Nets

Brent Nelson

Roughly speaking, nets are a generalization of sequences wherein the indexing set \( \mathbb{N} \) is replaced by a directed set. As the name suggests, these sets have a notion of direction much like \( \mathbb{N} \) does (1 \( \to \) 2 \( \to \) 3 \( \cdots \)), however they may be uncountable and may have multiple paths to “infinity.” The elements that are indexed by a directed set live in a topological space so that one can consider the notion of convergence of a net. Net are essential for general topology in the sense that they can characterize closedness, compactness, and continuity in the same way that sequences do in metric spaces.

1 Directed Sets

Definition 1.1. A directed set \( I \) is a set equipped with a binary relation \( \leq \) that satisfies:

(i) \( i \leq i \) for all \( i \in I \) (reflexive);

(ii) if \( i \leq j \) and \( j \leq k \), then \( i \leq k \) (transitive);

(iii) for any \( i,j \in I \) there exists \( k \in I \) with \( i,j \leq k \) (upper bound property).

Typically reflexivity and transitivity are obvious, whereas the upper bound property may need to be justified.

Example 1.2. \( \mathbb{N}, \mathbb{Z}, \mathbb{Q}, \mathbb{R} \) are all directed sets with the usual ordering. In fact, any subset of \( \mathbb{R} \) (even finite ones) are directed sets with the order they inherit from \( \mathbb{R} \).

Example 1.3. Let \( X \) be a set, and let \( \mathcal{F} \) denote the collection of all finite subsets of \( X \). For \( A, B \in \mathcal{F} \), write \( A \leq B \) if \( A \subset B \). This makes \( \mathcal{F} \) into a directed set. Note that \( A \cup B \) serves as an upper bound for both \( A \) and \( B \).

Example 1.4. Let \( X \) be a topological space, and fix \( x_0 \in X \). Let \( \mathcal{N}(x_0) \) denote the collection of open neighborhoods of \( x_0 \). For \( A, B \in \mathcal{N}(x_0) \), write \( A \leq B \) if \( A \supset B \). This makes \( \mathcal{N}(x_0) \) into a directed set where \( A \cap B \) is an upper bound for \( A \) and \( B \).

Example 1.5. Let \( X \) be a topological space. Then \( \{(\epsilon,K) : \epsilon > 0, K \subset X \text{ compact}\} \) is a directed set where

\[
(\epsilon,K) \leq (\epsilon',K')
\]

if and only if \( \epsilon \geq \epsilon' \) and \( K \subset K' \). (Exercise: determine a common upper bound for \( (\epsilon,K) \) and \( (\epsilon',K') \).)

2 Nets

Definition 2.1. Let \( X \) be a topological space. A net in \( X \) is a map \( x : I \rightarrow X \) where \( I \) is a directed set.

A net \( x : I \rightarrow X \) is usually denoted \( (x(i))_{i \in I} \) or \( (x_i)_{i \in I} \) where \( x_i := x(i) \). This is supposed to remind you of sequence notation. As with sequences in a metric space, there is a notion of convergence:

Definition 2.2. A net \( (x_i)_{i \in I} \) converges to \( x \in X \) if for every open subset \( U \subset X \) containing \( x \) there is \( i_0 \in I \) so that \( x_i \in U \) whenever \( i \geq i_0 \). In this case we call \( x \) the limit of the net and write

\[
x = \lim_{i \to \infty} x_i.
\]
When \( I = \mathbb{N} \), this is simply the usual notion of convergence for a sequence. When \( I = \mathbb{R} \) this is also capturing familiar behavior:

**Example 2.3.** Let \( f: \mathbb{R} \to \mathbb{R} \) be a function. Recall that we say \( f \) has a limit at \( \infty \) if there exists \( L \in \mathbb{R} \) so that for all \( \epsilon > 0 \) there exists \( t_0 \in \mathbb{R} \) so that

\[
|f(t) - L| < \epsilon \quad \forall t \geq t_0.
\]

But this is precisely saying that the net \((f(t))_{t \in \mathbb{R}}\) converges to \( L \).

**Example 2.4.** Let \( X \) be a topological space, let \( x_0 \in X \) and let \( \mathcal{N}(x_0) \) be as in Example 1.4. For each \( U \in \mathcal{N}(x_0) \) pick any point in \( U \) and label it \( x_U \). Then \((x_U)_{U \in \mathcal{N}(x_0)}\) is a net which converges to \( x_0 \). Indeed, let \( U \subset X \) be an open set containing \( x_0 \). Then \( U \in \mathcal{N}(x_0) \) and for any \( U' \in \mathcal{N}(x_0) \) with \( U' \supset U \), we have \( x_{U'} \in U' \subset U \).

**Example 2.5.** Let \( X \) be a topological space and let \( f: X \to \mathbb{C} \) be a function. For each pair \((\epsilon, K)\) as in Example 1.5, let \( f_{(\epsilon, K)} \) be any function \( g: X \to \mathbb{C} \) satisfying \(|f(x) - g(x)| < \epsilon \) for all \( x \in K \). Then the net \((f_{(\epsilon, K)})\) converges to \( f \) in the topology of uniform convergence on compact subsets. Indeed, fix \( K \subset X \) compact. Let \( \epsilon > 0 \), then for any \((\epsilon', K') \geq (\epsilon, K)\) we have \(|f(x) - f_{(\epsilon', K')}| < \epsilon' \leq \epsilon \) for all \( x \in K' \); in particular, for all \( x \in K \).

**Proposition 2.6.** Let \( X \) be a topological space. Then \( V \subset X \) is closed if and only if for every convergent net \((x_i)_{i \in I} \subset V \) one has \( \lim_i x_i \in V \).

**Proof.** (\( \Rightarrow \)): Let \((x_i)_{i \in I} \subset V \) be a convergent net. Suppose, towards a contradiction, that \( x := \lim_i x_i \) is not contained in \( V \). Then \( x \notin V \) which is an open set. Consequently, by definition of the convergence of a net, there exists \( i_0 \in I \) such that \( x_i \in V \) for all \( i \geq i_0 \). But this contradicts \( x_i \notin V \) for all \( i \in I \). Thus it must be that \( x \in V \).

(\( \Leftarrow \)): To show that \( V \) is closed, we will show that \( V^c \) is open. Suppose, towards a contradiction, that there exists \( x \in V^c \) such that for all open subsets \( U \) containing \( x \) one has \( U \cap V \neq \emptyset \). Let \( \mathcal{N}(x) \) be as in Example 1.4. For each \( U \in \mathcal{N}(x) \), let \( x_U \in U \cap V \). Then \((x_U)_{U \in \mathcal{N}(x)} \subset V \) and it converges to \( x \) by Example 2.4. By assumption we must have \( x \in V \), but this contradicts \( x \in V^c \). Thus for any \( x \in V^c \) there is an open set containing which does not intersect \( V \); that is, \( V^c \) is open.

We say a subset \( S \subset X \) in a topological space is **sequentially closed** if whenever \((x_n)_{n \in \mathbb{N}} \subset S \) is a convergent sequence one has \( \lim_n x_n \in S \). Since sequences are particular kinds of nets, the above proposition implies that closed sets are sequentially closed. In a metric space, the two notions are equivalent. However, for general topological spaces sequentially closed does not imply closed, as the following example illustrates.

**Example 2.7.** \(^1\) Consider \( \mathbb{R}^\mathbb{R} \) with the product topology, which we think of as arbitrary functions \( f: \mathbb{R} \to \mathbb{R} \). Recall that under the product topology, an open subset of \( \mathbb{R}^\mathbb{R} \) is a union of subsets of the form

\[
\prod_{t \in \mathbb{R}} U_t,
\]

where \( U_t \subset \mathbb{R} \) is open for all \( t \in \mathbb{R} \) and \( U_t \neq \mathbb{R} \) for only finitely many \( t \in \mathbb{R} \). Consequently, a net \((f_t)_{t \in I} \subset \mathbb{R}^\mathbb{R} \) converges to \( f \in \mathbb{R}^\mathbb{R} \) if and only if they converge pointwise as functions on \( \mathbb{R} \). Let \( B \) be the subset of Borel functions. Then \( B \) is sequentially closed because we know from measure theory that the pointwise limit of a sequence of Borel functions is Borel. \( B \) is also dense. Indeed, let \( f \in \mathbb{R}^\mathbb{R} \). Let \( \mathcal{F} \) be the collection of finite subsets of \( \mathbb{R} \), ordered by inclusion. Then for each \( F \in \mathcal{F} \) we can find a polynomial \( p_F \) such that \( p_F(t) = f(t) \) for each \( t \in F \). The net \((p_F)_{F \in \mathcal{F}} \) converges pointwise to \( f \) and consists of Borel functions. Therefore the closure of \( B \) is all of \( \mathbb{R}^\mathbb{R} \). On the other hand, we know there are non-Borel functions so \( B \) is not closed.

**Proposition 2.8.** Let \( X \) and \( Y \) be topological spaces. Then \( f: X \to Y \) is continuous if and only if for every convergent net \((x_i)_{i \in I} \subset X \) one has that \((f(x_i))_{i \in I} \subset Y \) is a convergent net with \( \lim_i f(x_i) = f(\lim_i x_i) \).

\(^1\)Thanks to Ben Hayes for supplying this example.

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Proof. \((\Rightarrow)\): Suppose \(f\) is continuous and \((x_i)_{i \in I} \subset X\) converges to some \(x \in X\). Let \(U \subset Y\) be an open subset containing \(f(x)\). Then \(f^{-1}(U) \subset X\) is an open subset containing \(x\). Consequently there exists \(i_0 \in I\) such that for all \(i \geq i_0\) we have \(x_i \in f^{-1}(U)\). Thus for all \(i \geq i_0\) we have \(f(x_i) \in U\). So \((f(x_i))_{i \in I}\) converges to \(f(x)\).

\((\Leftarrow)\): Let \(U \subset Y\) be an open subset. We must show \(f^{-1}(U)\) is open. If not, then there is an \(x \in f^{-1}(U)\) such that \(N \cap f^{-1}(U)^c \neq \emptyset\) for all \(N \in \mathcal{M}(x)\). We can then define a net by letting \(x_N \in N \cap f^{-1}(U)^c\) for each \(N \in \mathcal{M}(x)\). Then the net \((x_N)_{N \in \mathcal{M}(x)}\) converges to \(x\) by Example 2.4. By construction, \(f(x_N) \in U^c\) for all \(N \in \mathcal{M}(x)\). By assumption, \((f(x_N))_{N \in \mathcal{M}(x)}\) converges to \(f(x)\), and since \(U^c\) is closed the previous proposition implies \(f(x) \in U^c\). But this contradicts \(x \in f^{-1}(U)\). Thus \(f^{-1}(U)\) must be open and therefore \(f\) is continuous.

Let \((X,d)\) be a metric space. We say a net \((x_i)_{i \in I} \subset X\) is Cauchy if for all \(\epsilon > 0\) there exists \(i_0 \in I\) so that whenever \(i,j \geq i_0\) we have \(d(x_i,x_j) < \epsilon\). We conclude this section by examining Cauchy nets in a metric spaces. In particular, we will show that Cauchy nets in a complete metric space converge. The idea is to extract a Cauchy sequence from the Cauchy net, so as to use the completeness.

**Proposition 2.9.** Let \((X,d)\) be a complete metric space and let \((x_i)_{i \in I}\) be a Cauchy net. Then \((x_i)_{i \in I}\) converges.

**Proof.** Let \(i(1) \in I\) be such that \(d(x_i,x_j) < 1\). Let \(i(2) \in I\) be such that \(i(2) \geq i(1)\) and \(d(x_i,x_j) < \frac{1}{2}\) for all \(i,j \geq i(2)\). We inductively find \(i(n) \in I\) for each \(n \in \mathbb{N}\) such that \(i(n) \geq i(n-1)\) and \(d(x_i,x_j) < \frac{1}{n}\) for all \(i,j \geq i(n)\). We claim that the sequence \((x_{i(n)})_{n \in \mathbb{N}}\) is Cauchy. Indeed, let \(\epsilon > 0\). If \(N \in \mathbb{N}\) satisfies \(\frac{1}{N} < \epsilon\), then for \(n,m \geq N\) we have \(d(x_{i(n)},x_{i(m)}) < \frac{1}{N} < \epsilon\). Since \((X,d)\) is complete, \((x_{i(n)})_{n \in \mathbb{N}}\) converges to some \(x \in X\). We claim the original net also converges to this \(x\). Indeed, let \(\epsilon > 0\) and choose \(N \in \mathbb{N}\) such that for all \(n \geq N\) we have \(d(x_{i(n)},x) < \frac{\epsilon}{2}\). By choosing a larger \(N\) if necessary, we may assume \(\frac{1}{N} \leq \frac{\epsilon}{2}\). Then for any \(i \geq i(N)\) we have

\[
d(x_i,x) \leq d(x_i,x_{i(N)}) + d(x_{i(N)},x) < \frac{1}{N} + \frac{\epsilon}{2} \leq \epsilon.
\]

Hence the \((x_i)_{i \in I}\) converges to \(x\). 

**Remark 2.10.** When \((X,d)\) is a metric space, any Cauchy sequence \((x_n)_{n \in \mathbb{N}} \subset X\) is bounded. Indeed, let \(N \in \mathbb{N}\) be such that \(d(x_n,x_m) \leq 1\) for all \(n,m \geq N\). Then setting \(R := \max\{d(x_1,x_N),\ldots,d(x_{N-1},x_N),1\}\), we have \((x_n)_{n \in \mathbb{N}} \subset B(x_N,R)\). This same argument does not work for nets. We can still find \(i_0 \in I\) such that \(d(x_i,x_j) \leq 1\) for all \(i,j \geq i_0\), but then there are not necessarily finitely many \(i \leq i_0\). For example, the net \((x^{-t})_{t \in \mathbb{R}}\) converges in \(\mathbb{R}\) to zero but is not bounded.

# 3 Subnets

Subnets are the analogue of subsequences, though they are a bit more subtle.

**Definition 3.1.** Let \((x_i)_{i \in I}\) be a net in a topological space. Then \((y_j)_{j \in J}\) is a subnet of \((x_i)_{i \in I}\) if there exists a map \(\sigma: J \to I\) such that

1. \(x_{\sigma(j)} = y_j\) for all \(j \in J\);
2. \(j_1 \leq j_2\) then \(\sigma(j_1) \leq \sigma(j_2)\) (monotone);
3. for any \(i \in I\) there exists \(j \in J\) such that \(\sigma(j) \geq i\) (final).

**Example 3.2.** For a sequence \((x_n)_{n \in \mathbb{N}}\), any subsequence \((x_{n_k})_{k \in \mathbb{N}}\) is a subnet where \(\sigma(k) = n_k\). However, because we only require the map \(\sigma\) to be monotone (rather than strictly monotone) there are subnets of the sequence which are not subsequences. For example, \((x_1,x_1,x_1,x_2,x_3,\ldots)\) is a valid subnet, even though it is not a valid subsequence.

**Proposition 3.3.** Let \(X\) be a topological space. If a net \((x_i)_{i \in I} \subset X\) converges, then every subnet converges to the same limit.
\textbf{Proof.} Let \( x := \lim_i x_i \). Let \( (y_j)_{j \in J} \) be a subnet with monotone final map \( \sigma : J \to I \). Let \( U \subset X \) be an open subset containing \( x \). Then there exists \( i_0 \in I \) such that \( x_i \in U \) for all \( i \geq i_0 \). By finality there exists \( j_0 \in J \) such that \( \sigma(j_0) \geq i_0 \). Thus by monotonicity we for all \( j \geq j_0 \) that \( \sigma(j) \geq \sigma(j_0) \geq i_0 \) and hence \( y_j = x_{\sigma(j)} \in U \). That is, \( (y_j)_{j \in J} \) converges to \( x \).

Finally, we conclude this note by characterizing compactness in terms of convergent subnets. This is the analogue of the fact that in a metric space a set is compact if and only if every sequence in it has a convergent subsequence (which is sometimes called being sequentially compact).

\textbf{Proposition 3.4.} Let \( X \) be a topological space. Then \( K \subset X \) is compact if and only if every net \( (x_i)_{i \in I} \subset K \) has a convergent subnet.

\textbf{Proof.} \((\Rightarrow)\): Let \( K \) be a compact. We recall that it has the finite intersection property: if \( \{C_i\}_{i \in I} \) is a collection of closed subsets of \( K \) satisfying \( \bigcap_{i \in F} C_i \neq \emptyset \) for any finite subset \( F \subset I \), then \( \bigcap_{i \in I} C_i \neq \emptyset \). Indeed, otherwise \( \{C_i\}_{i \in I} \) is an open cover for \( K \) with no finite subcover.

Now, let \( (x_i)_{i \in I} \subset K \) be a net. Define \( C_i := \{x_j : j \geq i\} \). Then for \( F \subset I \) finite, we can find \( j \geq i \) for each \( i \in F \) and so
\[
x_j \in \bigcap_{i \in F} C_i \neq \emptyset
\]
By the finite intersection property we therefore have \( \bigcap_{i \in I} C_i \neq \emptyset \). Let \( y \) be an element of this set. Then for every \( i \in I \), \( y \in C_i \) which means for every neighborhood \( U \) of \( y \), \( U \cap \{x_j : j \geq i\} \neq \emptyset \). That is, for every \( i \in I \) and every neighborhood \( U \), there exists \( j \geq i \) such that \( x_j \in U \). Set \( y(U,j) := x_j \). Then \( (y(U,j)) \) is a net (where \((U,j) \leq (U',j') \) means \( U \supset U' \) and \( j \leq j' \)), which converges to \( y \). Defining \( \sigma(U,j) := j \) yields a monotone final map and so \( (y(U,j)) \) is a (convergent) subnet of \( (x_i)_{i \in I} \).

\((\Leftarrow)\): Towards a contradiction, let \( \{U_i : i \in I\} \) be an open cover of \( K \) with no finite subcover. Let \( \mathcal{F} \) be the collection of finite subsets of \( I \), which we make into a directed set by ordering by inclusion. For each \( F \in \mathcal{F} \) let \( x_F \) be any point in \( K \setminus \bigcup_{i \in F} U_i \) (which exists by virtue of there being no finite subcover). Then \( (x_F)_{F \in \mathcal{F}} \) is a net and consequently has a convergent subnet \( (x_{\sigma(j)})_{j \in J} \), say with limit \( x \). Then \( x \in U_i \) for some \( i \in I \) and consequently there is \( j_0 \in J \) such that \( x_{\sigma(j)} \in U_i \) for all \( j \geq j_0 \). Let \( j_1 \in J \) be such that \( \sigma(j_1) \geq \{i\} \in \mathcal{F} \). Then there exists \( j \geq j_1 \) and \( j \geq j_0 \). For this \( j \) we have \( x_{\sigma(j)} \in U_i \) but \( \sigma(j) \geq \sigma(j_1) \geq \{i\} \) implies \( x_{\sigma(j)} \notin U_i \), a contradiction. Thus every open cover of \( K \) has a finite subcover and \( K \) is therefore compact.

\(\square\)