4.1 The Dual Group

Throughout this chapter G will denote an abelian locally compact group. Recall from Corollary 3.24 that every irreducible representation $\pi: G \to \mathcal{U}(\mathcal{H})$ has $\dim(\mathcal{H}) = 1$. So identifying $\mathcal{H} \cong \mathbb{C}$ we get that π is a continuous homomorphism from G into $\mathcal{U}(\mathbb{C}) = \mathbb{T}$.

Def A character on G is a continuous homomorphism $\omega: G \to \mathbb{T}$. The set of characters, denoted $\hat{G}$, is called the dual of G and is equipped with: group operations determined by

$$(\omega \cdot \phi)(x) = \omega(x)\phi(x) \qquad \omega, \phi \in \hat{G} \quad x \in G$$
$$\omega^{-1}(x) = \overline{\omega(x)} = \omega(x^{-1});$$

an identity element given by $1(x) = 1$ for all $x \in G$; and the topology of compact convergence on G. □

One can check that the group operations on $\hat{G}$ are continuous with respect to its topology (Exercise: do so), and thus $\hat{G}$ is an abelian topological group. We will see that it is in fact a locally compact group, and toward this end we will first use $\hat{G} \subset L^\infty(G)$ and the results of section 3.3 to better understand the topology on $\hat{G}$.

Viewing $\omega \in \hat{G}$ as a unitary representation of G on $\mathbb{C}$, observe that

$$\omega(x) = \langle \omega(x) 1, 1 \rangle_{\mathbb{C}}$$

for all $x \in G$ and $\omega(1) = 1$. Thus $\omega \in \mathcal{P}_1(G)$ by Proposition 3.12. Moreover, π_ω is unitarily equivalent to ω by Corollary 3.18, and since the latter is irreducible we have $\omega \in \mathrm{ext}(\mathcal{P}_1(G))$ by Theorem 3.19.

Let us adopt the following notation

$$(x \mid \omega) := \omega(x) \qquad x \in G, \ \omega \in \hat{G}.$$

Note that

$$(x|\omega)(y|\omega) = (xy|\omega) \quad \text{and} \quad (x^{-1}|\omega) = (x|\omega^{-1}) = (x|\omega)^{-1} = \overline{(x|\omega)}$$
$$(x|\omega)(x|\phi) = (x|\omega\phi)$$

Viewing $\omega: G \to \mathcal{U}(\mathbb{C})$ as a unitary representation again, it induces the following $*$-representation of $L^1(G,\mu)$:

$$\omega(f) = \int_G f(x)\,(x|\omega)\,d\mu(x) \qquad f \in L^1(G,\mu).$$

Note that $\omega(f^*) = \omega(f)^* = \overline{\omega(f)}$. Since $\omega(f) \in B(\mathbb{C}) \cong \mathbb{C}$, we can view $f \mapsto \omega(f)$ as a multiplicative linear functional in $L^1(G,\mu)^*$. We also have the converse:

Theorem 4.1 Let G be a σ-compact abelian locally compact group. Then every non-zero multiplicative $\varphi \in L^1(G,\mu)^*$ is given by $\varphi(f) = \omega(f)$ a unique $\omega \in \hat{G}$. Consequently, one also has $\overline{\varphi(f)} = \varphi(f^*)$.

Proof Since μ is σ-compact we know $\varphi = \int \cdot\, \phi \, d\mu$ for some $\phi \in L^\infty(G)$ and will show $\phi \in \hat{G}$. Fix $f \in L^1(G,\mu)$ with $\varphi(f) \neq 0$. Then for any $g \in L^1(G,\mu)$ we have:

$$\begin{aligned}
\int g(x)\, \varphi(f)\, \phi(x)\, d\mu(x) &= \varphi(f)\varphi(g) = \varphi(f * g) \\
&= \int_G (f * g)(y)\, \phi(y)\, d\mu(y) \\
&= \int_G \int_G f(yx)\, g(x^{-1})\, d\mu(x)\, \phi(y)\, d\mu(y) \\
&= \int_G \int_G f(x^{-1}y)\, g(x)\, d\mu(x)\, \phi(y)\, d\mu(y) \\
&= \int_G g(x) \int_G (L_x f)(y)\, \phi(y)\, d\mu(y)\, d\mu(x) = \int_G g(x)\, \varphi(L_x f)\, d\mu(x)
\end{aligned}$$

G abelian (unimodular)

Thus $\varphi(f)\phi(x) = \varphi(L_x f)$ μ-almost everywhere. Replacing $\phi(x)$ with $\varphi(L_x f)/\varphi(f)$ we still have $\varphi = \int \cdot\, \phi\, d\mu$, but now ϕ is continuous by Proposition 2.27 and $\phi(1) = \varphi(L_1 f)/\varphi(f) = 1$. We also have

$$\phi(xy)\,\varphi(f) = \varphi(L_{xy} f) = \varphi(L_x(L_y f)) = \phi(x)\,\varphi(L_y f) = \phi(x)\phi(y)\varphi(f),$$

so that $\phi(xy) = \phi(x)\phi(y)$. Thus $\phi : G \to B(0, \|\phi\|_\infty)\setminus\{0\}$ is a continuous group homomorphism, and $\phi(x^n) = \phi(x)^n$ for all $n \in \mathbb{Z}$ implies $|\phi(x)| = 1$. Hence $\phi \in \hat{G}$.

To see that ϕ is unique, note that

$$\varphi(f^*) = \phi(f^*) = \overline{\phi(f)} = \overline{\varphi(f)}$$

So φ is a (necessarily nondegenerate) $*$-representation of $\mathbb{C}$, and therefore Theorem 3.9 implies ϕ is unique. □

Corollary 4.2 For a σ-compact abelian group G, $\hat{G}$ is a locally compact group.

Proof We saw in the discussion preceding Theorem 4.1 that $\hat{G} \subset \mathrm{ext}(P_1(G))$, and therefore the topology on $\hat{G}$ coincides with the weak* topology from $L^\infty(G) \cong L^1(G,\mu)^*$ by Theorem 3.25. Now, Theorem 4.1 implies

$$\hat{G} \cup \{0\} = \{\phi \in L^\infty(G) : \int_G \cdot\, \phi\, d\mu \text{ is multiplicative on } L^1(G,\mu)\}.$$

The right side is weak*-closed since $\varphi_i \to \varphi$ weak* with φ_i multiplicative implies

$$\varphi(f * g) = \lim_{i\to\infty} \varphi_i(f*g) = \lim_{i\to\infty} \varphi_i(f)\varphi_i(g) = \varphi(f)\varphi(g)$$

Thus $\hat{G} \cup \{0\}$ is weak* closed and bounded in $L^\infty(G)$, so Banach–Alaoglu implies it is weak* compact. Consequently $\hat{G} \subset \hat{G} \cup \{0\}$ is locally compact in this topology since it is an open subset. $\square$

$\boxed{\text{Remark}}$ The proof of Corollary 4.2 shows that $\hat{G} \cup \{0\} \subset L^\infty(G)$ is the one-point compactification of $\hat{G}$. $\square$

Before we look at some examples, let us consider two special cases.

4/1

$\boxed{\text{Lemma 4.3}}$ If G is compact with $\mu(G)=1$, then $\hat{G}$ is an orthonormal set in $L^2(G,\mu)$.

Proof For $\omega \in \hat{G}$ we have

$$\|\omega\|_2 = \left(\int_G |\omega(x)|^2 \, d\mu(x)\right)^{1/2} = \left(\int_G 1 \, d\mu(x)\right)^{1/2} = 1$$

For $\phi \in \hat{G}$ with $\phi \neq \omega$, there must be some $x_0 \in G$ satisfying $(x_0 \mid \omega\phi^{-1}) \neq 1$. Consequently,

$$\begin{aligned} \langle \omega, \phi \rangle_2 &= \int_G \omega(x) \overline{\phi(x)} \, d\mu(x) = \int_G (x \mid \omega\phi^{-1}) \, d\mu(x) \\ &= (x_0 \mid \omega\phi^{-1}) \int_G (x_0^{-1} x \mid \omega\phi^{-1}) \, d\mu(x) \\ &= (x_0 \mid \omega\phi^{-1}) \int_G (x \mid \omega\phi^{-1}) \, d\mu(x) = (x_0 \mid \omega\phi^{-1}) \langle \omega, \phi \rangle_2 . \end{aligned}$$

So we must have $\langle \omega, \phi \rangle_2 = 0$. $\square$

$\boxed{\text{Proposition 4.4}}$ If G is a countable discrete group then $\hat{G}$ is compact. If G is a compact group then $\hat{G}$ is discrete.

Proof First suppose G is countable and discrete. Then $\delta_1 \in L^1(G,\mu)$ is a unit and we have

$$\omega(\delta_1) = \int_G \delta_1(x) (x \mid \omega) \, d\mu(x) = (1 \mid \omega) = 1 \qquad \forall \omega \in \hat{G}.$$

This implies $\hat{G}$ is weak* closed inside $\hat{G} \cup \{0\}$ since $(\omega_i)_{i \in I} \subset \hat{G}$ converging to $\phi \in \hat{G} \cup \{0\}$ weak* gives

$$\phi(\delta_1) = \lim_{i \to \infty} \omega_i(\delta_1) = 1 \neq 0$$

Thus $\hat{G}$ is weak* compact as a weak* closed subset of the weak* compact set $\hat{G} \cup \{0\}$ (see the proof of Corollary 4.2 or the remark following it).

Next suppose G is compact. Then the constant function $1 \in \hat{G}$ lies in $L^1(G,\mu)$ and consequently

$$\mathcal{U} := \left\{ f \in L^\infty(G) : \left| \int_G f \, d\mu \right| > \tfrac{1}{2} \right\}$$

$= \int_G f \cdot 1 \, d\mu$

is a weak* open neighborhood of 1. Lemma 4.3 implies $\left| \int_G \omega \, d\mu \right| = |\langle \omega, 1 \rangle_2| = 0$ for $\omega \in \hat{G} \setminus \{1\}$, and thus $\{1\} = \mathcal{U} \cap \hat{G}$ is open in $\hat{G}$. But then $\{\omega\} = \omega \cdot \{1\}$ is open for all $\omega \in \hat{G}$ and hence $\hat{G}$ is discrete. $\square$

In the remainder of the section we compute examples of dual groups.

$\boxed{\text{Theorem 4.5}}$

(a) $\hat{\mathbb{R}} \cong \mathbb{R}$ with the pairing $(s|t) = e^{2\pi i s t}$.

(b) $\hat{\mathbb{T}} = \mathbb{Z}$ with the pairing $(n|\theta) = \theta^n$.

(c) $\hat{\mathbb{Z}} = \mathbb{T}$ with the pairing $(\theta|n) = \theta^n$.

(d) For each $n \in \mathbb{N}$, $\widehat{\mathbb{Z}/n\mathbb{Z}} \cong \mathbb{Z}/n\mathbb{Z}$ with pairing $(a|b) = e^{\frac{2\pi i ab}{n}}$.

Proof (a): For $\omega \in \hat{\mathbb{R}}$ we have $\omega(0) = 1$, and so continuity of ω implies there exists $\delta > 0$ so that

$$\alpha := \int_0^\delta \phi(t)\,dt \neq 0.$$

Then

$$\alpha \cdot \phi(s) = \int_0^\delta \phi(t+s)\,dt = \int_s^{s+\delta} \phi(t)\,dt$$

Thus for $0 < \varepsilon < \delta$ we have

$$\frac{\phi(s+\varepsilon) - \phi(s)}{\varepsilon} = \frac{1}{\alpha\varepsilon}\left(\int_{s+\varepsilon}^{s+\delta+\varepsilon} \phi(t)\,dt - \int_s^{s+\delta} \phi(t)\,dt\right)$$
$$= \frac{1}{\alpha\varepsilon}\left(\int_{s+\delta}^{s+\delta+\varepsilon} \phi(t)\,dt - \int_s^{s+\varepsilon} \phi(t)\,dt\right)$$

Therefore ϕ is differentiable with $\phi'(s) = \frac{1}{\alpha}(\phi(s+\delta) - \phi(s)) = \overbrace{\frac{1}{\alpha}[\phi(\delta) - 1]}^{=:c} \cdot \phi(s)$, which implies $\phi(s) = e^{cs}$. Since $|\phi(s)| = 1$ for all $s \in \mathbb{R}$, it must be that $c \in i\mathbb{R}$. So $t := \frac{c}{2\pi i} \in \mathbb{R}$ satisfies $\phi(s) = e^{2\pi i s t}$.

(b): We have $\mathbb{R}/\mathbb{Z} \cong \mathbb{T}$ via $s + \mathbb{Z} \mapsto e^{2\pi i s}$. In particular, we have a continuous, surjective homomorphism $q: \mathbb{R} \to \mathbb{T}$ with $\ker(q) = \mathbb{Z}$. So for any $\omega \in \hat{\mathbb{T}}$ one has $\omega \circ q \in \hat{\mathbb{R}}$ and by part (a) this means

$$e^{2\pi i s t} = \omega \circ q(s) = \omega(e^{2\pi i s})$$

for some $t \in \mathbb{R}$. So $\omega(\theta) = \theta^t$. In fact, we must have $t \in \mathbb{N}$ since

$$1 = \omega(1) = \omega \circ q(n) = e^{2\pi i n t}$$

for all $n \in \mathbb{Z}$.

(c): For $\omega \in \hat{\mathbb{Z}}$ set $\theta := \omega(1)$. Then $\omega(n) = \omega(1)^n = \theta^n$ for all $n \in \mathbb{Z}$.

(d): Let $q: \mathbb{Z} \to \mathbb{Z}/n\mathbb{Z}$ be the quotient map. Then for $\omega \in \widehat{\mathbb{Z}/n\mathbb{Z}}$, part (c) implies

$$\theta^k = \omega \circ q(k) = \omega(k \bmod n)$$

for some $\theta \in \mathbb{T}$. Since the above must be trivial for $k = n$, we see that θ is an nth root of unity: $\theta = e^{\frac{2\pi i b}{n}}$ for some $b = 0, 1, \ldots, n-1$. $\square$

Proposition 4.6 Let $\{G_i : i \in I\}$ be a family of locally compact groups with compact open subgroups $K_i \leq G_i$. Then the restricted direct product has the dual group

$$\widehat{\prod_{i\in I}(G_i, K_i)} \cong \{(\omega_i)_{i\in I} \in \prod_{i \in I} \hat{G}_i : \omega_i|_{K_i} \equiv 1 \text{ for all but finitely many } i \in I\}$$

Proof First each $\omega := (\omega_i)_{i\in I}$ as on the right-hand side above gives a well-defined element of the dual of the restricted direct product since

$$(x_i)_{i \in I} \in \prod_{i\in I}(G_i, K_i)$$

has $x_i \in K_i$ for all but finitely many $i \in I$ and hence

$$\prod_{i\in I} \omega_i(x_i)$$

is really a finite product. Conversely, let $\omega \in \widehat{\prod_i (G_i, K_i)}$. Using the canonical embedding of $G_i \hookrightarrow \prod_i (G_i, K_i)$, we set $\omega_i := \omega|_{G_i}$. It then suffices to show $\omega_i|_{K_i} \equiv 1$ for all but finitely many $i \in I$. Let $\mathcal{U}$ be a neighborhood of $1 \in \prod_i (G_i, K_i)$ such that $|\omega(x) - 1| < 1$ for all $x \in \mathcal{U}$. Recall that for each finite subset $F \subset I$,

$$H_F := \prod_{i\in F} G_i \times \prod_{i \in I \setminus F} K_i$$

is an open subgroup. Replacing $\mathcal{U}$ with $\mathcal{U} \cap H_F$ we may assume $\mathcal{U} \subset H_F$. Since H_F is endowed with the product topology, we have

$$\prod_{i\in I} V_i \subset \mathcal{U} \subset H_F$$

where $V_i = K_i$ for all but finitely many $i \in I \setminus F$. When $V_i = K_i$, we have $K_i \hookrightarrow \mathcal{U}$ and thus $|\omega_i(x) - 1| < 1$ for all $x \in K_i$. Since $\omega_i(K_i)$ is a subgroup of $\mathbb{T}$, it follows that $\omega_i|_{K_i} \equiv 1$. $\square$

Corollary 4.7 Let $\{K_i : i \in I\}$ be a family of compact groups. Then

$$\widehat{\prod_{i\in I} K_i} = \{(\omega_i)_{i\in I} \in \prod_{i \in I} \hat{K}_i : \omega_i \equiv 1 \text{ for all but finitely many } i \in I\} =: \bigoplus_{i\in I} \hat{K}_i$$

Corollary 4.8 For locally compact groups $G_1, \ldots, G_n$ one has

$$\widehat{G_1 \times \cdots \times G_n} \cong \hat{G}_1 \times \cdots \times \hat{G}_n$$

Corollary 4.9 $\widehat{\mathbb{R}^n} \cong \mathbb{R}^n$, $\widehat{\mathbb{T}^n} \cong \mathbb{Z}^n$, $\widehat{\mathbb{Z}^n} \cong \mathbb{T}^n$, and for any finite group G one has $\hat{G} \cong G$.

Ex For

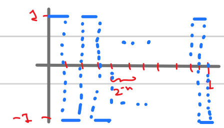
$$(\mathbb{Z}/2\mathbb{Z})^{\omega} := \prod_{n\in\mathbb{N}} (\mathbb{Z}/2\mathbb{Z})$$

and for each $n \in \mathbb{N}$ there is a unique $\xi_n \in \widehat{(\mathbb{Z}/2\mathbb{Z})^{\omega}}$ satisfying $\langle (a_n)_{n\in\mathbb{N}} \mid \xi_n \rangle = (-1)^{a_n}$.

Then by Corollary 4.7 we see that each element of $\widehat{(\mathbb{Z}/2\mathbb{Z})^\omega}$ is a finite product of the ξ_n, $n \in \mathbb{N}$. Recall by an example from Section 2 that

$$(\mathbb{Z}/2\mathbb{Z})^\omega \longrightarrow [0,1]$$
$$(x_n)_{n \in \mathbb{N}} \longmapsto \sum_{n=1}^{\infty} x_n 2^{-n}$$

is a continuous surjection that is almost everywhere injective. Under the identification $(\mathbb{Z}/2\mathbb{Z})^\omega \cong [0,1]$, ξ_n is identified with the nth Rademacher function r_n

r_n 

Finite products of Rademacher functions are known as Walsh functions. □

Dual of the p-adic numbers

Recall from Proposition 2.6 that each $x \in \mathbb{Q}_p$ can be written uniquely as a convergent series

$$x = \sum_{j \in \mathbb{Z}} c_j p^j$$

where $c_j \in \{0, 1, \dots, p-1\}$ and $c_j = 0$ for all $j < m$ for some $m \in \mathbb{Z}$. Also recall that $x \in \mathbb{Z}_p$ if and only if $c_j = 0$ for all $j < 0$. Define $\omega_1 \in \widehat{\mathbb{Q}_p}$ by $(x \mid \omega_1) = e^{2\pi i x}$. That is,

$$\Big(\sum_{j \in \mathbb{Z}} c_j p^j \,\Big|\, \omega_1\Big) := \exp\Big(2\pi i \sum_{j \in \mathbb{Z}} c_j p^j\Big) = \exp\Big(2\pi i \sum_{j < 0} c_j p^j\Big)$$

It follows that $(x+y \mid \omega_1) = (x \mid \omega_1)(y \mid \omega_1)$ for all $x, y \in \mathbb{Q}_p$. Also $\ker(\omega_1) = \mathbb{Z}_p$ so that $(x + \mathbb{Z}_p \mid \omega_1) = (x \mid \omega_1)$. This implies ω_1 is continuous since $\mathbb{Z}_p$ is open: $x + \mathbb{Z}_p$ is an open neighborhood of x on which ω_1 is close (equal) to $\omega_1(x)$.

Next, for $y \in \mathbb{Q}_p$ define $\omega_y \in \widehat{\mathbb{Q}_p}$ by

$$(x \mid \omega_y) := (xy \mid \omega_1)$$

Then

$$(x+z \mid \omega_y) = ((x+z)y \mid \omega_1) = (xy \mid \omega_1) + (zy \mid \omega_1) = (x \mid \omega_y) + (z \mid \omega_y)$$

and continuity follows from that of ω_1 and the map $x \mapsto xy$. Also note $\ker(\omega_y) = \{x \in \mathbb{Q}_p : |x|_p \le |y|_p^{-1}\}$. We will show every element of $\widehat{\mathbb{Q}_p}$ is of this form. We first require a lemma.

$\boxed{\text{Lemma 4.10}}$ If $\omega \in \widehat{\mathbb{Q}_p}$ satisfies $(1 \mid \omega) = 1$ and $(p^{-1} \mid \omega) \neq 1$, then there exists $y \in \mathbb{Q}_p$ with $|y|_p = 1$ such that $\omega = \omega_y$.

Proof We will find $y \in \mathbb{Q}_p$ of the form

$$y = \sum_{j=0}^{\infty} c_j p^j$$

where $c_0 \in \{1,\dots,p-1\}$ and $c_j = \{1,\dots,p-1\}$. Note that $c_0 \neq 0$ implies $|y|_p = p^{-0} = 1$. Toward this end, denote $\alpha_k := (p^{-k} \mid \omega)$ for each $k \in \mathbb{Z}$. Then $\alpha_0 = 1$ and $\alpha_1 \neq 1$ by hypothesis. Also observe that

$$\alpha_{k+1}^p = (p^{-k-1} \mid \omega)^p = (p \cdot p^{-k-1} \mid \omega) = (p^{-k} \mid \omega) = \alpha_k$$

so α_{k+1} and $\alpha_k^{1/p}$ agree up to a pth root of unity. In particular, α_1 is a non-trivial root of unity and so there exists $c_0 \in \{1,\dots,p-1\}$ satisfying $\alpha_1 = \exp(2\pi i c_0 p^{-1})$. Suppose we have found $c_1,\dots,c_{k-1} \in \{0,1,\dots,p-1\}$ satisfying

$$\alpha_k = \exp(2\pi i(c_0 p^{-k} + c_1 p^{-k+1} + \cdots + c_{k-1} p^{-1})) \qquad *$$

↪ consider $k=1$

Since $\alpha_{k+1} = \alpha_k^{1/p} \cdot \zeta$ for a pth root of unity $\zeta = \exp(2\pi i c_k p^{-1})$ for some $c_k \in \{0,1,\dots,p-1\}$, we then have

$$\alpha_{k+1} = \alpha_k^{1/p} \zeta = \exp(2\pi i(c_0 p^{-(k+1)} + \cdots + c_{k-1} p^{-2} + c_k p^{-1})).$$

So by induction we can find $(c_j)_{j \geq 1} \subset \{0,1,\dots,p-1\}$ satisfying (*). Letting y be as above, we have

$$(p^{-k} \mid \omega_y) = (p^{-k} y \mid \omega_1) = \Big(\sum_{j=0}^{\infty} c_j p^{j-k} \,\Big|\, \omega_1\Big)$$

$$= \exp(2\pi i(c_0 p^{-k} + c_1 p^{-k+1} + \cdots + c_{k-1} p^{-1})) \overset{*}{=} (p^{-k} \mid \omega)$$

Consequently, for $x \in \mathbb{Q}_p$ if we write $x = a_m p^{-m} + a_{m-1} p^{-m+1} + \cdots + a_1 p^{-1} + z$ for $a_j \in \{0,1,\dots,p-1\}$ and $z \in \mathbb{Z}_p$, we have

$$(x \mid \omega_y) = (p^{-m} \mid \omega_y)^{a_m} \cdots (p^{-1} \mid \omega_y)^{a_1} = (p^{-m} \mid \omega)^{a_m} \cdots (p^{-1} \mid \omega)^{a_1} = (x \mid \omega).$$

Thus $\omega_y = \omega$. □

Theorem 4.11 The map $\mathbb{Q}_p \ni y \mapsto \omega_y \in \widehat{\mathbb{Q}}_p$ is an isomorphism of topological groups.

Proof First note that

$$(x \mid \omega_y \omega_z) = (x \mid \omega_y)(x \mid \omega_z) = (xy \mid \omega_1)(xz \mid \omega_1) = (xy + xz \mid \omega_1) = (x(y+z) \mid \omega_1) = (x \mid \omega_{y+z}),$$

so that $y \mapsto \omega_y$ is a group homomorphism. Also, $\omega_y = \omega_z$ implies $(x(y-z) \mid \omega_1) = 0$ for all $x \in \mathbb{Q}_p$. Since $\ker(\omega_1) = \mathbb{Z}_p$, we must have have $y = z$. Thus $y \mapsto \omega_y$ is injective.

We next check surjectivity. Let $\omega \in \widehat{\mathbb{Q}}_p$. If $\omega = 1$ then $\omega = \omega_0$. Otherwise $\omega \neq 1$. Since ω is continuous, there exists k sufficiently large so that

$$B(0, p^{-k}) \subset \omega^{-1}(\{z \in \mathbb{T} : |z - 1| < 1\})$$

Noting that $B(0,p^{-k})$ is a subgroup of $\mathbb{Q}_p$, it follows that $\omega \equiv 1$ on $B(0,p^{-k})$. Thus we can find a smallest $k_0 \in \mathbb{Z}$ satisfying $(p^{k_0} \mid \omega) = 1$. Defining $\phi \in \widehat{\mathbb{Q}}_p$ by $(x \mid \phi) := (x p^{k_0} \mid \omega)$, we see that $(1|\phi) = (p^{k_0}|\omega) = 1$ but $(p^{-1}|\phi) = (p^{k_0-1}|\omega) \neq 1$ (else $\omega \equiv 1$ on $B(0, p^{-k_0+1})$, contradicting the minimality of k_0). Thus we can apply Lemma 4.10 to obtain $\phi = \omega_z$ for some $z \in \mathbb{Q}_p$ with $|z|_p = 1$. For $y := p^{-k_0} z$ we then have

$$(x \mid \omega_y) = (x p^{-k_0} z \mid \omega_1) = (x p^{-k_0} \mid \omega_z) = (x p^{-k_0} \mid \phi) = (x \mid \omega)$$

so that $\omega = \omega_y$. So $y \mapsto \omega_y$ is an isomorphism of groups. It remains to check that it is a homeomorphism.

Observe that the sets

$$N(j,k) := \{\omega \in \widehat{\mathbb{Q}_p} : |(x|\omega) - 1| < \tfrac{1}{k} \text{ for } |x|_p \le p^j\} \qquad j \in \mathbb{Z},\ k \in \mathbb{N}$$

form a neighborhood base at $1 \in \widehat{\mathbb{Q}_p}$. Indeed, using the notation of Section 8.3 for the neighborhood base for the topology of compact convergence, one has

$$N(j,k) \subset \mathcal{N}(1; \varepsilon, K)$$

whenever $k > \frac{1}{\varepsilon}$ and j is large enough so that $K \subset B(0, p^j)$. Now, the image of $\overline{B}(0, p^j)$ under ω_1 is $\{1\}$ if $j \le 0$ and otherwise is the group of p^j-th roots of unity. Consequently, $\omega_1 \in N(j,k)$ if and only if $j \le 0$. Consequently, $\omega_y \in N(j,k)$ if and only if $|y|_p \le p^{-j}$. Indeed, if $|y|_p \le p^{-j}$ then for $|x|_p \le p^j$ we have $|xy|_p \le p^0$ and thus

$$(x|\omega_y) = (xy|\omega_1) = 1$$

so that $\omega_y \in N(j,k)$. Conversely, if $|y|_p = p^{-m} > p^{-j}$ for some $m < j$ then we can find $c \in \{1, \ldots, p-1\}$ so that

$$(cp^{-(m+1)} \mid \omega_y) = (cp^{-(m+1)} y \mid \omega_1) \notin \{z \in \mathbb{T} : |z - 1| < \tfrac{1}{k}\},$$

and since $|cp^{-(m+1)}|_p = p^{m+1} \le p^j$ we see that $\omega_y \notin N(j,k)$. Thus

$$B(0, p^{-j}) \subset \{y \in \mathbb{Q}_p : \omega_y \in N(j,k)\}$$

shows $y \mapsto \omega_y$ is continuous at $1 \in \mathbb{Q}_p$, and

$$\{\omega_y : y \in B(0, p^{-j})\} \supset N(j-1, k)$$

shows the inverse is continuous at $1 \in \widehat{\mathbb{Q}_p}$. But then $y \mapsto \omega_y$ is a homeomorphism since we already established it was a group homomorphism. $\square$

4.2 The Fourier Transform

Let G be an abelian locally compact group with Haar measure μ. Recall that for $\omega \in \hat{G}$

$$L^1(G,\mu) \ni f \longmapsto \omega(f) = \int_G f(x)\,(x|\omega)\,d\mu(x) \in \mathbb{C}$$

is a $*$-homomorphism. For $f \in L^1(G,\mu)$ we define $\hat{f}: \hat{G} \to \mathbb{C}$ by

$$\hat{f}(\omega) := \int_G f(x)\,\overline{(x|\omega)}\,d\mu(x) = \overline{\omega}'(f).$$

Observe that $|\hat{f}(\omega)| \le \|f\|_1$. Also, for $\varepsilon > 0$ let $K \subset G$ be a compact subset with $\int_{G\setminus K} |f|\,d\mu < \frac{\varepsilon}{4}$. Then for $\omega \in \hat{G}$, if $\phi \in \hat{G}$ satisfies $|(x|\omega) - (x|\phi)| < \frac{\varepsilon}{2\|f\|_1}$ for all $x \in K$ then we have:

$$|\hat{f}(\omega) - \hat{f}(\phi)| \le \int_K |f(x)|\,|\overline{(x|\omega)} - \overline{(x|\phi)}|\,d\mu(x) + \int_{G\setminus K} |f| \cdot 2\,d\mu < \|f\|_1 \cdot \frac{\varepsilon}{2\|f\|_1} + \frac{\varepsilon}{4}\cdot 2 = \varepsilon$$

Hence $\hat{f}$ is bounded and continuous on $\hat{G}$ with $\|\hat{f}\|_\infty \le \|f\|_1$.

Def The map $\mathcal{F}: L^1(G,\mu) \to C_b(\hat{G})$ defined by $\mathcal{F}f = \hat{f}$ is called the <u>Fourier transform</u> of G. □

Recall that for a locally compact Hausdorff space X, $C_0(X)$ is a Banach $*$-algebra (in fact, a C^*-algebra) with pointwise operations and norm $\|f\|_\infty = \sup_{x\in X} |f(x)|$.

Proposition 4.12 Let G be a σ-compact group. The Fourier transform is a $*$-homomorphism $\mathcal{F}: L^1(G,\mu) \to C_0(\hat{G})$ with dense range and $\|\mathcal{F}f\|_1 \le \|f\|_\infty$. Moreover, for $x \in G$, $\omega, \phi \in \hat{G}$ one has

$$[\mathcal{F}(L_x f)](\omega) = \overline{(x|\omega)}\,\mathcal{F}(f)(\omega) \quad \text{and} \quad \mathcal{F}(\phi f)(\omega) = [L_\phi \mathcal{F}(f)](\omega)$$

Proof We have already seen $\mathcal{F}(f) \in C_b(\hat{G})$ with $\|\mathcal{F}f\|_\infty \le \|f\|_1$ for all $f \in L^1(G,\mu)$. To see that $\mathcal{F}f \in C_0(\hat{G})$, recall from the proof of Corollary 4.2 that $\hat{G} \cup \{0\} \subset L^\infty(G)$ is the one-point compactification of $\hat{G}$ and of course

$$\int f \cdot 0\,d\mu = 0.$$

Thus for $\varepsilon > 0$ we can find a weak* neighborhood $\mathcal{U}$ of $0 \in L^\infty(G)$ such that

$$\left| \int_G f \cdot \overline{\phi}\,d\mu \right| < \varepsilon$$

for all $\phi \in \mathcal{U}$. Then $K := \hat{G} \setminus \mathcal{U} = \hat{G} \cup \{0\} \setminus \mathcal{U}$ is a compact subset such that

$$|\hat{f}(\omega)| = \left| \int_G f(x)\,\overline{(x|\omega)}\,d\mu(x) \right| < \varepsilon$$

for all $\omega \in \hat{G} \setminus K$. Thus $\hat{f} \in C_0(\hat{G})$.

Now, $\mathcal{F}$ being a $*$-homomorphism follows from $f \mapsto \overline{\omega}'(f)$ be a $*$-representation. Indeed, $\mathcal{F}$ is clearly linear and

$$\widehat{(f*g)}(\omega) = \overline{\omega}'(f*g) = \overline{\omega}'(f)\,\overline{\omega}'(g) = \hat{f}(\omega)\,\hat{g}(\omega)$$

and

$$\widehat{f^*}(\omega) = \omega^{-1}(f^*) = \overline{\omega^{-1}(f)} = \overline{\hat f(\omega)}$$

Thus $\mathcal{F}(L^1(G,\mu))$ is a $*$-subalgebra of $C_0(X)$. For each $\omega\in\hat G$ we can find $f\in L^1(G,\mu)$ with $\hat f(\omega)\neq 0$ (for example, $\hat\phi_U(\omega)\to\omega^{-1}(1)=1$ for an approximate identity), and for distinct $\omega,\phi\in\hat G$ we can find $f\in L^1(G,\mu)$ with $\hat f(\omega)\neq\hat f(\phi)$ (for example, if $\omega(x)\neq\phi(x)$ then $\widehat{L_x\phi_U}(\omega)\neq\widehat{L_x\phi_U}(\phi)$ for large enough U). Therefore the Stone–Weierstrass theorem implies $\mathcal{F}$ has dense range.

Lastly, we compute

$$[\mathcal{F}(L_x f)](\omega) = \int_G f(x^{-1}y)\,\overline{(y|\omega)}\,d\mu(y) = \int_G f(y)\,\overline{(xy|\omega)}\,d\mu(y) = \overline{(x|\omega)}\,\mathcal{F}(f)(\omega)$$

and

$$\mathcal{F}(\phi f)(\omega) = \int_G \phi(x) f(x)\,\overline{(x|\omega)}\,d\mu(x) = \int_G f(x)\,\overline{(x|\phi^{-1}\omega)}\,d\mu(x) = \mathcal{F}(f)(\phi^{-1}\omega) = [L_\phi\mathcal{F}(f)](\omega)$$

Note that if G is countable and discrete, so that $\hat G$ is compact by Proposition 4.4, then $C_0(\hat G)=C(\hat G)$.

Remark For $G=\mathbb{R}=\hat G$, the inclusion $\mathcal{F}(L^1(\mathbb{R},m))\subset C_0(\mathbb{R})$ recovers the Riemann–Lebesgue lemma from classical Fourier analysis.

It is possible to extend the domain of the Fourier transform to the measure algebra $M(G)$ as follows: for $\nu\in M(G)$ define $\hat\nu\in C_b(\hat G)$ by

$$\hat\nu(\omega) := \int_G \overline{(x|\omega)}\,d\mu(x).$$

Then $\widehat{f\,d\mu}=\hat f$. One can also prove $\widehat{\nu*\sigma}=\hat\nu\hat\sigma$ by approximating ν,σ, and $\nu*\sigma$ by measures with compact support. (This approximation is needed since $\omega^{-1}\in C_b(G)$ rather than $C_0(G)$, and is possible since these measures are inner regular.)

One can also consider the dual version of the above: for $\nu\in M(\hat G)$ define $\phi_\nu: G\to\mathbb{C}$ by

$$\phi_\nu(x) := \int_{\hat G} (x|\omega)\,d\nu(\omega).$$

Note that $|\phi_\nu(x)|\leq\|\nu\|$ so that ϕ_ν is bounded. We also claim it is continuous. Indeed, since ν is finite and inner regular, for $\varepsilon>0$ we can find $\hat K\subset\hat G$ compact satisfying

$$|\nu|(\hat G\setminus\hat K)<\frac{\varepsilon}{4}$$

Next, fix $x\in G$ and a compact neighborhood K of x. Since $\hat K$ is compact in the topology of compact convergence, $\{\omega|_K : \omega\in\hat K\}\subset C(K)$ is a compact set with respect to $\|\cdot\|_\infty$. Consequently, the Arzela–Ascoli theorem implies the family $\{\omega|_K : \omega\in\hat K\}$ is equicontinuous on K, and hence we can find a neighborhood U of x in K so that

$$|c(x|\omega)-c(y|\omega)| < \frac{\varepsilon}{2\|\nu\|}$$

for all $\omega\in\hat{K}$ and $y\in U$. Thus for $y\in U$ we have

$$|\phi_\nu(x)-\phi_\nu(y)| \le \int_{\hat{K}} |c(x|\omega)-c(y|\omega)|\, d|\nu| + 2|\nu|(\hat{G}\setminus\hat{K}) < \frac{\varepsilon}{2\|\nu\|}\cdot\|\nu\| + 2\frac{\varepsilon}{4} = \varepsilon.$$

Thus ϕ_ν is continuous at x, and hence on G.

Proposition 4.13 The map $M(\hat{G})\ni\nu\mapsto\phi_\nu\in C_b(G)$ is a linear injection satisfying $\|\phi_\nu\|_\infty\le\|\nu\|$.

Proof Linearity is clear from the definition and we have already seen $|\phi_\nu(x)|\le\|\nu\|$. So it remains to check injectivity. Suppose $\phi_\nu\equiv 0$. Then for any $f\in L^1(G,\mu)$ we have

$$0=\int_G f(x)\,\phi_\nu(x)\,d\mu(x)=\int_G\int_{\hat{G}} f(x)\,c(x|\omega)\,d\nu(\omega)\,d\mu(x)=\int_{\hat{G}}\hat{f}(\omega^{-1})\,d\nu(\omega).$$

Recalling that $\mathcal{F}(L^1(G,\mu))$ is dense in $C_0(\hat{G})$, the above implies $\nu=0$. □

Observe that for positive $\nu\in M(\hat{G})$, ϕ_ν is of positive type: for $f\in L^1(G,\mu)$

$$\int_G\int_G f(x)\overline{f(y)}\,\phi_\nu(y^{-1}x)\,d\mu(x)\,d\mu(y)=\int_G\int_G\int_{\hat{G}} f(x)\overline{f(y)}\,c(y^{-1}x|\omega)\,d\nu(\omega)\,d\mu(x)\,d\mu(y)$$

$$=\int_{\hat{G}}\left(\int_G f(x)\,\overline{c(x|\omega^{-1})}\,d\mu(x)\right)\overline{\left(\int_G f(y)\,\overline{c(y|\omega^{-1})}\,d\mu(y)\right)}\,d\nu(\omega)$$

$$=\int_{\hat{G}}|\hat{f}(\omega^{-1})|^2\,d\nu(\omega)\ge 0.$$

We also have the converse:

Theorem 4.14 (Bochner Theorem) Let G be a σ-compact group. For each $\phi\in P(G)$ there is a unique positive $\nu\in M(\hat{G})$ with $\phi_\nu=\phi$.

Proof The uniqueness follows from Proposition 4.13, so it remains to prove existence. Note that, by rescaling, it suffices to consider $\phi\in P_0(G)$ ($\|\phi\|_\infty=\phi(1)\le 1$). Let M_0 denote the set of positive $\nu\in M(\hat{G})$ with $\|\nu\|=|\nu|(\hat{G})\le 1$, which we know is weak* compact by Banach–Alaoglu (recall $M(\hat{G})\cong C_0(\hat{G})^*$). We claim that $\mu\mapsto\phi_\mu$ is weak* to weak* continuous. Toward proving this, let us compute for $f\in L^1(G,\mu)$ and $\nu\in M(\hat{G})$:

$$\int_G f(x)\,\phi_\nu(x)\,d\mu(x)=\int_G\int_{\hat{G}} f(x)\,c(x|\omega)\,d\nu(\omega)\,d\mu(x)=\int_{\hat{G}}\hat{f}(\omega^{-1})\,d\nu(\omega).$$

Also recall that $\hat{f}\in C_0(\hat{G})$ by Proposition 4.12. Thus if $(\nu_i)_{i\in I}\subset M(\hat{G})$ converges weak* to some ν, then for all $f\in L^1(G,\mu)$ we have

$$\int_G f\phi_\nu\,d\mu=\int_{\hat{G}}\hat{f}(\omega^{-1})\,d\nu(\omega)=\lim_{i\to\infty}\int_{\hat{G}}\hat{f}(\omega^{-1})\,d\nu_i(\omega)=\lim_{i\to\infty}\int_G f\phi_{\nu_i}\,d\mu.$$

That is, $\phi_{\nu_i}\to\phi_\nu$ weak* in $L^\infty(G)\cong L^1(G,\mu)^*$, as claimed. It follows that $\Phi_0:=\{\phi_\nu:\nu\in M_0\}$ is weak* compact, and it is also convex by the convexity of M_0 and the linearity of

$\nu \mapsto \varphi_\nu$. Thus $\Phi_0 = \overline{\mathrm{conv}}(\Phi_0)$. Recall from Lemma 3.20 and the discussion preceding it that

$$P_0(G) = \overline{\mathrm{conv}}(\mathrm{ext}(P_0(G))) = \overline{\mathrm{conv}}(\mathrm{ext}(P_1(G)) \cup \{0\}) = \overline{\mathrm{conv}}(\hat{G} \cup \{0\})$$

where the last equality follows from Theorem 3.19 and the identification of $\hat{G}$ as the set of irreducible representations of G. Observing that $\varphi_{\delta_\omega} = \omega$ for all $\omega \in \hat{G}$ and $\varphi_0 = 0$, we see that $\hat{G} \cup \{0\} \subset \Phi_0$ and thus

$$P_0(G) \subset \overline{\mathrm{conv}}(\Phi_0) = \Phi_0 \subset P_0(G)$$

So $P_0(G) = \Phi_0 = \{\varphi_\nu : \nu \in M_0\}$. $\square$

4/8

Let us denote

$$\mathcal{B}(G) := \{\varphi_\nu : \mu \in M(\hat{G})\}$$

and

$$\mathcal{B}^p(G) := \mathcal{B}(G) \cap L^p(G, \mu) \qquad 1 \leq p < \infty.$$

Then Bochner's theorem states $\mathcal{B}(G) = \mathrm{span}\, P(G)$. Also note that by Proposition 3.26

$$\{f * g : f, g \in C_c(G)\} \subset \mathcal{B}(G)$$

and $\mathcal{B}(G)$ (resp. $\mathcal{B}^p(G)$) is dense in $C_0(G)$ under $\|\cdot\|_\infty$ (resp. $L^p(G,\mu)$ under $\|\ \|_p$).

Our next goal is to show that $\mathcal{F}$ can be inverted on $\mathcal{B}^1(G)$, but first we require a few lemmas.

$\boxed{\text{Lemma 4.15}}$ Let G be a σ-compact group. For $K \subset \hat{G}$ compact there exists $f \in C_c(G) \cap P(G)$ satisfying $\hat{f} \geq 0$ on $\hat{G}$ and $\hat{f} > 0$ on K.

Proof. Fix $h \in C_c(G)$ satisfying $\hat{h}(1) = \int h \, d\mu = 1$ and set $g := h^* * h$. Note $g \in C_c(G) \cap P(G)$ by Corollary 3.13. Also, recalling that $\mathcal{F}$ is a $*$-homomorphism we have $\hat{g} = \overline{\hat{h}}\hat{h} = |\hat{h}|^2$ so that $\hat{g} \geq 0$ with $\hat{g}(1) = 1$. Hence there exists a neighborhood V of $1 \in \hat{G}$ on which $\hat{g} > 0$. Let $\omega_1 V \cup \cdots \cup \omega_n V \supset K$ be a finite subcover of $\{\omega V : \omega \in K\}$ and set

$$f := \left(\sum_{j=1}^{n} \omega_j\right) \cdot g \in C_c(G)$$

Then by Proposition 4.12 we have

$$\hat{f}(\omega) = \sum_{j=1}^{n} \hat{g}(\omega_j^{-1}\omega),$$

and hence $\hat{f} \geq 0$. Also, for $\omega \in K$ let $\omega \in \omega_j V$. Then $\omega_j^{-1}\omega \in V$ so that $\hat{f}(\omega) \geq \hat{g}(\omega_j^{-1}\omega) > 0$. Finally, recalling that $g \in C_c(G) \cap P(G)$, we clearly have $f \in C_c(G)$, and for any $e \in L^1(G,\mu)$ we have

$$\int_G\int_G e(x)\overline{e(y)}\, f(y^{-1}x)\, d\mu(x)\, d\mu(y) = \sum_{j=1}^{n}\int_G\int_G e(x)\overline{e(y)}\,(y^{-1}x|\omega_j)\, g(y^{-1}x)\, d\mu(x)\, d\mu(y)$$
$$= \sum_{j=1}^{n}\int_G\int_G e(x)(x|\omega_j)\,\overline{e(y)(y|\omega_j)}\, g(y^{-1}x)\, d\mu(x)\, d\mu(y) \geq 0.$$

Hence $f\in P(G)$ as well. □

The map $M(\hat{G})\ni\nu\mapsto\phi_\nu\in B(G)$ is a bijection by Proposition 4.13 (and the definition of $B(G)$). Let $B(G)\ni\phi\mapsto\nu_\phi\in M(\hat{G})$ denote its inverse.

$\boxed{\text{Lemma 4.16}}$ Let G be a σ-compact group. For $\phi,\psi\in B^1(G)$ one has $\hat{\phi}\, d\nu_\psi = \hat{\psi}\, d\nu_\phi$.

Proof For $f\in L^1(G,\mu)$ we compute

$$\int_{\hat{G}}\hat{f}\, d\nu_\phi = \int_{\hat{G}}\int_G f(x)\overline{(x|\omega)}\, d\mu(x)\, d\nu_\phi(\omega)$$
$$= \int_G f(x)\int_{\hat{G}}(x^{-1}|\omega)\, d\nu_\phi(\omega)\, d\mu(x)$$
$$= \int_G f(x)\,\phi(x^{-1})\, d\mu(x) = f*\phi(1).$$

Thus we have

$$\int_{\hat{G}}\hat{f}\hat{\psi}\, d\nu_\phi = \int_{\hat{G}}\widehat{f*\psi}\, d\nu_\phi = (f*\psi)*\phi(1) = (f*\phi)*\psi(1) = \int_{\hat{G}}\hat{f}\hat{\phi}\, d\nu_\psi$$

(recall that convolution is commutative since G is abelian). Since $\mathcal{F}(L^1(G,\mu))$ is dense in $C_0(\hat{G})$ by Proposition 4.12, the above implies $\hat{\psi}\, d\nu_\phi = \hat{\phi}\, d\nu_\psi$. □

$\boxed{\text{Theorem 4.17}}$ (Fourier Inversion Theorem I) Let G be a σ-compact group. Then $\hat{G}$ admits a Haar measure $\hat{\mu}$ normalized in such a way that

$$f(x) = \int_{\hat{G}}\hat{f}(\omega)(x|\omega)\, d\hat{\mu}(\omega)$$

for all $f\in B^1(G)$. In particular, $\hat{f}\in L^1(\hat{G},\hat{\mu})$ and $d\nu_f = \hat{f}\, d\hat{\mu}$.

Proof We will first construct a linear functional Φ on $C_c(\hat{G})$ that will ultimately be integration against our desired Haar measure $\hat{\mu}$. For $\psi\in C_c(\hat{G})$, we use Lemma 4.15 to find $f\in C_c(G)\cap P(G)$ with $\hat{f}\geq 0$ and $\hat{f}>0$ on $\operatorname{supp}(\psi)$, and then we define

$$\Phi(\psi) := \int_{\hat{G}}\psi/\hat{f}\, d\nu_f$$

Observe that for $g\in B^1(G)$ satisfying $\hat{g}>0$ on $\operatorname{supp}(\psi)$, Lemma 4.16 implies

$$\Phi(\psi) = \int_{\hat{G}}\frac{\psi}{\hat{f}\hat{g}}\hat{g}\, d\nu_f = \int_{\hat{G}}\frac{\psi}{\hat{f}\hat{g}}\hat{f}\, d\nu_g = \int_{\hat{G}}\psi/\hat{g}\, d\nu_g$$

so that $\Phi(\psi)$ is independent of f. It follows that $\psi\mapsto\Phi(\psi)$ is linear. If $\psi\geq 0$, note that $\Phi(\psi)\geq 0$ since $\hat{f}>0$ on $\operatorname{supp}(\psi)$ and ν_f is a positive measure by virtue of $f\in P(G)$.

Next, for $g \in L^1(G,\mu) \cap P(G)$ observe that

$$\Phi(\psi \hat{g}) = \int_{\hat{G}} \psi / \hat{f}\, \hat{g}\, d\nu_f = \int_{\hat{G}} \psi \, d\nu_g \qquad (*)$$

for any $\psi \in C_c(\hat{G})$. Thus if $g \neq 0$ so that $\nu_g \neq 0$, then we can find $\psi \in C_c(\hat{G})$ so that the last integral above is non-zero. Hence $\Phi \neq 0$.

Now, toward showing Φ is translation invariant, observe that for $\phi \in \hat{G}$ and $x \in G$ we have

$$\int_{\hat{G}} (x|\omega)\, d\nu_f(\phi\omega) = \int_{\hat{G}} (x|\phi^{-1}\omega)\, d\nu_f(\omega) = (x|\phi^{-1}) f(x) = (\phi^{-1} \cdot f)(x)$$

Thus $d\nu_f(\phi \cdot) = d\nu_{\phi^{-1} f}$. Also recall from Proposition 4.12 that $\widehat{\phi^{-1} \cdot f} = L_{\phi^{-1}} \hat{f}$. So if we choose f so that $\hat{f} > 0$ on $\mathrm{supp}(\psi) \cup \mathrm{supp}(L_\phi \psi)$, then

$$\begin{aligned} \Phi(L_\phi \psi) &= \int_{\hat{G}} \psi(\phi^{-1}\omega)/\hat{f}(\omega)\, d\nu_f(\omega) \\ &= \int_{\hat{G}} \psi(\omega)/\hat{f}(\phi\omega)\, d\nu_f(\phi\omega) \\ &= \int_{\hat{G}} \psi(\omega)/(\widehat{\phi^{-1} f})(\omega)\, d\nu_{\phi^{-1} f}(\omega) = \Phi(\psi) \end{aligned}$$

Therefore Φ is a translation invariant, non-trivial, positive linear functional on $C_c(\hat{G})$, and consequently

$$\Phi(\psi) = \int_{\hat{G}} \psi \, d\hat{\mu}$$

for some Haar measure $\hat{\mu}$ on $\hat{G}$.

Finally, for $g \in B^1(G)$ $(*)$ implies

$$\int_{\hat{G}} \psi \, d\nu_g = \Phi(\psi \hat{g}) = \int_{\hat{G}} \psi \hat{g} \, d\hat{\mu},$$

so that $d\nu_g = \hat{g}\, d\hat{\mu}$. Consequently, $\|\hat{g}\|_1 = \|\nu_g\| < \infty$ and

$$g(x) = \int_{\hat{G}} (x|\omega)\, d\nu_g(\omega) = \int_{\hat{G}} (x|\omega)\, \hat{g}(\omega)\, d\hat{\mu}(\omega).$$

$\square$

$\boxed{\text{Def}}$ Let μ be a Haar measure on an abelian σ-compact group G. The Haar measure $\hat{\mu}$ on $\hat{G}$ satisfying

$$f(x) = \int_{\hat{G}} \hat{f}(\omega)(x|\omega)\, d\hat{\mu}(\omega)$$

for all $x \in G$ and $f \in B^1(G)$ is called the <u>dual measure</u> of μ. $\square$

Observe that that scaling μ by a constant $c > 0$ has the effect of scaling $\hat{f}$ by c. Consequently, $\widehat{c\mu} = \frac{1}{c}\hat{\mu}$.

$\boxed{\text{Corollary 4.18}}$ Let G be a σ-compact group. For $f \in L^1(G,\mu) \cap P(G)$, one has $\hat{f} \geq 0$.

~~Proof~~ We have $d\nu_f = \hat{f}\, d\hat{\mu}$ and since ν_f is a positive measure by Bochner's theorem, we must have $\hat{f}(\omega) \geq 0$ for $\hat{\mu}$-almost every $\omega \in \hat{G}$. But $\hat{f}$ is continuous, so $\hat{f} \geq 0$. $\square$

Ex ① Recall that $\hat{\mathbb{R}} \cong \mathbb{R}$ via the pairing $(s|t) = e^{2\pi i s t}$. We claim $\hat{m} = m$. Indeed for $g(s) = e^{-\pi s^2}$ observe that

$$\hat{g}(0) = \int_G g\, dm = 1$$

and

$$\hat{g}'(t) = \frac{d}{dt}\left(\int_{\mathbb{R}} e^{-\pi s^2} \cdot e^{-2\pi i s t}\, dm(s)\right) = \int_{\mathbb{R}} e^{-\pi s^2}(-2\pi i s) e^{-2\pi i s t}\, dm(s)$$

$u = e^{-2\pi i s t}$
$dv = -2\pi s e^{-\pi s^2}$

$$= 0 - i\int_{\mathbb{R}} e^{-\pi s^2}(-2\pi i t) e^{-2\pi i s t}\, dm(s) = -2\pi t\, \hat{g}(t)$$

Consequently, $\hat{g}(t) = e^{-\pi t^2}$. Since

$$\int_{\mathbb{R}} \hat{g}(t)(s|t)\, dm(t) = \int_{\mathbb{R}} e^{-\pi t^2} e^{2\pi i s t}\, dm(t) = \int_{\mathbb{R}} e^{-\pi t^2} e^{-2\pi i s t}\, dm(t) = \hat{\hat{g}}(s) = g(s)$$

by the above argument, we see that $m = \hat{m}$. Observe that if we use the pairing $(s|t) = e^{ist}$

$$\int_{\mathbb{R}} f(s) e^{-ist}\, dm(s) = \int_{\mathbb{R}} f(s)\, e^{2\pi i s\left(\frac{t}{2\pi}\right)}\, dm(s) = \hat{f}\left(\frac{t}{2\pi}\right)$$

is the new Fourier transform of

$$f(s) = \int_{\mathbb{R}} \hat{f}(t) e^{2\pi i s t}\, dm(t) = \int_{\mathbb{R}} \hat{f}\left(\frac{t}{2\pi}\right) e^{ist} \frac{1}{2\pi}\, dm(t)$$

so that the dual of m under this pairing is $\frac{1}{2\pi} m$. Consequently, $\widehat{\left(\frac{1}{\sqrt{2\pi}} m\right)} = \frac{1}{\sqrt{2\pi}} m$ under this pairing.

② Recall that $\hat{\mathbb{Q}}_p \cong \mathbb{Q}_p$ via $y \mapsto \omega_y$ where $(x|\omega_y) = (xy|\omega_1) = e^{2\pi i xy}$. If μ is such that $\mu(\mathbb{Z}_p) = 1$, then $\hat{\mu} = \mu$. Indeed, let $f := 1_{\mathbb{Z}_p}$. Since $\omega|_{\mathbb{Z}_p} \in \hat{\mathbb{Z}}_p$ for any $\omega \in \hat{\mathbb{Q}}_p$, Lemma 4.3 implies

$$\hat{f}(\omega) = \int_{\mathbb{Q}_p} f(x)\overline{(x|\omega)}\, d\mu(x) = \int_{\mathbb{Z}_p} \overline{\omega}\, d\mu = \langle 1, \omega|_{\mathbb{Z}_p}\rangle_2 = \delta_{\omega|_{\mathbb{Z}_p} = 1}$$

Recall that $\ker(\omega_y) = \{x \in \mathbb{Q}_p : |x|_p \le |y|_p^{-1}\}$, and hence $\omega_y|_{\mathbb{Z}_p} = 1$ if and only if $y \in \mathbb{Z}_p$. Thus $\hat{f}$ is the indicator function of $\{\omega_y : y \in \mathbb{Z}_p\}$. Viewing μ as a Haar measure on $\hat{\mathbb{Q}}_p$, and repeating the above computation gives

$$\int_{\hat{\mathbb{Q}}_p} \hat{f}(\omega)(x|\omega)\, d\mu(\omega) = 1_{\mathbb{Z}_p}(x) = f(x),$$

so that $\hat{\mu} = \mu$. $\square$

4/10

Proposition 4.19 If G is compact with Haar measure μ satisfying $\mu(G) = 1$, then $\hat{\mu}$ is the counting measure on $\hat{G}$. If G is a countable discrete group with counting measure $\#$, then $\hat{\#}(\hat{G}) = 1$.

Proof Suppose G is compact and denote $f := 1$. Then Lemma 4.3 implies

$$\hat{f}(\omega) = \int_G f(x)\overline{(x|\omega)}\, d\mu(x) = \int_G \overline{\omega}\, d\mu = \langle 1, \omega\rangle_2 = \delta_{\omega = 1} = 1_{\{1\}}(\omega).$$

Thus

$$\int_{\hat{G}} \hat{f}(\omega)\,(x|\omega)\,d\#(\omega) = \int_{\hat{G}} 1_{\{1\}}(\omega)\,(x|\omega)\,d\#(\omega) = (x|1) = 1 = f$$

so that $\hat{\mu} = \#$.

Next suppose G is a countable discrete group. For $g := 1_{\{1\}}$ we have
$$\hat{g}(\omega) = \int_G g(x)\,\overline{(x|\omega)}\,d\#(x) = \overline{(1|\omega)} = 1$$
so that $\hat{g} \equiv 1$ and
$$1 = g(1) = \int_{\hat{G}} \hat{g}(\omega)\,(1|\omega)\,d\hat{\#}(\omega) = \hat{\#}(\hat{G}).$$

□

Ex ① Recall $\hat{\mathbb{T}} \cong \mathbb{Z}$ and $\hat{\mathbb{Z}} \cong \mathbb{T}$. Identifying $[0,2\pi)$ with $\mathbb{T}$ via $\theta \mapsto e^{i\theta}$, $\frac{1}{2\pi}m$ corresponds to the normalized Haar measure on $\mathbb{T}$, and hence $\widehat{\frac{1}{2\pi}m} = \#$ by Proposition 4.19. Hence for $f \in B^1(\mathbb{T})\ (= B(\mathbb{T}))$ we have
$$\hat{f}(n) = \int_0^{2\pi} f(\theta)\,e^{-in\theta}\,\frac{1}{2\pi}\,dm(\theta) \qquad \text{and} \qquad f(\theta) = \sum_{n\in\mathbb{Z}} \hat{f}(n)\,e^{in\theta}$$

② Recall $\widehat{\mathbb{Z}/n\mathbb{Z}} \cong \mathbb{Z}/n\mathbb{Z}$. Proposition 4.19 implies $\hat{\#} = \frac{1}{n}\#$ and hence
$$\hat{f}(\ell) = \sum_{k=0}^{n} f(k)\,e^{-2\pi i k\ell/n} \qquad \text{and} \qquad f(k) = \frac{1}{n}\sum_{\ell=0}^{n} \hat{f}(\ell)\,e^{2\pi i k\ell/n}.$$

Note that $\frac{1}{\sqrt{n}}\#$ is self-dual.

□

Theorem 4.20 (The Plancherel Theorem) Let G be a σ-compact group. The Fourier transform on $L^1(G,\mu)\cap L^2(G,\mu)$ extends uniquely to a unitary $L^2(G,\mu)\to L^2(\hat{G},\hat{\mu})$.

Proof For $f \in L^1(G,\mu)\cap L^2(G,\mu)$, we have $f^* * f \in L^1(G,\mu)\cap P(G) \subset B^1(G)$ by Corollary 3.13. Hence the Fourier Inversion Theorem I (Theorem 4.17) implies
$$\|f\|_2^2 = f*f^*(1) = \int_{\hat{G}} \widehat{f*f^*}(\omega)\,(1|\omega)\,d\hat{\mu}(\omega) = \int_{\hat{G}} |\hat{f}(\omega)|^2\,d\hat{\mu}(\omega) = \|\hat{f}\|_2^2.$$
Thus $L^1(G,\mu)\cap L^2(G,\mu) \ni f \mapsto \hat{f} \in L^2(\hat{G},\hat{\mu})$ is isometric and extends uniquely to an isometry on $L^2(G,\mu)$ (Note $L^1(G,\mu)\cap L^2(G,\mu)$ is dense in $L^2(G,\mu)$ since, for example, it contains $C_c(G)$.) Toward showing the extension is a surjection (and hence unitary), suppose $\psi \in L^2(\hat{G},\hat{\mu})$ satisfies $\psi \perp \hat{f}$ for all $f\in L^1(G,\mu)\cap L^2(G,\mu)$. Proposition 4.12 implies that for all $x\in G$ and $f \in L^1(G,\mu)\cap L^2(G,\mu)$ that
$$0 = \int_{\hat{G}} \psi\,\overline{\mathcal{F}(\lambda_x f)}\,d\hat{\mu} = \int_{\hat{G}} \psi(\omega)\,(x|\omega)\,\overline{\hat{f}(\omega)}\,d\hat{\mu}(\omega)$$
Note that $\psi\overline{\hat{f}} \in L^1(\hat{G},\hat{\mu})$ implies $\nu := \psi\overline{\hat{f}}\,d\hat{\mu} \in M(\hat{G})$, and the above equals $\phi_\nu(x)$. Thus $\phi_\nu \equiv 0$, and so Proposition 4.13 implies $\nu = 0$. Hence $\psi\overline{\hat{f}} = 0$ $\hat{\mu}$-almost everywhere for all $f\in L^1(G,\mu)\cap L^2(G,\mu)$. For $K \subset \hat{G}$ compact, we can find $f\in C_c(G)\cap P(G)$ with $\hat{f} > 0$ on K by Lemma 4.15, and consequently $\psi|_K = 0$ $\hat{\mu}$-almost everywhere. We claim this implies $\psi = 0$ $\hat{\mu}$-almost everywhere. Indeed, otherwise $E_n := \{\omega\in\hat{G} : |\psi(\omega)| > \frac{1}{n}\}$ has positive measure for some $n\in\mathbb{N}$. Since $\psi \in L^2(\hat{G},\hat{\mu})$,

we also have $\hat{\mu}(E_n) < \infty$, and so by inner regularity there exists $K \subset E_n$ with $\hat{\mu}(K) > 0$. But $\psi|_K = 0$ $\hat{\mu}$-almost everywhere contradicts this. Thus $\psi = 0 \in L^2(\hat{G}, \hat{\mu})$, and therefore the extension of the Fourier transform is surjective. □

Corollary 4.21 For a compact group G with Haar measure μ satisfying $\mu(G) = 1$, $\hat{G}$ is an orthonormal basis for $L^2(G, \mu)$.

Proof Lemma 4.3 implies $\hat{G}$ is an orthonormal set. If $f \in L^2(G, \mu)$ satisfies $f \perp \omega$ for all $\omega \in \hat{G}$, then

$$0 = \int_G f \overline{\omega} \, d\mu = \hat{f}(\omega)$$

for all $\omega \in \hat{G}$. Thus $\|f\|_2 = \|\hat{f}\|_2 = 0$ by the Plancherel theorem. □

Def A locally compact group G is <u>second countable</u> if it admits a countable base; that is, there exists a countable family $\{U_i : i \in I\}$ of open subsets of G such that for all $U \subset G$ open and $x \in U$, one has $x \in U_i \subset U$ for some $i \in I$. □

Exercise Show that a locally compact second countable group is σ-compact. □

4/11

Lemma 4.22 A locally compact group G is second countable if and only if $L^2(G, \mu)$ is separable.

Proof ($\Rightarrow$) Let $\{U_i : i \in I\}$ be a countable base for G. Let $\Sigma = \{U_{i_1} \cup \cdots \cup U_{i_n} : n \in \mathbb{N},\ i_1, \ldots, i_n \in I\}$, which is still countable. For $E \subset G$ Borel with $\mu(E) < \infty$ and $\varepsilon > 0$, we can find $U \supset E$ open with $\|1_U - 1_E\|_2 = \mu(U \setminus E)^{1/2} < \varepsilon$. Since U is a countable union of U_i's, continuity from below implies we can find $V \in \Sigma$ satisfying $V \subset U$ and $\|1_V - 1_U\|_2 = \mu(U \setminus V)^{1/2} < \varepsilon$. Hence

$$\|1_V - 1_E\|_2 \leq \|1_V - 1_U\|_2 + \|1_U - 1_E\|_2 < \varepsilon.$$

It follows that the closure of $\operatorname{span}\{1_V : V \in \Sigma, \mu(V) < \infty\}$ in $L^2(G, \mu)$ contains all 1_E for $E \subset G$ Borel with $\mu(E) < \infty$. Thus $\overline{\operatorname{span}}\{1_V : V \in \Sigma, \mu(V) < \infty\} = L^2(G, \mu)$, which is separable since Σ is countable.

($\Leftarrow$) Let $\{f_i \in L^2(G, \mu) : i \in I\}$ be a countable dense subset. By approximating each f_i by a sequence of functions in $C_c(G)$, we can assume $f_i \in C_c(G)$ for all $i \in I$. Then

$$U_i := \{x \in G : 0 < |f_i(x)| < 1\}$$

forms a countable base for G (Exercise: show this). □

Corollary 4.23 If G is an abelian locally compact second countable group, then so is $\hat{G}$.

Proof Lemma 4.22 implies $L^2(G, \mu)$ is separable, and hence $L^2(\hat{G}, \hat{\mu})$ is separable by the Plancherel theorem. Hence $\hat{G}$ is second countable by the other direction of Lemma 4.22. □

Remark If G is an abelian σ-compact group, then $\hat{G}$ need not be σ-compact. Indeed, if G is compact but not second countable then $\hat{G}$ is discrete with $L^2(\hat{G}, \#) = \ell^2(\hat{G})$ non-separable. This requires $\hat{G}$ be uncountable, and hence not σ-compact. For a concrete example consider $\prod_{t \in \mathbb{R}} \mathbb{T}$, whose dual is $\bigoplus_{t \in \mathbb{R}} \mathbb{Z}$ by Corollary 4.7. $\square$

4.3 The Pontrjagin Duality Theorem

Let G be an abelian locally compact second countable group, and for $x \in G$ define $\check{x} : \hat{G} \to \mathbb{T}$ by $\check{x}(\omega) = (x|\omega)$. Then $\check{x} \in \hat{\hat{G}}$ with

$$(\omega | \check{x}) = (x|\omega)$$

Observe that $x \mapsto \check{x}$ is a group homomorphism:

$$(\omega | (xy)^{\vee}) = (xy|\omega) = (x|\omega)(y|\omega) = (\omega|\check{x})(\omega|\check{y}) = (\omega|\check{x}\check{y})$$

In this section, we will show this map is actually an isomorphism of topological groups. First, we require a few technical lemmas.

Lemma 4.24 For $\phi, \psi \in C_c(\hat{G})$ there exists $h \in B^1(G)$ satisfying $\hat{h} = \phi * \psi$. Consequently, $\mathcal{F}(B^1(G))$ is dense in $L^p(\hat{G}, \hat{\mu})$ for $1 \leq p < \infty$.

Proof Define

$$f(x) := \int_{\hat{G}} (x|\omega)\phi(\omega)\, d\hat{\mu}(\omega) \qquad g(x) := \int_{\hat{G}} (x|\omega)\psi(\omega)\, d\hat{\mu}(\omega) \qquad h(x) := \int_{\hat{G}} (x|\omega)\, (\phi*\psi)(\omega)\, d\hat{\mu}(\omega)$$

That is, f, g, h are the images of $\phi\, d\hat{\mu}$, $\psi\, d\hat{\mu}$, $\phi*\psi\, d\hat{\mu} \in M(\hat{G})$ under the map from Bochner's theorem (Theorem 4.14), and hence $f, g, h \in B(G)$. For $u \in L^1(G,\mu) \cap L^2(G,\mu)$ we also have

$$\left| \int_G f \bar{u}\, d\mu \right| = \left| \int_G \int_{\hat{G}} (x|\omega)\, \phi(\omega)\, \bar{u}(x)\, d\hat{\mu}(\omega)\, d\mu(x) \right| = \left| \int_{\hat{G}} \phi\, \bar{\hat{u}}\, d\hat{\mu} \right| \leq \|\phi\|_2 \|\hat{u}\|_2 = \|\phi\|_2 \|u\|_2$$

where the last equality follows from the Plancherel theorem (Theorem 4.20). Hence $f \in L^2(G,\mu)$ with $\|f\|_2 \leq \|\phi\|_2$. Similarly, $\|g\|_2 \leq \|\psi\|_2$ and $\|h\|_2 \leq \|\phi*\psi\|_2$. Next observe that

$$h(x) = \int_{\hat{G}} \int_{\hat{G}} (x|\omega)\, \phi(\omega\eta^{-1})\, \psi(\eta)\, d\hat{\mu}(\eta)\, d\hat{\mu}(\omega) = \int_{\hat{G}} \int_{\hat{G}} (x|\omega\eta)\, \phi(\omega)\, \psi(\eta)\, d\hat{\mu}(\eta)\, d\hat{\mu}(\omega) = f(x) g(x),$$

and hence $h \in L^1(G,\mu)$. Thus $h \in B^1(G)$, and so the Fourier Inversion Theorem I (Theorem 4.17) yields

$$h(x) = \int_{\hat{G}} (x|\omega)\, \hat{h}(\omega)\, d\hat{\mu}(\omega)$$

Thus h is also the image of $\hat{h}\, d\hat{\mu}$ under the map from Bochner's theorem, and so the injectivity of this map (Proposition 4.13) implies $\hat{h}\, d\hat{\mu} = (\phi * \psi)\, d\hat{\mu}$. Hence $\hat{h} = \phi*\psi$ $\hat{\mu}$-almost everywhere, but as continuous maps we therefore have $\hat{h} = \phi * \psi$ everywhere.

Hence $\{\phi * \psi : \phi, \psi \in C_c(\hat{G})\} \subset \mathcal{F}(B^1(G))$, and the former set is dense in $L^p(\hat{G}, \hat{\mu})$ for $1 \leq p < \infty$ (see the proof of Proposition 3.26). $\square$

4/15

Lemma 4.25 Let G be a locally compact group and let H be a subgroup equipped with the relative topology. Then H is locally compact if and only if H is closed.

Proof $(\Rightarrow)$ Let $V \subset H$ be a neighborhood of 1 whose closure K (in H) is compact in H. Then $V = H \cap U$ for $U \subset G$ a neighborhood of 1, and K is still compact in G (Exercise: show this).

But then K is closed in G and hence is the closure (in G) of $H \cap U$. Now, let $x \in \overline{H}$ and let $(x_i)_{i\in I} \subset H$ be a net converging to x. If $W \subset G$ is a symmetric neighborhood of 1 satisfying $WW \subset U$, then $Wx^{-1} \cap H \neq \emptyset$ (note $x^{-1} \in \overline{H}$ since $\overline{H}$ is a subgroup). Let $y \in Wx^{-1} \cap H$ and let $i_0 \in I$ be such that $x_i \in xW$ for all $i \geq i_0$. Then for $i \geq i_0$ we have

$$y x_i \in (Wx^{-1})(xW) = WW \subset U$$

Since $y, x_i \in H$, we further have $yx_i \in H \cap U \subset K$. Since $yx_i \to yx$, it follows that $yx \in K \subset H$. But then $x = y^{-1} y x \in H \cdot H = H$, and so H is closed.

($\Longleftarrow$) Let $K \subset G$ be a compact set with $1 \in K^\circ$. Since H is closed, $H \cap K$ is compact in H (Exercise: check this). Also, $1 \in H \cap K^\circ$, which is a relatively open set contained in $H \cap K$. Hence 1 lies in the the H-interior of $H \cap K$. That is, $H \cap K$ is a compact neighborhood of 1 and therefore H is locally compact. $\square$

$\boxed{\text{Theorem 4.26}}$ (The Pontrjagin Duality Theorem) Let G be an an abelian locally compact second countable group. The map

$$G \ni x \longmapsto \check{x} \in \hat{\hat{G}}$$

is an isomorphism of topological groups.

Proof We saw at the beginning of this section that $x \mapsto \check{x}$ is a group homomorphism. Recall that $\hat{G}$ separates points in G by the Gelfand-Raikov theorem (Theorem 3.27). Thus for $x, y \in G$ distinct there exists $\omega \in \hat{G}$ so that

$$(\omega \mid \check{x}) = (x \mid \omega) \neq (y \mid \omega) = (\omega \mid \check{y})$$

Therefore $\check{x} \neq \check{y}$ and the map $x \mapsto \check{x}$ is injective. Before showing the map is surjective, we will show it is a homeomorphism onto its image.

Suppose $(x_i)_{i\in I} \subset G$ converges to $x_0 \in G$. Then for all $f \in B^1(G)$, the Fourier Inversion Theorem I (Theorem 4.17) implies

$$\int_{\hat{G}} \check{x}_0 \hat{f} \, d\hat{\mu} = \int_{\hat{G}} (x_0 \mid \omega) \hat{f}(\omega) \, d\hat{\mu}(\omega) = f(x_0) = \lim_{i\to\infty} f(x_i) = \lim_{i\to\infty} \int_{\hat{G}} \check{x}_i \hat{f} \, d\hat{\mu}$$

Since $\|\check{x}_i\|_\infty = 1$ for all $i \in I$ and $\mathcal{F}(B^1(G))$ is dense in $L^1(\hat{G}, \hat{\mu})$ by Lemma 4.24, the above implies $\check{x}_i \to \check{x}_0$ in the weak* topology on $\hat{\hat{G}} \subset L^\infty(\hat{G}, \hat{\mu})$. Therefore $x \mapsto \check{x}$ is continuous. Conversely, suppose $\check{x}_i \to \check{x}_0$ in $\hat{\hat{G}}$. The computation above implies $f(x_i) \to f(x_0)$ for all $f \in B^1(G)$. So if we assume, towards a contradiction, that $x_i \not\to x_0$, then we can find a neighborhood $U \subset G$ of x and a subnet $(y_j)_J$ so that $y_j \notin U$ for all $j \in J$. Recall that $C_c(G) \cap P(G) \subset B^1(G)$ has dense span in $C_c(G)$ by Proposition 3.26, and so there exists $g \in B^1(G)$ with $\operatorname{supp}(g) \subset U$ and $g(x_0) \neq 0$. But then we obtain the contradiction

$$0 \neq g(x_0) = \lim_{j\to\infty} g(x_j) = 0.$$

So $\chi_i \to \chi_0$ and hence $x \mapsto \check{x}$ is a homeomorphism onto its image. Since G is locally compact, it follows that $\check{G} \subset \hat{\hat{G}}$ is a locally compact subgroup, and hence closed by Lemma 4.25.

Finally, suppose towards another contradiction that there exists $\xi \in \hat{\hat{G}} \setminus \check{G}$. Let $V \subset \hat{\hat{G}}$ be a symmetric neighborhood of 1 satisfying $\xi V V \subset \hat{\hat{G}} \setminus \check{G}$, and let $\phi, \psi \in C_c^+(\hat{\hat{G}}) \setminus \{0\}$ with $\operatorname{supp} \phi \subset \xi V$ and $\operatorname{supp} \psi \subset V$. Then $\phi * \psi$ is non-zero with

$$\operatorname{supp}(\phi * \psi) \subset \operatorname{supp}(\phi) \cdot \operatorname{supp}(\psi) \subset \xi V V \subset \hat{\hat{G}} \setminus \check{G}.$$

Also, $\phi * \psi = \hat{h}$ for some $h \in B^1(\hat{G})$ by Lemma 4.24. But then for all $x \in G$

$$0 = \hat{h}(\check{x}^{-1}) = \int_{\hat{G}} \overline{(\omega \mid \check{x}^{-1})}\, h(\omega)\, d\hat{\mu}(\omega) = \int_{\hat{G}} (x \mid \omega)\, h(\omega)\, d\hat{\mu}(\omega).$$

This implies, by Proposition 4.13, that $h\, d\hat{\mu} \in M(\hat{G})$ is zero. Thus $h = 0$ $\hat{\mu}$-almost everywhere, but this gives the contradiction $\phi * \psi = \hat{h} = 0$. Therefore $\check{G} = \hat{\hat{G}}$. □

We will henceforth identify $\hat{\hat{G}}$ with G. We will also write either $(x \mid \omega)$ or $(\omega \mid x)$ for pairings between $x \in G$ and $\omega \in \hat{G}$. The Pontrjagin duality theorem yields a number of important corollaries.

Theorem 4.27 (The Fourier Inversion Theorem II) Let G be an abelian locally compact second countable group. If $f \in L^1(G, \mu)$ satisfies $\hat{f} \in L^1(\hat{G}, \hat{\mu})$, then

$$f(x) = \hat{\hat{f}}(x^{-1}) = \int_{\hat{G}} (x \mid \omega)\, \hat{f}(\omega)\, d\hat{\mu}(\omega)$$

for μ-almost everywhere $x \in G$. If f is continuous, then the above holds for all $x \in G$.

Proof First note

$$\hat{f}(\omega) = \int_G \overline{(x \mid \omega)}\, f(x)\, d\mu(x) = \int_G (x^{-1} \mid \omega) f(x)\, d\mu(x) = \int_G (x \mid \omega) f(x^{-1})\, d\mu(x),$$

which implies $\hat{f} \in B^1(\hat{G})$ with $d\nu_{\hat{f}} = f(x^{-1})\, d\mu(x)$. The Fourier Inversion Theorem I (Theorem 4.17) then implies

$$\hat{f}(\omega) = \int_G (\omega \mid x)\, \hat{\hat{f}}(x)\, d\mu(x).$$

Thus $\hat{\hat{f}}\, d\mu = d\nu_{\hat{f}} = f(x^{-1})\, d\mu(x)$ so that $\hat{\hat{f}}(x) = f(x^{-1})$ for μ-almost every $x \in G$. Since $\hat{\hat{f}}$ is automatically continuous, if f is continuous then this equality holds everywhere. □

Corollary 4.28 (The Fourier Uniqueness Theorem) Let G be an abelian locally compact second countable group. The Fourier transform on $M(G)$, $\nu \mapsto \hat{\nu}$, is injective. In particular, if $\hat{f} = \hat{g}$ for $f, g \in L^1(G, \mu)$ then $f = g$ μ-almost everywhere.

Proof Reversing the roles of G and $\hat{G}$ in Proposition 4.13, we see that $\nu \mapsto \phi_\nu$ is injective. But

$$\phi_\nu(\omega) = \int_G (x \mid \omega)\, d\nu(x) = \int_G \overline{(x \mid \omega^{-1})}\, d\nu(x) = \hat{\nu}(\omega^{-1})$$

so $\nu \mapsto \hat{\nu}$ is injective. □

Proposition 4.29 Let G be an abelian locally compact second countable group. If $\hat{G}$ is compact (resp. discrete) then G is discrete (resp. compact).

Proof Since $G \cong \hat{\hat{G}}$, this follows from Proposition 4.4. $\square$

Proposition 4.30 Let G be an abelian locally compact second countable group. For $f, g \in L^2(G, \mu)$ one has $\widehat{(fg)} = \hat{f} * \hat{g}$.

Proof First suppose $f, g \in L^2(G,\mu) \cap \mathcal{F}(B^1(\hat{G}))$ with $f = \hat{\phi}$ and $g = \hat{\psi}$ for $\phi, \psi \in L^2(\hat{G}, \hat{\mu}) \cap B^1(\hat{G})$. Then $\widehat{\phi * \psi} = \hat{\phi}\hat{\psi} = fg$. Also $\phi(\omega^{-1}) = \hat{f}(\omega)$ and $\psi(\omega^{-1}) = \hat{g}(\omega)$ by Theorem 4.17. Since $\phi * \psi \in L^1(\hat{G}, \hat{\mu})$ and $\widehat{\phi * \psi} = fg \in L^1(G, \mu)$, Theorem 4.27 implies

$$\phi * \psi(\omega) = \widehat{\widehat{(\phi * \psi)}}(\omega^{-1}) = \widehat{(fg)}(\omega^{-1})$$

On the other hand,

$$\phi * \psi(\omega) = \int_{\hat{G}} \phi(\omega\eta^{-1})\psi(\eta)\, d\hat{\mu}(\eta) = \int_{\hat{G}} \hat{f}(\omega^{-1}\eta)\, \hat{g}(\eta^{-1})\, d\hat{\mu}(\eta) = \hat{f} * \hat{g}(\omega^{-1})$$

Thus $\widehat{fg} = \hat{f} * \hat{g}$.

It remains to remove the assumption that $f, g \in \mathcal{F}(B^1(\hat{G}))$. By Lemma 4.24 we can find sequences $(f_n)_{n\in\mathbb{N}}, (g_n)_{n\in\mathbb{N}} \subset L^2(G,\mu) \cap \mathcal{F}(B^1(\hat{G}))$ with $\|f - f_n\|_2, \|g - g_n\|_2 \to 0$. Then $\|fg - f_n g_n\|_1 \to 0$ so that $\widehat{f_n g_n} \to \widehat{fg}$ uniformly. Also

$$\begin{aligned}
|\hat{f} * \hat{g}(\omega) - \hat{f}_n * \hat{g}_n(\omega)| &\le |(\hat{f} - \hat{f}_n) * \hat{g}(\omega) + \hat{f}_n * (\hat{g} - \hat{g}_n)(\omega)| \\
&\le \int_{\hat{G}} |(\hat{f} - \hat{f}_n)(\omega\eta^{-1})| \cdot |\hat{g}(\eta)| + |\hat{f}_n(\omega\eta^{-1})|\, |(\hat{g} - \hat{g}_n)(\eta)|\, d\hat{\mu}(\eta) \\
&\le \|\hat{f} - \hat{f}_n\|_2 \cdot \|\hat{g}\|_2 + \|\hat{f}_n\|_2 \cdot \|\hat{g} - \hat{g}_n\|_2 \\
&= \|f - f_n\|_2 \|g\|_2 + \|f_n\|_2 \|g - g_n\|_2 \to 0,
\end{aligned}$$

where the last equality uses the Plancherel theorem (Theorem 4.20). $\square$

For a closed subgroup $H \le G$, denote

$$H^\perp := \{\omega \in \hat{G} : (x|\omega) = 1 \text{ for all } x \in H\}.$$

Observe that $H^\perp$ is a closed subgroup of $\hat{G}$.

Proposition 4.31 $(H^\perp)^\perp = H$ for any closed subgroup $H \le G$.

Proof $H \subset (H^\perp)^\perp$ follows from definition. Let $q : G \to G/H$ be the quotient map. For $x_0 \notin H$, the Gelfand–Raikov theorem (Theorem 3.27) applied to G/H yields $\eta \in \widehat{G/H}$ with $(q(x_0)|\eta) \ne 1$. But then $\eta \circ q \in H^\perp$ with $(x_0 | \eta\circ q) \ne 1$ so that $x_0 \notin (H^\perp)^\perp$. Thus $G \setminus H \subset G \setminus (H^\perp)^\perp$ which yields $H = (H^\perp)^\perp$. $\square$

Theorem 4.32 For a closed subgroup $H \le G$ let $q : G \to G/H$ be the quotient map. Then

$$\Phi : \widehat{G/H} \to H^\perp \qquad \text{and} \qquad \Psi : \hat{G}/H^\perp \to \hat{H}$$
$$\eta \mapsto \eta \circ q \qquad\qquad\qquad \omega H^\perp \mapsto \omega|_H$$

are isomorphisms of topological groups.

Proof A routine computation shows Φ is a group homomorphism, and it is injective since q is surjective. Also for $\omega \in H^\perp$, if $x = ya$ for $y \in G$ and $a \in H$ then $\omega(x) = \omega(y)\omega(a) = \omega(y)$. Thus $\eta(xH) := \omega(x)$ is a well-defined character on $\widehat{G/H}$ and it satisfies $\eta \circ q(x) = \eta(xH) = \omega(x)$. Hence Φ is surjective. Now suppose $(\eta_i)_{i \in I} \subset \widehat{G/H}$ converge to $\eta \in \widehat{G/H}$. Then for $K \subset G$ compact we have $q(K) \subset G/H$ is compact and hence $\eta_i \circ q \to \eta \circ q$ uniformly on K. Hence $\eta_i \circ q \to \eta \circ q$ in $H^\perp \leq \hat{G}$. Conversely, suppose $\eta_i \circ q \to \eta \circ q$ in $\hat{G}$ and let $F \subset G/H$ be compact. We claim there exists $K \subset G$ compact with $q(K) = F$. Indeed, let $V \subset G$ be a precompact open neighborhood of 1. Since q is open, $q(xV)$ is open for all $x \in G$ and therefore $\{q(xV) : x \in G\}$ is an open cover of F. Let $q(x_1V) \cup \cdots \cup q(x_nV)$ be a finite subcover. Then

$$K := \overline{x_1V} \cup \cdots \cup \overline{x_nV} \cap q^{-1}(F)$$

is compact with $q(K) = F$. Since $\eta_i \circ q \to \eta \circ q$ uniformly on K, it follows that $\eta_i \to \eta$ uniformly on F. Thus $\eta_i \to \eta$ in $\widehat{G/H}$ and therefore Φ is a homeomorphism.

The above with G replaced by $\hat{G}$ and H replaced by $H^\perp$ yields

$$\widehat{(\hat{G}/H^\perp)} \cong (H^\perp)^\perp = H,$$

where the equality follows from Proposition 4.31. Recalling how we showed Φ was surjective, this means for $x \in H$ that the corresponding $\eta \in \widehat{(\hat{G}/H^\perp)}$ is given by

$$(\omega H^\perp \mid \eta) := (x \mid \omega).$$

Pontrjagin duality (Theorem 4.26) yields $\hat{G}/H^\perp \cong \widehat{\widehat{(\hat{G}/H^\perp)}} \cong \hat{H}$. The above pairing implies $\omega H^\perp$ corresponds to $\omega|_H = \Psi(\omega)$. Hence Ψ is an isomorphism of topological groups. □

Since $\Psi : \hat{G}/H^\perp \to \hat{H}$ is surjective, for any $\omega \in \hat{H}$ we have $\omega = \Psi(\tilde{\omega} H^\perp) = \tilde{\omega}|_H$ for some $\tilde{\omega} \in \hat{G}$. That is, ω admits an extension to G. Note, however, that the extension is not unique: any $\tilde{\tilde{\omega}} \in \hat{G}$ with $\tilde{\omega} H^\perp = \tilde{\tilde{\omega}} H^\perp$ will also extend ω. We record this Hahn-Banach type result below.

Corollary 4.33 Every character on a closed subgroup $H \leq G$ extends to a character on G.

Ex Recall $\hat{\mathbb{Q}}_p \cong \mathbb{Q}_p$ via $\omega_y \leftrightarrow y$. Then $\mathbb{Z}_p^\perp = \mathbb{Z}_p$ (Exercise: check this), and so $\hat{\mathbb{Z}}_p \cong \mathbb{Q}_p/\mathbb{Z}_p$ by Theorem 4.32. Noting that $\ker(\omega_1) = \mathbb{Z}_p$ so that

$$\begin{tikzcd}
\mathbb{Q}_p \arrow[r, "\omega_1"] \arrow[d, "q"'] & \operatorname{ran}(\omega_1) \leq \mathbb{T} \\
\hat{\mathbb{Z}}_p \cong \mathbb{Q}_p/\mathbb{Z}_p \arrow[ur, dashed, "\cong"'] &
\end{tikzcd}$$

Thus $\hat{\mathbb{Z}}_p$ is isomorphic as a group to $\operatorname{ran}(\omega_1)$, which is $U_p := \{\theta \in \mathbb{T} : \theta^{p^k} = 1, k \geq 1\}$. Since $\hat{\mathbb{Z}}_p$ is discrete by virtue of $\mathbb{Z}_p$ being compact, we actually have $\hat{\mathbb{Z}}_p \cong U_p$ when U_p is given its discrete topology. □