\mathbf{u}\end{matrix}\right\|_{C^{1+\alpha}(\partial\Omega)}\Big\} \\
\le& C(\Omega,R,\alpha)\Big\{\|\mathrm{div}\,\mathbf{u}\|_{C^\alpha(\bar\Omega_R)} + \|\mathrm{curl}\,\mathbf{u}\|_{C^\alpha(\bar\Omega_R)} + \|\mathbf{u}\|_{C^\alpha(\bar\Omega_R)} \\
&+ \left\|\begin{matrix}\nu\cdot\mathbf{u}\\ \nu\times\mathbf{u}\end{matrix}\right\|_{C^{1+\alpha}(\Gamma_R)} + \|\mathbf{u}\|_{C^\alpha(\Gamma_R)}\Big\} \\
\le& C(\Omega,R,\alpha)\Big\{\|\mathrm{div}\,\mathbf{B}\|_{C^{1+\alpha}(\bar\Omega_R)} + \|\mathrm{curl}\,\mathbf{B}\|_{C^{1+\alpha}(\bar\Omega_R)} + \|\mathbf{B}\|_{C^{1+\alpha}(\bar\Omega_R)} \\
&+ \left\|\begin{matrix}\nu\cdot\mathbf{u}\\ \nu\times\mathbf{u}\end{matrix}\right\|_{C^{1+\alpha}(\Gamma_R)}\Big\},
\end{aligned} \tag{A.6}$$

where we have used the fact $\|\mathbf{B}\|_{C^{1+\alpha}(\Gamma_R)} \le \|\mathbf{B}\|_{C^{1+\alpha}(\bar\Omega_R)}$. As in step 2 we can control the term $\|\mathbf{B}\|_{C^{1+\alpha}(\bar\Omega_R)}$ by using (A.3). Note that $\nu\cdot\mathbf{u} = \partial_{y_i}(\nu\cdot\mathbf{B}) - (\partial_{y_i}\nu)\cdot\mathbf{B}$ and

$\nu \times \mathbf{u} = \partial_{y_i}(\nu \times \mathbf{B}) - (\partial_{y_i}\nu) \times \mathbf{B}$. So

$$
\begin{aligned}
\|\nu \cdot \mathbf{u}\|_{C^{1+\alpha}(\Gamma_R)} \le& \|\partial_{y_i}(\nu \cdot \mathbf{B})\|_{C^{1+\alpha}(\Gamma_R)} + \|(\partial_{y_i}\nu) \cdot \mathbf{B}\|_{C^{1+\alpha}(\Gamma_R)} \\
\le& \|\nu \cdot \mathbf{B}\|_{C^{2+\alpha}(\Gamma_R)} + C(\Omega, R, \alpha)\|\mathbf{B}\|_{C^{1+\alpha}(\Gamma_R)}, \\
\|\nu \times \mathbf{u}\|_{C^{1+\alpha}(\Gamma_R)} \le& \|\partial_{y_i}(\nu \times \mathbf{B})\|_{C^{1+\alpha}(\Gamma_R)} + \|(\partial_{y_i}\nu) \times \mathbf{B}\|_{C^{1+\alpha}(\Gamma_R)} \\
\le& \|\nu \times \mathbf{B}\|_{C^{2+\alpha}(\Gamma_R)} + C(\Omega, R, \alpha)\|\mathbf{B}\|_{C^{1+\alpha}(\Gamma_R)}.
\end{aligned}
$$

Plugging these back into (A.6) and using $\|\mathbf{B}\|_{C^{1+\alpha}(\Gamma_R)} \le \|\mathbf{B}\|_{C^{1+\alpha}(\bar\Omega_R)}$, we get

$$
\begin{aligned}
\|\partial_{y_i}\mathbf{B}\|_{C^{1+\alpha}(\bar\Omega_{R/2})} \le C(\Omega, R, \alpha)\Big\{&\|\operatorname{div}\mathbf{B}\|_{C^{1+\alpha}(\bar\Omega_R)} + \|\operatorname{curl}\mathbf{B}\|_{C^{1+\alpha}(\bar\Omega_R)} \\
&+ \|\mathbf{B}\|_{C^{1+\alpha}(\bar\Omega_R)} + \left\|\begin{matrix}\nu \cdot \mathbf{B} \\ \nu \times \mathbf{B}\end{matrix}\right\|_{C^{2+\alpha}(\Gamma_R)}\Big\}.
\end{aligned}
\tag{A.7}
$$

Step 4. Now we consider the normal derivative. Let $\mathbf{w} = \partial_z\mathbf{B}$. Then $\mathbf{w} = -\partial_\nu\mathbf{B}$ on Γ_R. For the cut-off function used above, we apply (2.3) to $\eta\mathbf{w}$ to get

$$
\begin{aligned}
\|\nabla(\eta\mathbf{w})\|_{C^\alpha(\bar\Omega)} \le C(\Omega, R, \alpha)\Big\{&\|\operatorname{div}\mathbf{B}\|_{C^{1+\alpha}(\bar\Omega_R)} + \|\operatorname{curl}\mathbf{B}\|_{C^{1+\alpha}(\bar\Omega_R)} + \|\mathbf{B}\|_{C^{1+\alpha}(\bar\Omega_R)} \\
&+ \left\|\begin{matrix}\nu \times \mathbf{w} \\ \nu \cdot \mathbf{w}\end{matrix}\right\|_{C^{1+\alpha}(\Gamma_R)}\Big\}.
\end{aligned}
\tag{A.8}
$$

We claim that there exists $C = C(\Omega, R, \alpha)$ such that, for all $\mathbf{B} \in C^{2+\alpha}(\bar\Omega, \mathbb{R}^3)$,

$$
\begin{aligned}
&\|\nu \times \partial_\nu\mathbf{B}\|_{C^{1+\alpha}(\Gamma_R)} \le C\{\|\nu \times \operatorname{curl}\mathbf{B}\|_{C^{1+\alpha}(\Gamma_R)} + \|\nu \cdot \mathbf{B}\|_{C^{2+\alpha}(\Gamma_R)} + \|\mathbf{B}\|_{C^{1+\alpha}(\Gamma_R)}\}, \\
&\|\nu \cdot \partial_\nu\mathbf{B}\|_{C^{1+\alpha}(\Gamma_R)} \le C\{\|\operatorname{div}\mathbf{B}\|_{C^{1+\alpha}(\Gamma_R)} + \|\nu \times \mathbf{B}\|_{C^{2+\alpha}(\Gamma_R)} + \|\mathbf{B}\|_{C^{1+\alpha}(\Gamma_R)}\}.
\end{aligned}
\tag{A.9}
$$

We will prove (A.9) in step 5. From (A.8) and (A.9) (either all choose the first option or all choose the second option), and using the obvious facts

$$
\begin{aligned}
&\|\operatorname{div}\mathbf{B}\|_{C^{1+\alpha}(\Gamma_R)} \le \|\operatorname{div}\mathbf{B}\|_{C^{1+\alpha}(\bar\Omega_R)}, \\
&\|\nu \times \operatorname{curl}\mathbf{B}\|_{C^{1+\alpha}(\Gamma_R)} \le C(\Omega, R, \alpha)\|\operatorname{curl}\mathbf{B}\|_{C^{1+\alpha}(\bar\Omega_R)},
\end{aligned}
$$

we get

$$
\begin{aligned}
\|\nabla\partial_z\mathbf{B}\|_{C^\alpha(\bar\Omega_R)} \le C(\Omega, R, \alpha)\Big\{&\|\operatorname{div}\mathbf{B}\|_{C^{1+\alpha}(\bar\Omega_R)} + \|\operatorname{curl}\mathbf{B}\|_{C^{1+\alpha}(\bar\Omega_R)} \\
&+ \|\mathbf{B}\|_{C^{1+\alpha}(\bar\Omega_R)} + \left\|\begin{matrix}\nu \cdot \mathbf{B} \\ \nu \times \mathbf{B}\end{matrix}\right\|_{C^{2+\alpha}(\Gamma_R)}\Big\}.
\end{aligned}
\tag{A.10}
$$

From (A.5), (A.7), (A.10) (either all choose the first option or all choose the second option) we conclude that

$$\|\nabla\mathbf{B}\|_{C^{1+\alpha}(\bar\Omega)}\le C(\Omega,\alpha)\Big\{\|\mathrm{div}\,\mathbf{B}\|_{C^{1+\alpha}(\bar\Omega)}+\|\mathrm{curl}\,\mathbf{B}\|_{C^{1+\alpha}(\bar\Omega)}+\|\mathbf{B}\|_{C^{1+\alpha}(\bar\Omega)}$$
$$+\left\|\begin{matrix}\nu\cdot\mathbf{B}\\ \nu\times\mathbf{B}\end{matrix}\right\|_{C^{2+\alpha}(\partial\Omega)}\Big\}.$$

Combing this with (A.3) we finally get (2.4).

Step 5. Proof of (A.9). In Ω_R we write $\mathbf{B}$ as in (A.1). Then on Γ_R we have

$$\nu\times\partial_\nu\mathbf{B}=-\partial_\nu\tilde B_1\mathbf{E}_2+\partial_\nu\tilde B_2\mathbf{E}_1+\sum_{j=1}^3\tilde B_j\nu\times\partial_\nu\mathbf{E}_j.$$

Since

$$\begin{aligned}\nu\times\mathrm{curl}\,\mathbf{B}&=\sum_{j=1}^2\partial_z\tilde B_j\mathbf{E}_j-\sum_{j=1}^2\Big(\frac{\partial_{y_j}\tilde B_3}{\sqrt{g_{jj}}}+\kappa_j(y)\tilde B_j\Big)\mathbf{E}_j\\&=-\sum_{j=1}^2\partial_\nu\tilde B_j\mathbf{E}_j+\sum_{j=1}^2\Big(\frac{\partial_{y_j}(\nu\cdot\mathbf{B})}{\sqrt{g_{jj}}}-\kappa_j(y)\tilde B_j\Big)\mathbf{E}_j,\end{aligned}$$

we have

$$\partial_\nu\tilde B_1\mathbf{E}_1+\partial_\nu\tilde B_2\mathbf{E}_2=-\nu\times\mathrm{curl}\,\mathbf{B}+\sum_{j=1}^2\Big(\frac{\partial_{y_j}(\nu\cdot\mathbf{B})}{\sqrt{g_{jj}}}-\kappa_j(y)\tilde B_j\Big)\mathbf{E}_j,$$

$$\begin{aligned}&\|\nu\times\partial_\nu\mathbf{B}\|_{C^{1+\alpha}(\Gamma_R)}\le C_1\{\|\partial_\nu\tilde B_1\mathbf{E}_2-\partial_\nu\tilde B_2\mathbf{E}_1\|_{C^{1+\alpha}(\Gamma_R)}+\|\mathbf{B}\|_{C^{1+\alpha}(\Gamma_R)}\}\\ \le&C_2\{\|\partial_\nu\tilde B_1\mathbf{E}_1+\partial_\nu\tilde B_2\mathbf{E}_2\|_{C^{1+\alpha}(\Gamma_R)}+\|\mathbf{B}\|_{C^{1+\alpha}(\Gamma_R)}\}\\ \le&C_3\{\|\nu\times\mathrm{curl}\,\mathbf{B}\|_{C^{1+\alpha}(\Gamma_R)}+\|\partial_{y_1}(\nu\cdot\mathbf{B})\|_{C^{1+\alpha}(\Gamma_R)}+\|\partial_{y_2}(\nu\cdot\mathbf{B})\|_{C^{1+\alpha}(\Gamma_R)}\\ &\quad+\|\mathbf{B}\|_{C^{1+\alpha}(\Gamma_R)}\}\\ \le&C_4\{\|\nu\times\mathrm{curl}\,\mathbf{B}\|_{C^{1+\alpha}(\Gamma_R)}+\|\nu\cdot\mathbf{B}\|_{C^{2+\alpha}(\Gamma_R)}+\|\mathbf{B}\|_{C^{1+\alpha}(\Gamma_R)}\},\end{aligned}$$

where the $C_j's$ depend only on Ω, R and α. This verifies the first inequality in (A.9). Since

$$\mathrm{div}\,\mathbf{B}=\frac{1}{\sqrt G}\Big[\sum_{j=1}^2\partial_{y_j}(\sqrt G\tilde B_j)+\partial_z(\sqrt G\tilde B_3)\Big],$$

we have

$$\nu\cdot\partial_\nu\mathbf{B}=\partial_z\tilde B_3=\mathrm{div}\,\mathbf{B}-\partial_{y_1}\tilde B_1-\partial_{y_2}\tilde B_2-\frac{1}{\sqrt G}\Big[\sum_{j=1}^2\tilde B_j\partial_{y_j}\sqrt G+\tilde B_3\partial_z\sqrt G\Big].$$

From this and $\nu \times \mathbf{B} = \tilde{B}_2 \mathbf{E}_1 - \tilde{B}_1 \mathbf{E}_2$, we find

$$\begin{aligned}\|\nu \cdot \partial_\nu \mathbf{B}\|_{C^{1+\alpha}(\Gamma_R)} \leq & \|\operatorname{div} \mathbf{B}\|_{C^{1+\alpha}(\Gamma_R)} + \|\partial_{y_1} \tilde{B}_1 + \partial_{y_2} \tilde{B}_2\|_{C^{1+\alpha}(\Gamma_R)} + C_5 \|\mathbf{B}\|_{C^{1+\alpha}(\Gamma_R)} \\ \leq & C_6 \{\|\operatorname{div} \mathbf{B}\|_{C^{1+\alpha}(\Gamma_R)} + \|\nu \times \mathbf{B}\|_{C^{2+\alpha}(\Gamma_R)} + \|\mathbf{B}\|_{C^{1+\alpha}(\Gamma_R)}\},\end{aligned}$$

where the C_j's depend only on Ω, R and α. The second inequality in (A.9) follows from this inequality. $\square$

§A.4. Proof of Lemma 2.3.

Let us define

$$\begin{aligned}&\mathcal{H}_t(\Omega, \operatorname{curl}, \operatorname{div} 0) \\ &\quad = \{\mathbf{H} \in L^2(\Omega, \mathbb{R}^3) : \operatorname{curl} \mathbf{H} \in L^2(\Omega, \mathbb{R}^3),\ \operatorname{div} \mathbf{H} = 0 \text{ in } \Omega,\ \mathbf{H}_T = \mathcal{H}_T^e \text{ on } \partial\Omega\}, \\ &\mathcal{H}_{t0}(\Omega, \operatorname{curl}, \operatorname{div} 0) = \{\mathbf{H} \in \mathcal{H}_t(\Omega, \operatorname{curl}, \operatorname{div} 0) : \mathbf{H}_T = \mathbf{0} \text{ on } \partial\Omega\}, \\ &H_{t0}^1(\Omega, \mathbb{R}^3) = \{\mathbf{B} \in H^1(\Omega, \mathbb{R}^3) : \mathbf{B}_T = 0 \text{ on } \partial\Omega\}.\end{aligned}$$

Consider the following minimization problem:

$$\min_{\mathbf{H} \in \mathcal{H}_t(\Omega, \operatorname{curl}, \operatorname{div} 0)} \|\operatorname{curl} \mathbf{H}\|_{L^2(\Omega)}^2.$$

From (2.1) we see that a minimizing sequence is bounded in $H^1(\Omega, \mathbb{R}^3)$. Thus it is easy to see that a minimizer exists. Let $\mathbf{H}$ be a minimizer. Then for all $\mathbf{B} \in \mathcal{H}_{t0}(\Omega, \operatorname{curl}, \operatorname{div} 0)$ it holds that

$$\int_\Omega \operatorname{curl} \mathbf{H} \cdot \operatorname{curl} \mathbf{B}\, dx = 0. \tag{A.11}$$

It was observed in [Mon, p.924] that $\operatorname{curl} \mathcal{H}_{t0}(\Omega, \operatorname{curl}, \operatorname{div} 0) = \operatorname{curl} H_{t0}^1(\Omega, \mathbb{R}^3)$. Hence (A.11) holds for all $\mathbf{B} \in H_{t0}^1(\Omega, \mathbb{R}^3)$, that is, $\mathbf{H}$ is a weak solution of the equation

$$\operatorname{curl}^2 \mathbf{H} = \mathbf{0} \quad \text{and} \quad \operatorname{div} \mathbf{H} = 0 \quad \text{in } \Omega, \qquad \mathbf{H}_T = \mathcal{H}_T^e \quad \text{on } \partial\Omega.$$

Hence (2.7) follows from (2.4).

Now assume that (2.11) holds. Then $\mathbf{J} = \operatorname{curl} \mathbf{H}$ satisfies $\operatorname{curl} \mathbf{J} = \mathbf{0}$ and $\operatorname{div} \mathbf{J} = 0$ in Ω, and $\nu \cdot \mathbf{J} = 0$ on $\partial\Omega$. Hence $\mathbf{J} = \mathbf{0}$. Since Ω is simply-connected, there exists a scalar function ϕ such that $\nabla\phi = \mathbf{H}$. The regularity of ϕ follows from the regularity of $\mathbf{H}$. $\square$

§A.5. Proof of Lemma 2.4 (i).

We keep the notation used in §A.1. Note that $\nu = -\mathbf{n}$. So on $\partial\Omega$ we have

$$\nu \cdot \operatorname{curl} \mathbf{B}(x) = -\frac{1}{\sqrt{g_{11}(y) g_{22}(y)}} [\partial_1 b_2(y,0) - \partial_2 b_1(y,0)].$$

For $x \in \partial\Omega$ near x_0, we represent the tangential component $\mathbf{B}_T$ by

$$\mathbf{B}_T(y,0) = \mathbf{B}_T(\mathcal{F}(y,0)) = \sum_{j=1}^{2} \frac{b_j(y,0)}{\sqrt{G_{jj}(y,0)}} \mathbf{E}_j(y,0).$$

From these two equalities we see immediately that $\nu \cdot \operatorname{curl} \mathbf{B}$ is determined by $\mathbf{B}_T$. Thus, conclusion (i) in Lemma 2.4 is true. □

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