V.1 Duality

Notation For an LCS X, let X^* denote the collection of continuous linear functionals on X. Then X^* is a vector space with pointwise operations.
For $x \in X$ and $x^* \in X^*$ we write

$$(x, x^*) := x^*(x) =: (x^*, x)$$

Note that

$$p_{x^*}(\cdot) := |(\cdot, x^*)| \quad \text{and} \quad p_x(\cdot) := |(x, \cdot)|$$

define seminorms on X and X^*, respectively.

Def Let X be an LCS. The weak topology on X, denoted $\sigma(X, X^*)$, is the topology induced by the family of seminorms $\{p_{x^*} : x^* \in X^*\}$.
The weak* topology on X^*, denoted $\sigma(X^*, X)$, is the topology induced by the family of seminorms $\{p_x : x \in X\}$.

- $U \subseteq X$ is weakly open iff $\forall x_0 \in X$ $\exists \varepsilon > 0$ and $x_1^*, \ldots, x_n^* \in X^*$ such that
$$U \supseteq \bigcap_{j=1}^{n} \{x \in X : p_{x_j^*}(x - x_0) < \varepsilon\} = \{x \in X : |(x - x_0, x_j^*)| < \varepsilon \quad j = 1, \ldots, n\}$$

A net $(x_i)_{i \in I} \subseteq X$ converges weakly to $x_0 \in X$ iff $\forall x^* \in X^*$
$$0 = \lim_{i \to \infty} p_{x^*}(x_i - x_0) = \lim_{i \in I} |(x_i - x_0, x^*)| = \lim_{i \to \infty} |(x_i, x^*) - (x_0, x^*)| \iff \lim_{i \to \infty} (x_i, x^*) = (x_0, x^*).$$

Likewise, $(x_i^*)_{i \in I} \subseteq X^*$ converges weak* to $x_0^* \in X^*$ iff $\forall x \in X$ $\lim_{i \to \infty} x_i^*(x) = x_0^*(x)$. i.e. pointwise convergence.

- X with the weak topology is an LCS, but X was assumed to be an LCS so X has two topologies on it. Denote the original topology by $\mathcal{T}$, then we claim $\sigma(X, X^*) \subseteq \mathcal{T}$. Indeed, if $x_i \xrightarrow{\mathcal{T}} 0$, then $x^*(x_i) \to 0$ $\forall x^* \in X^*$ since x^* is $\mathcal{T}$-continuous. Thus $x_i \xrightarrow{\sigma(X, X^*)} 0$. Since addition is continuous in both topologies, this yields $x_i \xrightarrow{\mathcal{T}} x \Rightarrow x_i \xrightarrow{\sigma(X, X^*)} x$ and so $\sigma(X, X^*) \subseteq \mathcal{T}$.

- A priori, $\sigma(X^*, X)$ is the only topology on X^*. Of course, if X is a normed space, then X^* is a Banach space and it is easy to see that $\sigma(X^*, X) \subseteq$ norm-topology.

Thm If X is a LCS, then $(X, \sigma(X, X^*))^* = X^*$.

Proof The containment $\supseteq$ follows from the definition of $\sigma(X, X^*)$. Conversely if $f : X \to \mathbb{F}$ is linear and $\sigma(X, X^*)$-continuous, then the preceding discussion implies
$$f^{-1}(U) \in \sigma(X, X^*) \subseteq \mathcal{T} \qquad \forall U \subseteq \mathbb{F} \text{ open}$$
where $\mathcal{T}$ is the original topology on X. Hence f is $\mathcal{T}$-continuous and therefore $f \in X^*$. $\square$

Thm If X is an LCS, then $(X^*, \sigma(X^*;X))^* = X$

Proof Again the inclusion $\supseteq$ follows from the definition of $\sigma(X^*;X)$. Conversely, let $f: X^* \to \mathbb{F}$ be linear and $\sigma(X^*,X)$-continuous. Since $\sigma(X^*,X)$ is determined by the seminorms $p_x(\cdot) = |\langle x, \cdot\rangle|$ for $x \in X$, the first theorem in Section IV.3 implies $\exists\, x_1,\ldots,x_n \in X$ and $\alpha_1,\ldots,\alpha_n > 0$ such that

$$|f(x^*)| \le \sum_{j=1}^n \alpha_j p_{x_j}(x^*) = \sum_{j=1}^n |\langle \alpha_j x_j, x^*\rangle|$$

By replacing x_j with $\alpha_j x_j$ we may assume $\alpha_1 = \cdots = \alpha_n = 1$. The above implies

$$* \qquad \bigcap_{j=1}^n \ker(\hat{x}_j) \subseteq \ker f$$

We claim $\exists\, c_1,\ldots,c_n \in \mathbb{F}$ such that $f = \sum c_j \hat{x}_j$. Removing some $1 \le j \le n$ if necessary, we may assume

$$\bigcap_{j \ne k} \ker(\hat{x}_j) \supsetneq \bigcap_{j=1}^n \ker(\hat{x}_j)$$

for all $1 \le k \le n$, while still retaining $(*)$. For each $1 \le k \le n$, let

$$x_k^* \in \bigcap_{j\ne k} \ker(\hat{x}_j) \setminus \bigcap_{j=1}^n \ker(\hat{x}_j)$$

so $\hat{x}_j(x_k^*) = 0$ for $j \ne k$, but $\hat{x}_k(x_k^*) \ne 0$. By normalizing, we may assume $\hat{x}_k(x_k^*) = 1$.
Define $c_k := f(x_k^*)$. For $x^* \in X^*$ consider

$$y^* := x^* - \sum_{k=1}^n \hat{x}_k(x^*)\, x_k^*$$

Then

$$\hat{x}_j(y^*) = \hat{x}_j(x^*) - \sum_{k=1}^n \hat{x}_k(x^*)\hat{x}_j(x_k^*) = \hat{x}_j(x^*) - \hat{x}_j(x^*) = 0$$

Thus $y^* \in \bigcap_{j=1}^n \ker(\hat{x}_j) \subseteq \ker(f)$. Therefore

$$\begin{aligned} 0 = f(y^*) &= f(x^*) - \sum_{k=1}^n \hat{x}_k(x^*) f(x_k^*) \\ &= f(x^*) - \sum_{k=1}^n c_k \hat{x}_k(x^*) \end{aligned}$$

Thus $f = \sum_{k=1}^n c_k \hat{x}_k^* \in X^*$. $\square$

⚠ This does not mean every Banach space X is reflexive. Instead, it says that after changing the topology on X^* from the norm topology to $\sigma(X^*,X)$, one has X as the dual. We note that typically $\sigma(X^*,X) \subsetneq$ norm topology on X^*.

Thm Let X be a LCS. For $C \subseteq X$ convex, $\overline{C} = \overline{C}^{wk}$ ← weak closure

Proof Let $\mathcal{J}$ denote the original topology on X. Then $\sigma(X,X^*) \subseteq \mathcal{J}$ implies $\mathcal{J}$ has more closed sets. Consequently, $\overline{C} \subseteq \overline{C}^{wk}$. Let $x_0 \in X \setminus \overline{C}$. The complex Hahn-Banach separation theorem from Section IV.3 implies $\exists\, x^* \in X^*$, $\alpha \in \mathbb{R}$, and $\varepsilon > 0$ such that

$$\mathrm{Re}\langle x, x^*\rangle \le \alpha < \alpha + \varepsilon \le \mathrm{Re}\langle x_0, x^*\rangle \qquad \forall x \in \overline{C}.$$

Let $B := \{x \in X : \mathrm{Re}\langle x, x^*\rangle \le \alpha\}$. Then $\overline{C} \subseteq B$ and $x_0 \notin B$. Note that B is weakly closed since x^* is $\sigma(X,X^*)$-continuous. Thus $C \subseteq \overline{C} \subseteq B$ implies $\overline{C}^{wk} \subseteq B$, and so $x_0 \notin \overline{C}^{wk}$. $\square$

Cor In a LCS, a convex subset is closed if and only if it is weakly closed.

- Note that if X is over $\mathbb{C}$, then $x^* \longleftrightarrow \operatorname{Re} x^*$ gives a 1-1 correspondence between the dual of X as a $\mathbb{C}$-linear space and X as an $\mathbb{R}$-linear space. It follows that $\sigma(X_{\mathbb{C}}, X_{\mathbb{C}}^*) = \sigma(X_{\mathbb{R}}, X_{\mathbb{R}}^*)$.

Def For $A \subset X$, the polar of A is

$$A^\circ := \{x^* \in X^* : |(a, x^*)| \leq 1 \quad \forall a \in A\}$$

For $B \subseteq X^*$, the prepolar of B is

$$^\circ B := \{x \in X : |(x, b)| \leq 1 \quad \forall b \in B\}$$

The bipolar of A is $^\circ A^\circ := {}^\circ(A^\circ)$

Ex ① Let X be a normed space. Then for $\bar{B}(0,1) \subseteq X$, $\bar{B}(0,1)^\circ$ is the unit ball in X^* and $^\circ\bar{B}(0,1)^\circ = \bar{B}(0,1)$. □

② Let $Y \subseteq X$ be a subspace and let $y^* \in Y^\circ$. For $y \in Y$, we have $ty \in Y$ $\forall t > 0$. Thus

$$1 \geq |(ty, y^*)| \iff \frac{1}{t} \geq |(y, y^*)|$$

and so $(y, y^*) = 0$. It follows that

$$Y^\circ = Y^\perp := \{y^* \in X^* : (y, y^*) = 0 \quad \forall y \in Y\}.$$

Similarly, for a subspace $Z \subseteq X^*$ we have

$$^\circ Z = {}^\perp Z := \{z \in X : (z, z^*) = 0 \quad \forall z^* \in Z\}.$$ □

Thm Let $A \subseteq X$. Then

① A° is convex and balanced.

② If $A_0 \subseteq A$ then $A^\circ \subseteq A_0^\circ$.

③ If $\alpha \in \mathbb{F} \setminus \{0\}$, then $(\alpha A)^\circ = \frac{1}{\alpha} A^\circ$.

④ $A \subset {}^\circ A^\circ$

⑤ $A^\circ = ({}^\circ A^\circ)^\circ$.

Proof (1) - (4): exercise.

(5): Note that (4) and (2) imply $({}^\circ A^\circ)^\circ \subseteq A^\circ$. (4) also implies $A^\circ \subseteq {}^\circ(A^\circ)^\circ$ by switching the roles of polars and prepolars. But ${}^\circ(A^\circ)^\circ = ({}^\circ(A^\circ))^\circ = ({}^\circ A^\circ)^\circ$, so $A^\circ = ({}^\circ A^\circ)^\circ$. □

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Thm (Bipolar Theorem)

Let X be a LCS. For $A \subseteq X$, ${}^\circ A^\circ$ is the intersection of all closed convex balanced sets containing A (i.e. the closed convex balanced hull of A).

Proof Let A_1 be the closed convex balanced hull of A. The analogue of the previous theorem for prepolars implies ${}^\circ A^\circ$ is convex and balanced. It is also easy to see that it is closed. Since $A \subseteq {}^\circ A^\circ$, we therefore have $A_1 \subseteq {}^\circ A^\circ$.

Now let $x_0 \in A_1^c$. Since A_1 is closed and convex, the complex Hahn-Banach separation theorem implies $\exists x^* \in X^*$, $\alpha \in \mathbb{R}$, and $\varepsilon > 0$ such that

$$\text{Re}\,(x, x^*) \leq \alpha < \alpha + \varepsilon \leq \text{Re}\,(x_0, x^*) \qquad \forall x \in A_1.$$

Let $\beta \in (\alpha, \alpha+\varepsilon)$. Now, since A_1 is balanced $0 \in A_1$. Thus

$$0 = \text{Re}\,(0, x^*) \leq \alpha < \beta.$$

Replacing x^* with $\frac{1}{\beta} x^*$ we have

$$\text{Re}\,(x, x^*) < 1 < \text{Re}\,(x_0, x^*) \qquad \forall x \in A_1.$$

For $x \in A_1$, let $(x, x^*) = e^{i\theta}|(x,x^*)|$ so that $|(x,x^*)| = (e^{-i\theta}x, x^*)$. Since A_1 is balanced, we have $e^{-i\theta} \in A_1$ and so

$$|(x,x^*)| = \text{Re}\,(e^{-i\theta}x, x^*) < 1$$

Thus $x^* \in A_1^\circ \subseteq A^\circ$. So $\text{Re}\,(x,x^*) > 1$ implies $x_0 \notin {}^\circ A^\circ$. $\square$

Cor Let X be a LCS. For $B \subseteq X^*$, $({}^\circ B)^\circ$ is the weak* closed convex balanced hull of B.

Def Let X be a LCS. We say $A \subseteq X$ is <u>weakly bounded</u> if $\forall\, U \in \sigma(X, X^*)$ with $0 \in U$ $\exists\, \varepsilon > 0$ such that $\varepsilon A \subseteq U$. (That is, A is bounded when viewing X as a TVS with the $\sigma(X, X^*)$ topology.)

- Observe that for $U \in \sigma(X, X^*)$ with $0 \in U$, $\exists\, x_1^*, \dots, x_n^* \in X^*$ and $\varepsilon > 0$ such that
$$U \supseteq \{x \in X : |(x, x_j^*)| < \varepsilon \quad j = 1, \dots, n\}.$$
Moreover, for any $x^* \in X^*$,
$$\{x \in X : |(x, x^*)| < 1\} \in \sigma(X, X^*).$$
Thus $A \subseteq X$ is weakly bounded iff for all $x^* \in X^*$
$$\sup_{x \in A} |(x, x^*)| < \infty.$$

- This allows us to translate one of the corollaries to the Principle of Uniform Boundedness as follows:

Thm Let X be a Banach space and Y a normed space. For $\mathcal{C} \subseteq B(X, Y)$, suppose that for all $x \in X$, $\{Ax : A \in \mathcal{C}\}$ is weakly bounded in Y. Then $\mathcal{C}$ is norm bounded.

IV.2 Duals of Subspaces and Quotients

• In this section we will generalize the results of Section III.10 to LCS's.

Prop Let X be a vector space and $Y \leq X$ a subspace. Given a seminorm p on X, one can define a seminorm $\bar{p}$ on X/Y by

$$\bar{p}(x+Y) = \inf\{p(x+y) : y \in Y\}.$$

If X is an LCS and $\mathcal{P}$ is the family of seminorms defining the topology on X, then for $Y \leq X$ the topology on X/Y defined by

$$\bar{\mathcal{P}} := \{\bar{p} : p \in \mathcal{P}\}$$

equals the quotient topology $\{U \subseteq X/Y : Q^{-1}(U) \subseteq X \text{ is open}\}$.

Proof Exercise. $\square$

• Thus if X is an LCS, then X/Y is an LCS with the quotient topology.

Notation Just as with normed spaces, for an LCS X and $Y \leq X$ a subspace we write

$$Y^\perp := \{f \in X^* : Y \subseteq \ker f\}.$$

Exercise show $Y^\perp \leq X^*$.

Thm Let X be an LCS and $Y \leq X$. If $Q : X \to X/Y$ is the quotient map, then

$$\rho : (X/Y)^* \to Y^\perp$$
$$f \mapsto f \circ Q$$

is a linear bijection. Moreover, this is a homeomorphism when $(X/Y)^*$ is equipped with its weak* topology, $\sigma((X/Y)^*, X/Y)$, and $Y^\perp$ is equipped with the relative weak* topology on X^*, $\sigma(X^*, X)|_{Y^\perp}$.

Proof Given $f \in (X/Y)^*$, $f \circ Q \in X^*$ and clearly $Y \subseteq \ker(f \circ Q)$. So ρ is indeed valued in $Y^\perp$. ρ is also clearly linear. Since Q is surjective, ρ is injective by Exercise 3 on Homework 4. If $g \in Y^\perp$, then whenever $x_1 + Y = x_2 + Y$ we have $0 = g(x_1 - x_2) \Rightarrow g(x_1) = g(x_2)$. Hence $f : x+Y \mapsto g(x)$ is a well-defined linear functional on X/Y. Clearly $f \circ Q = g$. To see that f is continuous, observe that for $U \subseteq \mathbb{F}$ open

$$Q^{-1}(f^{-1}(U)) = \{x \in X : \underbrace{f \circ Q(x)}_{g} \in U\} = g^{-1}(U)$$

is open since g is continuous. Hence $f^{-1}(U)$ is open (in the quotient topology) and so f is continuous. Thus $\rho(f) = g$ and ρ is surjective.
Now, suppose $(f_i)_{i \in I} \subseteq (X/Y)^*$ converges to $f_0 \in (X/Y)^*$ in the weak* topology. Then for each $x \in X$,

$$\lim_{i \to \infty} [\rho(f_i)](x) = \lim_{i \to \infty} f_i(Q(x)) = f_0(Q(x)) = [\rho(f_0)](x).$$

Thus $\rho(f_i) \to \rho(f_0)$ weak*. So ρ is continuous. Conversely, if $(g_i)_{i\in I} \subseteq Y^\perp$ converges weak* to some $g_0 \in Y^\perp$. Let $f_i := \rho^{-1}(g_i)$ and $f_0 := \rho^{-1}(g_0)$. Then for each $x+Y \in X/Y$

$$\lim_{i\to\infty} f_i(x+Y) = \lim_{i\to\infty} f_i \circ Q(x) = \lim_{i\to\infty} g_i(x) = g_0(x) = f_0 \circ Q(x) = f_0(x+Y)$$

Thus $f_i \to f_0$ weak* and so ρ^{-1} is also continuous. $\square$

$\boxed{\text{Thm}}$ Let X be a LCS and $Y \leq X$. Then

$$\rho : X^*/Y^\perp \to Y^*$$
$$f + Y^\perp \mapsto f|_Y$$

is a linear bijection. Moreover, this is a homeomorphism when $X^*/Y^\perp$ is equipped with the quotient topology induced by the weak* topology on X^*, $\sigma(X^*, X)$, and Y^* is equipped with the weak* topology $\sigma(Y^*, Y)$.

Proof First note that ρ is well-defined since $f_1 + Y^\perp = f_2 + Y^\perp$ implies $f_1 - f_2 \in Y^\perp$ and so $f_1|_Y = f_2|_Y$. If $0 = \rho(f+Y^\perp) = f|_Y$, then $f \in Y^\perp$ and so $f + Y^\perp = 0 + Y^\perp$. Thus ρ is injective. Given $g \in Y^*$, the Hahn-Banach theorem implies $\exists f \in X^*$ with $f|_Y = g$. Then $\rho(f+Y^\perp) = g$ and ρ is surjective.

Now, let $Q : X^* \to X^*/Y^\perp$ be the quotient map. Recall that $\sigma(Y^*, Y)$ is generated by sets of the form

$$\{y^* \in Y^* : |(y, y^*)| < \varepsilon\} \qquad y \in Y,\ \varepsilon > 0$$

So to show ρ is continuous, we must show ρ^{-1} of the above sets are open in the quotient topology. Fix $y \in Y$ and $\varepsilon > 0$. Then

$$\begin{aligned} Q^{-1}(\rho^{-1}(\{y^* \in Y^* : |(y, y^*)| < \varepsilon\})) &= \{x^* \in X^* : |(y, \rho \circ Q(x^*))| < \varepsilon\} \\ &= \{x^* \in X^* : |(y, x^*|_Y)| < \varepsilon\} \\ &= \{x^* \in X^* : |(y, x^*)| < \varepsilon\} \in \sigma(X^*, X). \end{aligned}$$

Thus ρ is continuous.

To see ρ^{-1} is continuous, we recall that the above proposition implies the topology on $X^*/Y^\perp$ is determined by seminorms $\bar{p}_x$ where

$$p_x(x^*) = |(x, x^*)| \qquad x^* \in X^*.$$

We claim $\bar{p}_x \equiv 0$ for $x \notin Y$. Indeed, fix $x \notin Y$ and let $Z := \operatorname{span}\{x\} \cup Y$. Then Z is closed (Exercise). For $x^* \in X^*$ we must show $\bar{p}_x(x^* + Y^\perp) = 0$.

Define $f : Z \to \mathbb{F}$ by

$$f(\alpha x + y) := x^*(y).$$

We claim f is cts. Indeed, if $(\alpha_i x + y_i)_{i\in I} \subseteq Z$ converges to some $z_0 \in X$, then $z_0 \in Z$ since Z is closed and hence $z_0 = \alpha_0 x + y_0$ for some $y_0 \in Y$. Using the quotient map $Z \to Z/Y$, one can show $\alpha_i \to \alpha_0$ and consequently $y_i \to y_0$. Therefore

$$\lim_{i\to\infty} f(\alpha_i x + y_i) = \lim_{i\to\infty} x^*(y_i) = x^*(y_0) = f(\alpha_0 x + y_0).$$

Thus $f \in Z^*$ and so the Hahn-Banach theorem $\exists x_1^* \in X^*$ such that $x_1^*|_Z = f$. In particular, $(x, x_1^*) = 0$. Also note that $x_1^*|_Y = x^*|_Y \Rightarrow x^* - x_1^* \in Y^\perp \Rightarrow x^* + Y^\perp = x_1^* + Y^\perp$.

Thus
$$\bar{p}_x(x^*+Y^\perp) = \bar{p}_x(x_i^*+Y^\perp) \le p_x(x_i^*) = |(x, x_i^*)| = 0$$
This establishes the claim.
Now, suppose $(x_i^*+Y^\perp)_{i\in I} \subseteq X^*/Y^\perp$ is a net such that $\rho(x_i^*+Y^\perp) = x_i^*|_Y \to 0$ weak*. For $x\in X$, if $x\notin Y$ then the claim implies
$$\lim_{i\to\infty} \bar{p}_x(x_i^*+Y^\perp) = 0.$$
If $x\in Y$, then
$$\bar{p}_x(x_i^*+Y^\perp) \le |(x, x_i^*)| = |(x, x_i^*|_Y)| \to 0.$$
Thus $x_i^*+Y^\perp \to 0$ in $X^*/Y^\perp$ and so ρ^{-1} is continuous. $\square$

- For $Y\subseteq X$, define
$$\sigma(X,X^*)|_Y := \{U\cap Y : U\in\sigma(X,X^*)\}.$$

$\boxed{\text{Cor}}$ For X a LCS and $Y\subseteq X$, $\sigma(X,X^*)|_Y = \sigma(Y,Y^*)$.

<u>Proof</u> Fix $U\in\sigma(X,X^*)$ and let $y_0\in U\cap Y$. Then by definition of $\sigma(X,X^*)$, $\exists\, x^*\in X^*$ and $\varepsilon>0$ such that $B := \{x\in X : |(x-y_0, x^*)|<\varepsilon\} \subseteq U$. Observe that
$$U\cap Y \supseteq B\cap Y = \{y\in Y : |(y-y_0, x^*)|<\varepsilon\} = \{y\in Y \mid |(y-y_0, x^*|_Y)|<\varepsilon\}$$
Since $x^*|_Y\in Y^*$, this shows $\sigma(X,X^*)|_Y \subseteq \sigma(Y,Y^*)$. To show the other inclusion, it suffices to show that $\forall y^*\in Y^*$ and $\varepsilon>0$ that
$$B := \{y\in Y : |(y,y^*)|<\varepsilon\} \in \sigma(X,X^*)|_Y.$$
By the previous theorem, $\exists\, x^*\in X^*$ such that $x^*|_Y = y^*$. Thus
$$B = \{y\in Y : |(y,x^*)|<\varepsilon\} = \{x\in X : |(x,x^*)|<\varepsilon\}\cap Y \in \sigma(X,X^*)|_Y.$$ $\square$

V.3 The Banach–Alaoglu Theorem

Thm (The Banach–Alaoglu Theorem)
For a normed space X, the closed unit ball in X^* is weak* compact.

Proof Consider
$$D := \prod_{\substack{x\in X\\ \|x\|=1}} \{\alpha \in \mathbb{F} : |\alpha| \le 1\}$$
Then D is compact by Tychonoff's theorem. Denote $B^* := \{x^* \in X^* : \|x^*\| \le 1\}$ and define
$$\phi : B^* \longrightarrow D$$
$$x^* \mapsto (x^*(x))_x$$
We will first show $\phi(B^*)$ is closed. Suppose $(\phi(x_i^*))_{i\in I}$ converges to some $(c_x)_x \in D$. Then $x_i^*(x) \longrightarrow c_x$ for all $x \in X$ with $\|x\| = 1$. Define $f : X \to \mathbb{F}$ by $f(0) = 0$ and
$$f(x) = \|x\| \, c_{x/\|x\|} \qquad x \in X \setminus \{0\}.$$
Then for all $x \in X$,
$$f(x) = \|x\| \lim_{i\to\infty} x_i^*(x/\|x\|) = \lim_{i\to\infty} x_i^*(x).$$
From this it follows that f is linear. Since
$$|f(x)| = \|x\| \, |c_{x/\|x\|}| \le \|x\|$$
we see $f \in B^*$. Since $\phi(f) = (c_x)_x$, we see that $\phi(B^*) \subseteq D$ is closed and hence compact.
Now, we claim $\phi : B^* \to \phi(B^*)$ is a homeomorphism when B^* inherits the weak* topology on X^*. This will complete the proof since since $B^* = \phi^{-1}(\phi(B^*))$ will be the continuous image of a compact set. The linearity of $x^* \in X^*$ implies it is completely determined by its values on $x \in X$ with $\|x\| = 1$. Hence ϕ is injective. If $(x_i)_{i\in I} \subseteq B^*$ converges to some $x_0^* \in B^*$ weak*, then $\forall x \in X$ $x_i^*(x) \longrightarrow x_0^*(x)$. This implies
$$\phi(x_i^*) = (x_i^*(x))_x \longrightarrow (x_0^*(x))_x = \phi(x_0^*).$$
So ϕ is continuous. Conversely, if $(\phi(x_i^*))_{i\in I} \subseteq \phi(B^*)$ converges to some $\phi(x_0^*) \in B^*$, then $x_i^*(x) \to x_0^*(x)$ $\forall x \in X$ with $\|x\| = 1$. But then for any $x \in X \setminus \{0\}$
$$x_i^*(x) = \|x\| \, x_i^*\left(\tfrac{x}{\|x\|}\right) \to \|x\| \, x_0^*\left(\tfrac{x}{\|x\|}\right) = x_0^*(x).$$
Thus $x_i^* \to x_0^*$ weak* and ϕ^{-1} is continuous. □

Ex Let $\mathcal{H}$ be a Hilbert space. Since $\mathcal{H}$ is reflexive, it has a weak* topology $\sigma(\mathcal{H}^{**}, \mathcal{H}^*) = \sigma(\mathcal{H}, \mathcal{H}^*)$, namely its weak topology. For $C \subset \mathcal{H}$ bounded and convex, let $\overline{C}$ be its norm closure. Then we showed in Section V.1 that it is weakly closed. It is of course also bounded. By scaling, we can assume $\overline{C} \subseteq B(0,1) \subseteq \mathcal{H} = \mathcal{H}^{**}$. So by the Banach–Alaoglu theorem, it is compact with respect to the $\sigma(\mathcal{H}^{**}, \mathcal{H}^*) = \sigma(\mathcal{H}, \mathcal{H}^*)$ topology. So any net $(h_i)_{i\in I} \subseteq \overline{C}$ has a weakly convergent subnet. □

⚠ This does not imply the subnet converges in norm. E.g. $(e_n)_{n\in\mathbb{N}} \subseteq \ell^2(\mathbb{N})$ converges weakly to zero. see also HW6 Ex 4

V.4 Reflexivity Revisited

- For a normed space X, we can embed $X \hookrightarrow X^{**}$ isometrically by the natural mapping. We claim $\sigma(X^{**}, X^*)|_X = \sigma(X, X^*)$. Indeed, this is simply because

$$\langle x^*, \hat{x}\rangle = \hat{x}(x^*) = x^*(x) = \langle x, x^*\rangle$$

so $\hat{x}_i \to \hat{x}_0$ weak* in X^{**} is equivalent to $x_i \to x$ weakly in X.

Notation For a normed space X, $(X)_1 := \{x \in X : \|x\| \le 1\}$.

Prop If X is a normed space, then $(X)_1$ is $\sigma(X^{**}, X^*)$ dense in $(X^{**})_1$.

Proof Let B be the $\sigma(X^{**}, X^*)$-closure of $(X)_1$. Then $(X)_1 \subseteq (X^{**})_1$ implies $B \subseteq (X^{**})_1$. Suppose, towards a contradiction, that $\exists x_0^{**} \in (X^{**})_1 \setminus B$. Then B and $\{x_0^{**}\}$ are both $\sigma(X^{**}, X^*)$-closed and convex, and $\{x_0^{**}\}$ is compact. The Hahn-Banach theorem implies there exists linear $f: X^{**} \to \mathbb{F}$ which is $\sigma(X^{**}, X^*)$-continuous, $\alpha \in \mathbb{R}$, and $\varepsilon > 0$ such that

$$\operatorname{Re} f(x^{**}) \le \alpha < \alpha + \varepsilon \le \operatorname{Re} f(x_0^{**}) \qquad \forall x^{**} \in B.$$

Recall that in Section V.1 we showed that $(X^{**}, \sigma(X^{**}, X^*))^* = X^*$. Thus $f = x^*$ and so the above implies

$$\operatorname{Re}\langle x, x^*\rangle \le \alpha < \alpha + \varepsilon \le \operatorname{Re}\langle x^*, x_0^{**}\rangle \qquad \forall x \in (X)_1.$$

Since $0 \in (X)_1$, we have $\alpha \ge 0$. By replacing α with $\beta \in (\alpha, \alpha+\varepsilon)$ and rescaling x^* by β^{-1}, we have

$$\operatorname{Re}\langle x, x^*\rangle < 1 < 1 + \delta < \operatorname{Re}\langle x^*, x_0^{**}\rangle$$

for all $x \in (X)_1$ and some $\delta > 0$. If $\langle x, x^*\rangle = e^{i\theta}|\langle x, x^*\rangle|$, then

$$|\langle x, x^*\rangle| = \langle e^{-i\theta}x, x^*\rangle < 1$$

so that $\|x^*\| \le 1$. However,

$$1 + \delta \le \operatorname{Re}\langle x^*, x_0^{**}\rangle \le |\langle x^*, x_0^{**}\rangle| \le \|x_0^{**}\| \le 1,$$

a contradiction. Thus we must have $B = (X^{**})_1$. $\square$

Thm Let X be a Banach space. The following statements are equivalent:

1. X is reflexive.
2. X^* is reflexive.
3. $\sigma(X^*, X) = \sigma(X^*, X^{**})$.
4. $(X)_1$ is weakly compact.

Proof (1) $\Rightarrow$ (4): By the Banach-Alaoglu theorem $(X^{**})_1 = (X)_1$ is $\sigma(X^{**}, X^*)$-compact, hence $\sigma(X, X^*)$-compact.

(4) $\Rightarrow$ (1): Note that the natural map is weakly continuous. Thus $(X)_1$ in X^{**} is compact and hence closed. The previous proposition then implies $(X)_1 = (X^{**})_1$, and thus $X = X^{**}$.

(1) $\Rightarrow$ (3): since $X = X^{**}$, this is immediate.

(3) $\Rightarrow$ (2): By the Banach–Alaoglu theorem, $(X^*)_1$ is $\sigma(X^*,X)$-compact. But then (3) implies it is $\sigma(X^*,X^{**})$-compact. Using (1) $\Leftrightarrow$ (4) applied to X^* shows X^* is reflexive.

(2) $\Rightarrow$ (1): Since $(X)_1$ is norm closed in X^{**} and convex, it is $\sigma(X^{**},X^{***})$-closed. Hence it is weak* closed since $X^{***}=X^*$. By the previous proposition, we have $(X)_1=(X^{**})_1$ and hence $X=X^{**}$. □

Cor If X is a reflexive Banach space and $Y\subseteq X$, then Y is a reflexive Banach space.

Proof We have $(Y)_1 = Y\cap(X)_1$. So $(Y)_1$ is $\sigma(X,X^*)$-compact. Since $(Y)_1\cap Y=(Y)_1$, we have that $(Y)_1$ is $\sigma(X,X^*)|_Y$-compact and hence $\sigma(Y,Y^*)$-compact by the last corollary in Section V.2. □

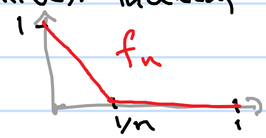
Def For an LCS X, a <u>weakly Cauchy sequence</u> is a sequence $(x_n)_{n\in\mathbb{N}}\subseteq X$ such that $(\langle x_n,x^*\rangle)_{n\in\mathbb{N}}\subseteq\mathbb{F}$ is Cauchy for all $x^*\in X^*$.

Cor If X is a reflexive Banach space, then every weakly Cauchy sequence is weakly convergent. That is, X is weakly sequentially complete.

Proof Let $(x_n)_{n\in\mathbb{N}}\subset X$ be a weakly Cauchy sequence. Then
$$\sup_{n\in\mathbb{N}}|\langle x_n,x^*\rangle|<\infty\qquad\forall x^*\in X^*.$$
By a corollary of the P.U.B., $(x_n)_{n\in\mathbb{N}}$ is bounded. Since X is reflexive, $(X)_1$ is weakly compact and so is any scaling of it. Hence $\{x_n : n\in\mathbb{N}\}$ has a weak cluster point $x\in X$. Since $\lim_{n\to\infty}\langle x_n,x^*\rangle$ exists for all $x^*\in X$, we must have that the limit is $\langle x,x^*\rangle$. Thus $x_n\to x$ weakly. □

⚠ In the above proof, we <u>cannot</u> use the weak compactness to find a weakly convergent subsequence of $(x_n)_{n\in\mathbb{N}}$. Instead, we get a weakly convergent <u>subnet</u>, which need <u>not</u> be a subsequence. However, the limit of this subnet is still a weak cluster point.

Ex $C[0,1]$ is not weakly sequentially complete (and hence not reflexive). Indeed, consider



$$f_n(t):=\begin{cases}1-nt & \text{if } 0\le t\le 1/n\\ 0 & \text{if } 1/n<t\le 1\end{cases}$$

Then
$$\lim_{n\to\infty}f_n(t)=\begin{cases}1 & \text{if } t=0\\ 0 & \text{otherwise}\end{cases}=:g(t)\notin C[0,1]$$
and the convergence is monotone. Hence $\forall\mu\in M(X)=C[0,1]^*$ we have
$$\lim_{n\to\infty}\int f_n\,d\mu=\mu(\{0\})$$
by the monotone convergence theorem. So $(f_n)_{n\in\mathbb{N}}$ is weakly Cauchy, but <u>not</u> weakly convergent. Indeed, if $f\in C[0,1]$ were a weak limit, then for all $t\in[0,1]$ we have

$$f(t) = \int f \, d\delta_t = \lim_{n\to\infty} \int f_n \, d\delta_t = \lim_{n\to\infty} f_n(t) = g(t)$$

So that $f(t) = g(t)$, but $g \notin C[0,1]$. $\square$

$\boxed{\text{Cor}}$ If X is a reflexive Banach space and $Y \subseteq X$, then for all $x_0 \in X \setminus Y$ there exists $y_0 \in Y$ such that

$$\operatorname{dist}(x_0, Y) = \|x_0 - y_0\|.$$

Proof Let $d = \operatorname{dist}(x_0, Y)$. Then $Y \cap \{x : \|x - x_0\| \le 2d\}$ is closed and bounded. So X being reflexive implies it is weakly compact by the Banach–Alaoglu theorem. Finally, $x \mapsto \|x - x_0\|$ is weakly lower semicontinuous (Homework 6). Since lower semicontinuous functions attain their minimum on compact sets, we are done. $\square$

4/3

V.5 Separability and Metrizability

- For X an infinite dimensional Banach space, $\sigma(X,X^*)$ and $\sigma(X^*,X)$ are never metrizable. However, $\sigma(X,X^*)|_B$ and $\sigma(X^*,X)|_{B^*}$ are sometimes metrizable for $B\subseteq X$, $B^*\subseteq X^*$ bounded.

Thm If X is a Banach space, then $(X^*)_1$ is weak*-metrizable if and only if X is separable.

Proof $(\Leftarrow)$ Let $\{x_n : n\in\mathbb{N}\}\subseteq (X)_1$ be dense. Let

$$T := \prod_{n\in\mathbb{N}} \{\alpha\in\mathbb{F} : |\alpha|\le 1\}$$

which is a compact metric space with metric

$$d((\alpha_n)_{n\in\mathbb{N}}, (\beta_n)_{n\in\mathbb{N}}) = \sum_{n=1}^{\infty} 2^{-n}|\alpha_n-\beta_n|.$$

It suffices to show $(X^*)_1$ with the weak* topology is homeomorphic to a subset of T. Define

$$\phi : (X^*)_1 \to T$$
$$x^* \mapsto (x^*(x_n))_{n\in\mathbb{N}}.$$

If $\phi(x^*)=\phi(y^*)$, then $(x^*-y^*)(x_n)=0$ $\forall n\in\mathbb{N}$. Since $\{x_n\}_{n\in\mathbb{N}}$ is dense in $(X)_1$, we have $(x^*-y^*)|_{(X)_1}\equiv 0$ $\Rightarrow$ $x^*=y^*$. So ϕ is injective. If $(x_i^*)_{i\in I}\subseteq (X^*)_1$ converges to some $x_0^*\in X$, then $\forall n\in\mathbb{N}$ $\lim_{i\to\infty} x_i^*(x_n)=x_0^*(x_n)$. This implies $\phi(x_i^*)\to\phi(x_0^*)$. Thus ϕ is continuous. Since $(X^*)_1$ is compact by the Banach-Alaoglu theorem, ϕ is a closed map and hence a homeomorphism onto $\phi((X^*)_1)$.

$(\Leftarrow)$ Let d be a metric on $(X^*)_1$ inducing the weak*-topology. Then $\forall n\in\mathbb{N}$

$$U_n := \{x^*\in(X^*)_1 : d(0,x^*)<\tfrac{1}{n}\}\in\sigma(X^*,X)|_{(X^*)_1}.$$

Thus $U_n = W_n\cap (X^*)_1$ for some $W_n\in\sigma(X^*,X)$. By definition of $\sigma(X^*,X)$, $\exists\, F_n\subseteq X$ finite and $\varepsilon>0$ such that

$$\{x^*\in X^* : |\langle x,x^*\rangle|<\varepsilon \quad \forall x\in F_n\}\subseteq W_n$$

and so

$$\{x^*\in (X^*)_1 : |\langle x,x^*\rangle|<\varepsilon \quad \forall x\in F_n\}\subseteq W_n\cap (X^*)_1 = U_n.$$

Set $F := \bigcup_{n=1}^{\infty} F_n$, which is countable. We claim the countable set

$$\mathbb{Q}F + i\,\mathbb{Q}F = \{\sum_{j=1}^{k}(\alpha_j+i\beta_j)x_j : k\in\mathbb{N},\ \alpha_j,\beta_j\in\mathbb{Q},\ x_j\in F\}$$

is dense in X. Its closed span equals

$$\overline{\operatorname{span}}\,F \overset{\text{convexity}}{=} \overline{\operatorname{span} F}^{\,wk} = \bigcap_{x^*\in(\operatorname{span}F)^\perp} \ker x^*$$

Note $(\operatorname{span}F)^\perp = F^\perp$. For $x^*\in F^\perp$ we have

$$|\langle x, \tfrac{x^*}{\|x^*\|}\rangle| = 0 \qquad \forall x\in F_n$$

for all $n\in\mathbb{N}$. Thus

$$\frac{x^*}{\|x^*\|}\in\bigcap_{n=1}^{\infty} U_n = \{y^*\in (X_1)^* : d(0,0)<\tfrac{1}{n}\ \forall n\in\mathbb{N}\} = \{0\}.$$

So $F^\perp=\{0\}$ and hence $\overline{\operatorname{span}}\,F = X$. So X is separable. $\square$

- If X^* is separable, then $(X^{**})_1$ is $\sigma(X^{**},X^*)$-metrizable. Since $(X)_1 \subseteq (X^{**})_1$ and $\sigma(X^{**},X^*)|_X = \sigma(X,X^*)$, it follows that $(X)_1$ is weakly-metrizable.

- Does X being separable imply $(X)_1$ is weakly-metrizable?

Ex ① If X is reflexive and separable, then $X^{**}=X$ is separable. Exercise 10 on Homework 4 then implies X^* is separable and so $(X)_1$ is weakly-metrizable by the above discussion.

② $X=c_0(\mathbb{N})$ is separable but not reflexive. However $X^*=\ell^1(\mathbb{N})$ is separable and so $(X)_1$ is weakly metrizable.

③ $X=\ell^1(\mathbb{N})$ is separable, but $(X)_1$ is not weakly metrizable. This will follow from the next proposition. □

Prop. Any weakly convergent sequence in $\ell^1(\mathbb{N})$ converges in norm.

Proof Since $\ell^1(\mathbb{N})$ is separable, $\ell^\infty(\mathbb{N}) = (\ell^1(\mathbb{N}))^*$ has a weak*-metrizable unit ball by the previous theorem. We also know $(\ell^\infty(\mathbb{N}))_1$ is weak*-compact. Hence $(\ell^\infty(\mathbb{N}))_1$ with the weak* topology is a complete metric space. In fact, an equivalent metric is given by

$$d(\phi,\psi) := \sum_{j=1}^\infty 2^{-j}|\phi(j)-\psi(j)| \qquad \phi,\psi\in(\ell^\infty(\mathbb{N}))_1.$$

Suppose $(f_n)_{n\in\mathbb{N}}\subseteq \ell^1(\mathbb{N})$ converges to 0 weakly. Let $\varepsilon>0$ and for $m\in\mathbb{N}$ define

$$F_m := \{\phi\in(\ell^\infty(\mathbb{N}))_1 : |(f_n,\phi)| \le \tfrac{\varepsilon}{3} \text{ for } n\ge m\}$$

Then F_m is weak* closed and $f_n\to 0$ weakly implies

$$\bigcup_{m=1}^\infty F_m = (\ell^\infty(\mathbb{N}))_1$$

The Baire Category Theorem implies some F_m is not nowhere dense. That is, F_m has non-empty weak* interior. Thus $\exists\, \phi\in F_m$ and $\delta>0$ such that

$$\{\psi\in(\ell^\infty(\mathbb{N}))_1 : d(\phi,\psi)<\delta\}\subseteq F_m.$$

Let $J\in\mathbb{N}$ be such that $2^{-J}\cdot 2<\delta$. For $n\ge m$ define $\psi_n\in(\ell^\infty(\mathbb{N}))_1$ by

$$\psi_n(j) = \begin{cases}\phi(j) & \text{if } 1\le j\le J\\ \operatorname{sgn} f_n(j) & \text{otherwise}\end{cases}$$

Then $\psi_n(j) f_n(j) = |f_n(j)|$ $\forall j>J$ and

$$d(\phi,\psi_n) = \sum_{j=J+1}^\infty 2^{-j}|\phi(j)-\psi_n(j)| \le \sum_{j=J+1}^\infty 2^{-j}\cdot 2 = 2\cdot 2^{-J}<\delta$$

So $\psi_n\in F_m$ and therefore $|(\psi_n, f_k)|\le \varepsilon/3$ for $k\ge m$. In particular

$$\ast \quad \frac{\varepsilon}{3}\ge |(\psi_n,f_n)| = \left|\sum_{j=1}^J \phi(j)f_n(j) + \sum_{j=J+1}^\infty |f_n(j)|\right| \qquad \forall n\ge m.$$

Now, $f_n\to 0$ weakly implies $f_n(j)\to 0$ for all $j\in\mathbb{N}$. We can therefore find $m_1\ge m$ so that for $k\ge m_1$

$$|f_k(j)| \le \tfrac{1}{J}\tfrac{\varepsilon}{3} \qquad 1\le j\le J$$

Consequently, $\sum_{j=1}^{J} |f_k(j)| < \frac{\varepsilon}{3}$ for $k \geq n_1$. Finally, using $(*)$ we obtain for $n \geq n_1$

$$
\begin{aligned}
\|f_n\|_1 &= \sum_{j=1}^{\infty} |f_n(j)| \\
&< \frac{\varepsilon}{3} + \sum_{j=J+1}^{\infty} |f_n(j)| + \sum_{j=1}^{J} \phi(j) f_n(j) - \sum_{j=1}^{J} \phi(j) f_n(j) \\
&\leq \frac{\varepsilon}{3} + \left| \sum_{j=J+1}^{\infty} |f_n(j)| + \sum_{j=1}^{J} \phi(j) f_n(j) \right| + \sum_{j=1}^{J} |f_n(j)| \\
&\leq \frac{\varepsilon}{3} + \frac{\varepsilon}{3} + \frac{\varepsilon}{3} = \varepsilon.
\end{aligned}
$$

Thus $\lim_{n\to\infty} \|f_n\|_1 = 0$. $\square$

- So if $(\ell^1(\mathbb{N}))_1$ was weakly metrizable then convergence of sequences (rather than nets) would determine the topology. The previous Proposition would then imply the weak and norm topologies are the same. Since $\ell^1(\mathbb{N})$ is infinite dimensional, this is false (Homework 6, Exercise 7).

V.6 The Stone-Čech Compactification

- In this section we construct a compactification of topological space X by consider the we weak* topology on $C_b(X)^*$

- Fix a Hausdorff space X. For $x \in X$, define $\delta_x \in C_b(X)^*$ by $\delta_x(f) = f(x)$. Then define $\Delta: X \to C_b(X)^*$ by $\Delta(x) = \delta_x$. Observe that if $(x_i)_{i \in I} \subseteq X$ is a net converging to $x \in X$, then $f(x_i) \to f(x)$ $\forall f \in C_b(X)$. Hence $\delta_{x_i} \to \delta_x$ weak*. Thus $\Delta: X \to C_b(X)^*$ is continuous when $C_b(X)^*$ is given the weak* topology.

- When is Δ a homeomorphism onto $\Delta(X)$?

Def A topological space X is called completely regular if for any $V \subseteq X$ closed and $x \in X \setminus V$ there exists $f \in C_b(X)$ such that $f(x) = 1$ and $f|_V \equiv 0$. A Tychonoff space is a completely regular Hausdorff space

Prop The map $\Delta: X \to C_b(X)^*$ is a homeomorphism onto $\Delta(X)$ (with the weak* topology) if and only if X is a Tychonoff space.

Proof ($\Leftarrow$) If $x_1, x_2 \in X$ are distinct then $\exists f \in C_b(X)$ with $f(x_1) = 1$ and $f(x_2) = 0$. Consequently $\delta_{x_1}(f) \neq \delta_{x_2}(f)$ and so Δ is injective. We've already seen Δ is continuous, so it suffices to show it is an open map. Let $U \subseteq X$ be open. Fix $x_0 \in U$. Then $\exists f \in C_b(X)$ such that $f(x_0) = 1$ and $f|_{U^c} \equiv 0$. Set

$$W := \{\mu \in C_b(X)^* : |\langle \mu, f \rangle| > 0\} \in \sigma(C_b(X)^*, C_b(X)).$$

Thus $V := W \cap \Delta(X)$ is weak* open in $\Delta(X)$. Also $\delta_{x_0} \in V \subseteq \Delta(U)$. Thus $\Delta(U)$ is open.

($\Rightarrow$) Since $(C_b(X)^*)_1$ is weak* compact and Hausdorff, it is Tychonoff. Then $\Delta(X) \subseteq (C_b(X)^*)_1$ is Tychonoff. Since Δ is a homeomorphism, it follows that X is a Tychonoff space. $\square$

Thm (The Stone-Čech Compactification)

For a Tychonoff space X, there exists a compact Hausdorff space βX such that:

1. there is a continuous map $\Delta: X \to \beta X$ that is a homeomorphism onto its image;
2. $\Delta(X)$ is dense in βX;
3. if $f \in C_b(X)$, then there is $f^\beta \in C(\beta X)$ such that $f^\beta \circ \Delta = f$.

Moreover, βX is unique up to homeomorphism.

Proof Let $\Delta: X \to C_b(X)^*$ be as above. Define βX to be the weak* closure of $\Delta(X)$. Then βX is compact as closed subset of $(C_b(X)^*)_1$. So 1 holds by the previous proposition and ② holds by definition. It remains to show ③. For $f \in C_b(X)$, define $f^\beta : \beta X \to \mathbb{F}$ by $f^\beta(x^*) := x^*(f)$ (recall $\beta X \subseteq C_b(X)^*$). Then f^β is clearly continuous and

$$f^\beta \circ \Delta(x) = f^\beta(\delta_x) = \delta_x(f) = f(x)$$

Checking uniqueness is left as an exercise. □

Def For a Tychonoff space X, βX as above is called the Stone–Čech compactification of X.

- We typically view $\beta X \supset X$ and Δ as the inclusion map. Then $\overline{X} = \beta X$ by ② and every $f \in C_b(X)$ extends continuously to βX by ③.

Ex ① If X is a compact Hausdorff space then it is Tychonoff. Since $\Delta(X)$ is homeomorphic to X, it is compact and hence closed. Thus $\beta X = \Delta(X) \cong X$.

② For $X := (0,1]$, $\beta X \neq [0,1]$. Indeed, $\sin(\frac{1}{t}) \in C_b(0,1]$ but has no continuous extension to $[0,1]$, but does extend continuously to βX.

③ Let $X = \mathbb{N}$ with the discrete topology so that $C_b(\mathbb{N}) = \ell^\infty(\mathbb{N})$. Hence $\beta\mathbb{N} \subseteq \ell^\infty(\mathbb{N})^*$. Suppose $\omega \in \beta\mathbb{N}$ is the weak* limit of $(\delta_{n_i})_{i\in I}$ for $(n_i)_{i\in I} \subseteq \mathbb{N}$. Then for any $S \subseteq \mathbb{N}$

$$\langle 1_S, \omega\rangle = \lim_{i\to\infty} \langle 1_S, \delta_{n_i}\rangle = \lim_{i\to\infty} 1_S(n_i) \in \{0,1\}$$

Since simple functions are dense in $\ell^\infty(\mathbb{N})$, ω is determined by its values on 1_S for $S \subseteq \mathbb{N}$. Consequently, we can think of it as a map $\omega : 2^{\mathbb{N}} \to \{0,1\}$ where $\omega(S) := \langle 1_S, \omega\rangle$.
Observe that

$$\omega(\mathbb{N}) = \lim_{i\to\infty} 1_{\mathbb{N}}(n_i) = 1$$
$$\omega(\emptyset) = \lim_{i\to\infty} 1_{\emptyset}(n_i) = 0.$$

Also, if $\omega(S) = 1$, then $\exists i_0 \in I$ s.t. $\forall i \geq i_0$ $n_i \in S$. Consequently $n_i \notin S^c$ $\forall i \geq i_0$ and so $\omega(S^c) = 0$. This can be used to show that for any $d \in \mathbb{N}$ and $S_1, \ldots, S_d \subseteq \mathbb{N}$ disjoint

$$\omega(S_1 \cup \cdots \cup S_d) = \omega(S_1) + \cdots + \omega(S_d).$$

Thus $\omega \in \beta\mathbb{N}$ defines an finitely additive $\{0,1\}$-valued measure on $\mathbb{N}$. From this point of view, $n \in \mathbb{N} \subseteq \beta\mathbb{N}$ corresponds to δ_n. Dirac mass □

Prop For $\omega \in \beta\mathbb{N}$, $\omega \in \mathbb{N}$ if and only if there exists $F \subset \mathbb{N}$ finite with $\omega(F) = 1$.

Proof ($\Longrightarrow$) If $\omega = n \in \mathbb{N}$, then $\omega(\{n\}) = 1$.

($\Longleftarrow$) Suppose $F \subset \mathbb{N}$ is finite with $\omega(F) = 1$. Then

$$1 = \omega(F) = \sum_{n \in F} \omega(\{n\})$$

So $\exists n_0 \in F$ with $\omega(\{n_0\}) = 1$. But then if ω is the weak* limit of $(\delta_{n_i})_{i \in I} \subset \mathbb{N}$ then this means $\exists i_0 \in I$ s.t. $\forall i \geq i_0$ $n_i \in \{n_0\}$. That is $n_i = n_0$ $\forall i \geq i_0$, and so $\omega = \lim_{i \to \infty} \delta_{n_i} = \delta_{n_0}$. $\square$

- $\omega \in \beta\mathbb{N}$ is sometimes called an <u>ultrafilter</u> (on $\mathbb{N}$). $\omega \in \beta\mathbb{N} \setminus \mathbb{N}$ is called a <u>free</u> (or <u>non-principal</u>) ultrafilter.

Prop Let $(a_n)_{n \in \mathbb{N}} \subset \mathbb{F}$ be a convergent sequence. For any $\omega \in \beta\mathbb{N} \setminus \mathbb{N}$

$$\omega((a_n)_{n \in \mathbb{N}}) = \lim_{n \to \infty} a_n$$

Proof Let $a_0 := \lim_{n \to \infty} a_n$. Let $\varepsilon > 0$ and let $N \in \mathbb{N}$ be such that $|a_n - a_0| < \varepsilon$ $\forall n \geq N$. Define

$$b_n := \begin{cases} a_n & \text{if } n \leq N \\ a_0 & \text{otherwise} \end{cases}$$

Then

$$\|(a_n)_{n \in \mathbb{N}} - (b_n)_{n \in \mathbb{N}}\|_\infty < \varepsilon$$

Consequently

$$* \quad |\omega((a_n)_{n \in \mathbb{N}}) - \omega((b_n)_{n \in \mathbb{N}})| \leq \|\omega\| \varepsilon.$$

Since $\omega \notin \mathbb{N}$, the previous proposition implies $\omega(\{n\}) = 0$ $\forall n \in \mathbb{N}$. Consequently

$$\omega((a_0)_{n \in \mathbb{N}} - (b_n)_{n \in \mathbb{N}}) = \sum_{n=1}^{N} (a_0 - a_n)\, \omega(\{n\}) = 0$$

So

$$\omega((b_n)_{n \in \mathbb{N}}) = \omega((a_0)_{n \in \mathbb{N}}) = a_0\, \omega(\mathbb{N}) = a_0 \cdot 1 = a_0.$$

Thus $(*)$ implies

$$|\omega((a_n)_{n \in \mathbb{N}}) - a_0| \leq \|\omega\| \cdot \varepsilon$$

Letting $\varepsilon \to 0$ completes the proof. $\square$

- In light of the above proposition, for $\omega \in \beta\mathbb{N} \setminus \mathbb{N}$ and $(a_n)_{n \in \mathbb{N}} \subset \ell^\infty(\mathbb{N})$ we write

$$\lim_{n \to \omega} a_n := \omega((a_n)_{n \in \mathbb{N}})$$

which is a "limit" that always exists. We can define this for $\omega = \delta_{n_0} \in \mathbb{N}$, but in this case

$$\lim_{n \to \omega} a_n = a_{n_0},$$

and so this need not agree with the usual limit.

Cor Let X be a Tychonoff space. For $\mu \in M(\beta X)$, define $L_\mu \in C_b(X)^*$ by
$$L_\mu(f) := \int_{\beta X} f^\beta \, d\mu \qquad f \in C_b(X).$$
Then the map
$$\rho : M(\beta X) \longrightarrow C_b(X)^*$$
$$\mu \mapsto L_\mu$$
is an isometric isomorphism.

Proof First consider $V : C_b(X) \to C(\beta X)$ defined by $Vf = f^\beta$. Since f^β is the unique extension of f to βX (by density of X in βX), one can has
$$V(\alpha f + g) = (\alpha f + g)^\beta = \alpha f^\beta + g^\beta = \alpha Vf + Vg.$$
So V is linear. Also, by density of X in βX,
$$\|f\| = \sup_{x \in X} |f(x)| = \sup_{x \in \beta X} |f^\beta(x)| = \|Vf\|.$$
So V is an isometry. Lastly, for any $g \in C(\beta X)$ we have $V(g|_X) = g$, so V is surjective.
Now, for $\mu \in M(\beta X) = C(\beta X)^*$, we have $L_\mu = \mu \circ V$. So $\|L_\mu\| \le \|\mu\| \, \|V\| = \|\mu\|$, since V is an isometry, and in fact $\|L_\mu\| = \|\mu\|$ since V is surjective. So ρ is an isometry. To see that ρ is surjective, let $L \in C_b(X)^*$. Then $L \circ V^{-1} \in C(\beta X)^*$ with $\|L \circ V^{-1}\| = \|L\|$. Since $C(\beta X)^* = M(\beta X)$, we have $L \circ V^{-1} = \mu$ for some $\mu \in M(\beta X)$, and so $L_\mu = \mu \circ V = L$. $\square$

Thm Let X be a Tychonoff space. Then $C_b(X)$ is separable if and only if X is a compact metric space.

Proof ($\Longrightarrow$) By a theorem in Section II.5, $(C_b(X)^*)_1$ is weak* metrizable. By the first theorem in this section, X is homeomorphic to a subset of $(C_b(X)^*)_1$ and hence is metrizable. Also note that $\beta X \subset (C_b(X)^*)_1$ is metrizable. To see that X is compact, we'll show $X = \beta X$.
Suppose, towards a contradiction, that $\exists\, \omega \in \beta X \setminus X$. Since βX is metrizable, we can find a sequence $(x_n)_{n \in \mathbb{N}} \subseteq X$ converging to ω. Without loss of generality, we may assume $x_n \neq x_m$ for $n \neq m$. Define
$$A := \{x_n : n \text{ odd}\} \qquad B := \{x_n : n \text{ even}\}$$
Then as subsets of X, they are closed since they contain all of their limit points (in X). They are also disjoint. So using Urysohn's Lemma (X is normal as a metric space) we can find $f \in C_b(X)$ with $f|_A \equiv 1$ and $f|_B \equiv 0$. But then
$$1 = \lim_{n\to\infty} f(x_{2n+1}) = f^\beta(\omega) = \lim_{n\to\infty} f(x_{2n}) = 0$$
a contradiction. So $X = \beta X$ and is compact.

($\Longleftarrow$) Let d be the metric on d. Since X is totally bounded, for each $n\in\mathbb{N}$ we can find an open cover of X of the form

$$\mathcal{U}^{(n)} := \{B(x_j^{(n)}, \tfrac{1}{n}) : 1\le j\le d_n\} \qquad x_1^{(n)},\dots,x_{d_n}^{(n)}\in X$$

Note that X is normal as a compact metric space. We will use the following fact (see V.6.5 in Conway for a proof) applied to each $\mathcal{U}^{(n)}$:

$\boxed{\text{Fact}}$ For X a normal space and $\mathcal{U}=\{U_1,\dots,U_d\}$ a finite open cover, there exists a partition of unity subordinate to $\mathcal{U}$: continuous functions $f_1,\dots,f_d : X\to[0,1]$ satisfying

1. $f_j(x)=0$ for $x\in U_j^c$, $j=1,\dots,d$
2. $\sum_{j=1}^{d} f_j \equiv 1$

For each $n\in\mathbb{N}$, let $\{f_1^{(n)},\dots,f_{d_n}^{(n)}\}$ be a partition of unity subordinate to $\mathcal{U}^{(n)}$. Let $\mathcal{F}$ be the complex-rational span of

$$\bigcup_{n=1}^{\infty}\{f_1^{(n)},\dots,f_{d_n}^{(n)}\}$$

Then $\mathcal{F}$ is countable and so it suffices to show it is dense in $C_b(X)=C(X)$. Fix $g\in C(X)$ and $\varepsilon>0$. Then g is uniformly continuous: $\exists\,\delta>0$ such that

$$d(x_1,x_2)<\delta \implies |g(x_1)-g(x_2)|<\varepsilon/2$$

Let $n>2/\delta$ and consider $\mathcal{U}^{(n)}$. If $x_1,x_2\in B(x_j^{(n)},\tfrac{1}{n})$, then

$$d(x_1,x_2)\le d(x_1,x)+d(x,x_2) < \tfrac{2}{n}<\delta$$

so $|g(x_1,x_2)|<\frac{\varepsilon}{2}$. Let $\alpha_j,\beta_j\in\mathbb{Q}$ be such that

$$|\alpha_j+i\beta_j - g(x_j^{(n)})|<\tfrac{\varepsilon}{2}.$$

Define

$$f := \sum_{j=1}^{d_n}(\alpha_j+i\beta_j)f_j^{(n)} \in \mathcal{F}$$

For any $x\in X$ we have

$$\begin{aligned}
|g(x)-f(x)| &= \Big|g(x)\Big(\sum_{j=1}^{d_n} f_j^{(n)}(x)\Big) - \sum_{j=1}^{n}(\alpha_j+i\beta_j)f_j^{(n)}(x)\Big| \\
&\le \sum_{j=1}^{d_n} f_j^{(n)}(x)\,|g(x)-(\alpha_j+i\beta_j)| \\
&\le \sum_{j=1}^{d_n} f_j^{(n)}(x)\big(|g(x)-g(x_j^{(n)})| + |g(x_j^{(n)})-(\alpha_j+i\beta_j)|\big) \\
&< \sum_{j:\,x\in B(x_j^{(n)},\frac{1}{n})} f_j^{(n)}(x)\,\Big(\tfrac{\varepsilon}{2}+\tfrac{\varepsilon}{2}\Big) \le \varepsilon\sum_{j=1}^{d_n} f_j^{(n)}(x) = \varepsilon.
\end{aligned}$$

Thus $\mathcal{F}$ is dense. $\square$

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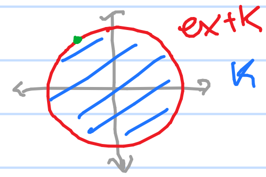
V.7 The Krein–Milman Theorem

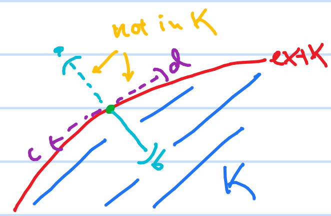
Def Let X be a vector space and $K \subseteq X$ convex. A point $x_0 \in K$ is called an extreme point if $x_0 \notin (a,b)$ for all $a,b \in K$. The set of extreme points in K is denoted $\operatorname{ext} K$.

Ex ① $X = \mathbb{R}^2$

$K = \{ \binom{x}{y} \in \mathbb{R}^2 : x^2+y^2 \le 1\}$

$\operatorname{ext} K = \{ \binom{x}{y} \in \mathbb{R}^2 : x^2+y^2 = 1\} = \partial K$

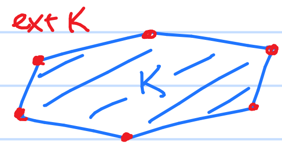




② $X = \mathbb{R}^2$, $K = \{ \binom{x}{y} \in \mathbb{R}^2 : y \le 0\}$, $\operatorname{ext} K = \emptyset$

③ $X = \mathbb{R}^2$, $K = \{ \binom{x}{y} \in \mathbb{R}^2 : y < 0\} \cup \{\binom{0}{0}\}$, $\operatorname{ext} K = \{\binom{0}{0}\}$.

④ $X = \mathbb{R}^2$



⑤ X a normed space and $K = (X)_1$, then $\operatorname{ext} K \subseteq \{x \in X : \|x\| = 1\}$, but can be empty:

Consider $X = L^1[0,1]$ and $K = \{ f \in L^1[0,1] : \|f\|_1 \le 1\}$. Then $\operatorname{ext} K = \emptyset$.

Indeed, if $\|f\|_1 = 1$ let $t_0 \in (0,1)$ be s.t.

$$\int_0^{t_0} |f(t)|\,dt = \tfrac{1}{2}$$

Define $g := 1_{[0,t_0]} 2f$ and $h = 1_{[t_0,1]} 2f$. Then $\|g\|_1 = \|h\|_1 = 1$ so $g, h \in K$ and $f = \tfrac{1}{2} g + \tfrac{1}{2} h$. So f is not extreme. □

Prop Let X be a vector space. For $K \subseteq X$ convex and $x_0 \in K$, the following statements are equivalent:

① $x_0 \in \operatorname{ext} K$.

② If $x_0 = \tfrac{1}{2}x_1 + \tfrac{1}{2}x_2$ for some $x_1, x_2 \in X$, then either $x_1 \notin K$, $x_2 \notin K$, or $x_1 = x_2 = x_0$.

③ If $x_0 \in (x_1, x_2)$ for some $x_1, x_2 \in X$, then either $x_1 \notin K$, $x_2 \notin K$, or $x_1 = x_2 = x_0$.

④ If $x_1, \dots, x_n \in K$ and $x_0 \in \operatorname{co}\{x_1, \dots, x_n\}$, then $x_0 = x_j$ for some $1 \le j \le n$.

⑤ $K \setminus \{x_0\}$ is convex.

Proof Exercise □

Thm (The Krein-Milman Theorem)

Let X be a LCS. For $K \subseteq X$ a nonempty compact convex subset, $\operatorname{ext} K \ne \emptyset$ and $K = \overline{\operatorname{co}}(\operatorname{ext} K)$.

Proof If $K = \{x\}$, then $\operatorname{ext} K = \{x\}$ and $\overline{\operatorname{co}}(\operatorname{ext} K) = \{x\} = K$. So we shall assume K is not a singleton. Note that by the previous proposition, $x \in \operatorname{ext} K$

if and only if $K \setminus \{x\}$ is convex. Since K is closed, $K \setminus \{x\}$ is a relatively open convex strict subset. Thus, in order to show $\operatorname{ext} K$ is non-empty we'll construct a maximal relatively open convex strict subset $U \subseteq K$ and show $K = U \cup \{x\}$ for some $x \in \operatorname{ext} K$.

Let $\mathcal{U}$ be the collection of relatively open convex strict subsets of K, ordered by inclusion. Then $\mathcal{U}$ is nonempty: let $x, y \in K$ be distinct points, then by the Hahn-Banach separation theorem $\exists x^* \in X^*$ and $\alpha \in \mathbb{R}$ such that

$$x \in \{ x \in X : \operatorname{Re} \langle x, x^* \rangle < \alpha \} \not\ni y,$$

and consequently $\{x \in K : \operatorname{Re} \langle x, x^* \rangle < \alpha\} \in \mathcal{U}$.

Now, suppose $\mathcal{C} \subseteq \mathcal{U}$ is a chain. Define

$$U_0 := \bigcup_{U \in \mathcal{C}} U.$$

Then U_0 is relatively open and convex (since $\mathcal{C}$ is a chain). If $U_0 = K$, then $\mathcal{C}$ is an open cover for K. Thus $K = U$ for some $U \in \mathcal{C}$ by the compactness of K and the fact that $\mathcal{C}$ is a chain. But this contradicts $U \in \mathcal{U}$ being a strict subset. Thus we must have $U_0 \subsetneq K$, and so $U_0 \in \mathcal{U}$. Clearly U_0 is an upper bound for $\mathcal{C}$, so Zorn's Lemma implies $\mathcal{U}$ has a maximal element, which we denote U.

$\boxed{\text{Claim 1}}$ For any relatively open convex subset $V \subseteq K$, either $V \cup U = U$ or $V \cup U = K$.

Indeed, for $x \in K$ and $t \in [0,1)$, define $T_{x,t} : K \to K$ by

$$T_{x,t}(y) = tx + (1-t)y$$

Then $T_{x,t}$ is continuous and "affine":

$$T_{x,t}(sy_1 + (1-s)y_2) = s\, T_{x,t}(y_1) + (1-s)\, T_{x,t}(y_2) \qquad s \in [0,1],\ y_1, y_2 \in K.$$

Now, suppose $x \in U$. Then $T_{x,t}(U) \subseteq U$ by convexity of U, so $U \subseteq T_{x,t}^{-1}(U)$. Also, $T_{x,t}^{-1}(U)$ is open and convex since $T_{x,t}$ is continuous and affine (respectively). By the maximality of U, $T_{x,t}^{-1}(U)$ is either U or K. Let $y \in \overline{U}$, then $T_{x,t}(y) \in [x, y) \in U$ by a proposition from Section IV.1. Thus $\overline{U} \subseteq T_{x,t}^{-1}(U)$ and so we must have $T_{x,t}^{-1}(U) = K$ since $K \setminus U \neq \emptyset$ and the convexity of K implies $(K \cap \overline{U}) \setminus U \neq \emptyset$. We have shown that

$$T_{x,t}(K) \subseteq U \quad \text{for} \quad x \in U \text{ and } t \in [0,1).$$

Let $V \subset K$ be open and convex. For $x \in U$, $y \in V$, and $t \in (0,1)$ we have

$$tx + (1-t)y = T_{x,t}(y) \in T_{x,t}(K) \subseteq U \subseteq U \cup V.$$

So $U \cup V$ is convex. By maximality of U, either $U \cup V = U$ or $U \cup V = K$. $\square$

With Claim 1 in hand, we now show $|K\setminus\mathcal{U}|=1$. Suppose, towards a contradiction that $\exists\, a,b\in K\setminus\mathcal{U}$ distinct. Using Hahn-Banach separation, we can find $x^*\in X^*$ and $\alpha\in\mathbb{R}$ such that

$$a\in V := \{x\in K : \mathrm{Re}\langle x,x^*\rangle<\alpha\}\not\ni b$$

so $V\subseteq K$ is relatively open and convex. Since $a\notin\mathcal{U}$, the claim implies $\mathcal{U}\cup V=K$. But $b\notin\mathcal{U}\cup V$, a contradiction. Thus $|K\setminus\mathcal{U}|\le 1$, and since $\mathcal{U}$ is a strict subset we obtain $|K\setminus\mathcal{U}|=1$. Let $x\in K\setminus\mathcal{U}$. Then $K\setminus\{x\}=\mathcal{U}$, which is convex and so $x\in\mathrm{ext}\,K$ by the previous proposition. Consequently $\mathrm{ext}\,K\ne\emptyset$.

Claim 2 For $V\subseteq X$ open and convex, if $\mathrm{ext}\,K\subseteq V$, then $K\subseteq V$.
Indeed, if not then $\exists\, V$ open convex with $\mathrm{ext}\,K\subseteq V$ and $K\not\subseteq V$. Then $K\cap V\subsetneq K$ and so $K\cap V\in\mathcal{U}$. Let $\mathcal{U}\in\mathcal{U}$ be a maximal element containing $K\cap V$. Then by the above, $K\setminus V\supseteq K\setminus\mathcal{U}=\{x\}$ for $x\in\mathrm{ext}\,K$, which contradicts $\mathrm{ext}\,K\subseteq V$. $\square$

Finally, we let $E:=\overline{\mathrm{co}}(\mathrm{ext}\,K)\subseteq K$. If $x_0\in E^c$, then Hahn-Banach separation yields $x^*\in X^*$ and $\alpha\in\mathbb{R}$ such that

$$E\subseteq V:=\{x\in X : \mathrm{Re}\langle x,x^*\rangle<\alpha\}\not\ni x_0$$

But $\mathrm{ext}\,K\subseteq E\subseteq V$, so Claim 2 implies $K\subseteq V$ and so $x_0\in K^c$. Thus $K=E$. $\square$

- Recall that for a Banach space X, $(X^*)_1$ is weak* compact by the Banach-Alaoglu theorem. It is also convex, so we obtain:

Cor For a Banach space X, $(X^*)_1=\overline{\mathrm{co}}(\mathrm{ext}\,(X^*)_1)$.

Ex ① Recall that $\mathrm{ext}\,(L^1[0,1])_1=\emptyset$. Thus there does <u>not</u> exist a Banach space X such that $L^1[0,1]$ is isometrically isomorphic to X^*.

② Likewise, $c_0(\mathbb{N})$ is not the dual of any Banach space because $\mathrm{ext}\,(c_0(\mathbb{N}))_1=\emptyset$. Indeed, for $(x_n)_{n\in\mathbb{N}}\in(c_0(\mathbb{N}))_1$ let $N\in\mathbb{N}$ be s.t. $|x_n|<\frac12$ $\forall n\ge N$. Define

$$y_n:=\begin{cases}x_n & \text{if } n\le N\\ x_n+2^{-n} & \text{if } n>N\end{cases}\qquad z_n:=\begin{cases}x_n & \text{if } n\le N\\ x_n-2^{-n} & \text{if } n>N\end{cases}$$

Then $(y_n)_{n\in\mathbb{N}},(z_n)_{n\in\mathbb{N}}\in(c_0(\mathbb{N}))_1$ with $(x_n)_{n\in\mathbb{N}}=\frac12(y_n)_{n\in\mathbb{N}}+\frac12(z_n)_{n\in\mathbb{N}}$, but $(y_n)_{n\in\mathbb{N}}\ne(x_n)_{n\in\mathbb{N}}\ne(z_n)_{n\in\mathbb{N}}$. Hence $(x_n)_{n\in\mathbb{N}}\notin\mathrm{ext}\,(c_0(\mathbb{N}))_1$. $\square$

Thm Let X be a LCS and $K\subseteq X$ a compact convex subset. If $F\subseteq K$ satisfies $\overline{\mathrm{co}}(F)=K$, then $\mathrm{ext}\,K\subseteq\overline{F}$.

Proof Since $F \subset K$ and $\overline{co}(F) = \overline{co}(\overline{F})$, we may assume F is closed. Suppose, towards a contradiction, that $\exists x_0 \in ext\, K \setminus F$. Hence $x_0 \in F^c$, an open set, and so we can find a continuous seminorm p on X such that

$$\{x \in X : p(x - x_0) < 1\} \subseteq F^c.$$

Define

$$U_0 := \{x \in X : p(x) < \tfrac{1}{3}\}$$

so that $(x_0 + U_0) \cap (F + U_0) = \emptyset$. Consequently, $x_0 \notin \overline{F + U_0}$. Now $\{y + U_0 : y \in F\}$ is an open cover for F, which is compact since K is. So there exist $y_1, \ldots, y_n \in F$ such that

$$F \subseteq \bigcup_{j=1}^{n} y_j + U_0.$$

For each $j = 1, \ldots, n$ define

$$K_j := \overline{co}(F \cap (y_j + U_0)).$$

Then $K_j \subseteq K$ is compact convex and satisfies

Homework 7

$$(*) \quad K = \overline{co}(F) = \overline{co}(K_1 \cup \cdots \cup K_n) = co(K_1 \cup \cdots \cup K_n)$$

~~Moreover,~~ Also note that for $y_j + x_1, y_j + x_2 \in y_j + U_0$ and $0 \leq t \leq 1$

$$t(y_j + x_1) + (1-t)(y_j + x_2) = y_j + tx_1 + (1-t)x_2 \in y_j + U_0.$$

U_0 is convex

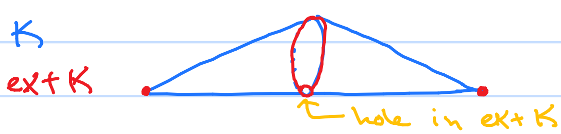
Thus $K_j \subseteq y_j + \overline{U_0} \subseteq \overline{F + U_0}$. Now, since $x_0 \in K$ we have by $(*)$ that

$$x_0 = \sum_{j=1}^{n} \alpha_j x_j$$

for $x_j \in K_j$ and $\alpha_j > 0$ satisfying $\sum \alpha_j = 1$. Since $x_0 \in ext\, K$, it must be that $x_0 = x_j$ for some $j = 1, \ldots, n$. But then $x_0 \in K_j \subseteq \overline{F + U_0}$, a contradiction. $\square$

Ex Note that $ext\, K$ need not be closed, even for compact convex K:

$X = \mathbb{R}^3$



$\square$

Def Let X, Y be LCS's. A map $T: X \to Y$ is *affine* if

$$T\left(\sum_{j=1}^{n} \alpha_j x_j\right) = \sum_{j=1}^{n} \alpha_j T(x_j)$$

for $x_1, \ldots, x_n \in X$ and $\alpha_1, \ldots, \alpha_n > 0$ satisfying $\alpha_1 + \cdots + \alpha_n = 1$.

- Note that any linear map is affine, but the converse is not true (e.g. $T: \mathbb{R} \to \mathbb{R}$ defined by $T(x) = x + 1$)

Prop Let X, Y be LCS's, let $K \subseteq X$ be compact convex, and let $T: K \to Y$ be a continuous affine map. Then $T(K)$ is compact convex and $ext\, T(K) \subseteq T(ext\, K)$.

Proof $T(K)$ is compact by the continuity of T and is convex since T is affine. Now, let $y \in ext\, T(K)$. Note that $K_0 := T^{-1}(\{y\})$ is compact as a closed subset of K, and is convex since T is affine. Let $x \in ext\, K_0$. We claim $x_0 \in ext\, K$. Indeed, if $x_0 = \frac{1}{2}x_1 + \frac{1}{2}x_2$ for $x_1, x_2 \in K$

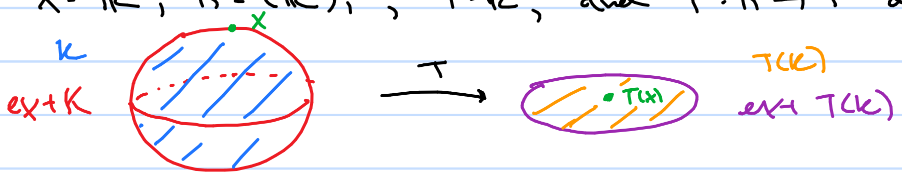
then we have

$$y = T(x_0) = T(\tfrac{1}{2}x_1 + \tfrac{1}{2}x_2) = \tfrac{1}{2}T(x_1) + \tfrac{1}{2}T(x_2)$$

Since $y \in \operatorname{ext} T(K)$, we have $T(x_1) = T(x_2) = y$. So $x_1, x_2 \in K_0$ and so $x_0 \in K_0$ implies $x_1 = x_2 = x_0$. Hence $x \in \operatorname{ext} K$. $\square$

- The inclusion $\operatorname{ext} T(K) \subseteq T(\operatorname{ext} K)$ can be strict, as in the following example.

$\boxed{\text{Ex}}$ Let $X = \mathbb{R}^3$, $K = (\mathbb{R}^3)_1$, $Y = \mathbb{R}^2$, and $T : K \to Y$ defined by $T\begin{pmatrix} x \\ y \\ z \end{pmatrix} = \begin{pmatrix} x \\ y \end{pmatrix}$



$\square$

V.8 The Stone–Weierstrass Theorem

Def Let X be a locally compact Hausdorff space. A subset $A \subseteq C_0(X)$ is called a subalgebra if it is a subspace and $f \cdot g \in A$ for all $f, g \in A$. We say A is unital if $1 \in A$. We say A is a $*$-subalgebra if it also satisfies $\bar{f} \in A$ for all $f \in A$. We say A separates points in X if for all $x, y \in X$ with $x \neq y$ there exists $f \in A$ with $f(x) \neq f(y)$.

Thm (The Stone–Weierstrass Theorem)
Let X be a compact Hausdorff space. If $A \subseteq C(X)$ is a closed unital $*$-subalgebra that separates points in X, then $A = C(X)$.

Proof It suffices to show $A^{\perp} = \{0\}$. Suppose, towards a contradiction, that $A^{\perp} \neq \{0\}$. Since $A^{\perp}$ is a subspace, it follows that $(A^{\perp})_1 \neq \{0\}$. Moreover, this is weak* compact by the Banach–Alaoglu theorem. By the Krein–Milman theorem, $\exists\, \mu \in \text{ext}\,(A^{\perp})_1$. Define

$$\text{supp}(\mu) := X \setminus \left(\bigcup \{ U \subseteq X : U \text{ open and } |\mu|(U) = 0 \} \right)$$

It follows that $x \in \text{supp}(\mu)$ iff for every open set $U \ni x$, $|\mu|(U) > 0$. Since $\text{supp}(\mu)^c$ is open, we therefore must have $|\mu|(\text{supp}(\mu)^c) = 0$. Also

$$\int_X f\, d\mu = \int_{\text{supp}(\mu)} f\, d\mu \qquad \forall f \in C(X).$$

Now, since $\mu \in \text{ext}\,(A^{\perp})_1$, we must have $\|\mu\| = 1$ because otherwise

$$\mu = \|\mu\| \left(\tfrac{1}{\|\mu\|} \mu \right) + (1 - \|\mu\|) \cdot 0$$

and $\frac{1}{\|\mu\|} \mu \neq \mu \neq 0$. Consequently

$$1 = \|\mu\| = |\mu|(X) = |\mu|(\text{supp}(\mu)),$$

and thus $\text{supp}(\mu) \neq \emptyset$. Fix $x_0 \in \text{supp}(\mu)$.

Claim $\text{supp}(\mu) = \{x_0\}$.

Let $x \in X \setminus \{x_0\}$ (if $X = \{x_0\}$ then we are done). We will construct $f \in C(X)$ such that $f(x) = 0$ while $f|_{\text{supp}(\mu)} \equiv \alpha > 0$.

Since A separates points in X, $\exists f_1 \in A$ with $f_1(x_0) \neq f_1(x) =: \beta$. Since A is a unital $*$-subalgebra

$$f_2 := |f_1 - \beta|^2 = \overline{(f_1 - \beta)} \cdot (f_1 - \beta) \in A$$

Note that $f_2(x) = 0$ while $f_2(x_0) > 0$. Define

$$f := \frac{1}{1 + \|f_2\|} f_2 \in A$$

Then $0 \leq f < 1$, $f(x) = 0$, and $f(x_0) > 0$. Towards showing $f|_{\text{supp}(\mu)}$ is constant, consider $f \cdot \mu,\ (1-f) \cdot \mu \in M(X)$ defined by

$$(f \cdot \mu)(E) := \int_E f\, d\mu \qquad ((1-f) \cdot \mu)(E) := \int_E (1-f)\, d\mu$$

Note that $f\cdot\mu, (1-f)\cdot\mu \in A^\perp$ since $gf, g(1-f)\in A$ $\forall g\in A$ and $\mu\in A^\perp$. Let

$$\alpha := \|f\cdot\mu\| = \int_X f\, d|\mu|.$$

Since $f(x_0)>0$ and $f\in C(X)$, $\exists\, U \ni x_0$ open and $\varepsilon>0$ such that $f(y)\geq\varepsilon$ for all $y\in U$. Since $x_0\in \text{supp}(\mu)$, we have $|\mu|(U)>0$. Thus

$$\alpha \geq \int_U f\, d|\mu| \geq \varepsilon\, |\mu|(U) > 0.$$

So $\alpha>0$. Similarly, using $f(x_0)<1$ we can show $\alpha<1$. Also,

$$1-\alpha = 1-\int_X f\, d|\mu| = \|\mu\| - \int_X f\,d|\mu| = \int (1-f)\, d|\mu| = \|(1-f)\cdot\mu\|.$$

Consequently,

$$\mu = \alpha \underbrace{\frac{f\cdot\mu}{\|f\cdot\mu\|}}_{\in (A^\perp)_1} + (1-\alpha)\underbrace{\frac{(1-f)\cdot\mu}{\|(1-f)\cdot\mu\|}}_{\in(A^\perp)_1}$$

Since $\mu\in \text{ext}\,(A^\perp)_1$, we must have $\mu = \frac{f\cdot\mu}{\|f\cdot\mu\|} = \alpha^{-1}f\cdot\mu$. That is,

$$\mu(E) = \int_E \alpha^{-1}\cdot f\, d\mu \qquad \forall E\subseteq X \text{ Borel}.$$

It follows that $\alpha^{-1}\cdot f = 1$ μ-a.e. Since f is continuous and $|\mu|(\text{supp}(\mu))>0$, we must have $f|_{\text{supp}(\mu)}\equiv\alpha$. Since $f(x)=0<\alpha$, it must be that $x\notin\text{supp}(\mu)$. Since $x\in X\setminus\{x_0\}$ was arbitrary, we have $\text{supp}(\mu)=\{x_0\}$. $\square$

The claim implies $\mu=\gamma\cdot\delta_{x_0}$ where $|\gamma|=1$. However, $\mu\in A^\perp$ and $1\in A$ imply

$$\gamma = \int 1\, d\mu = 0,$$

a contradiction. $\square$

- If $A\subseteq C(X)$ is a $*$-subalgebra, then so is $\overline{A}$ (Exercise). Thus the above theorem implies that if $A\subseteq C(X)$ is a unital $*$-subalgebra that separates points in X, then A is dense in $C(X)$. In fact, so long as $1\in\overline{A}$ then we do not need to assume A is unital.

Ex Let $A = C[0,1]$ be the set of polynomials. Then A is dense in $C[0,1]$. Indeed, the above discussion implies it suffices to show A separates points in $[0,1]$. But $p(t)=t\in A$ separates all points. $\square$

- A being unital was crucial in the above proof of the Stone–Weierstrass theorem. The following theorem shows this is a necessary hypothesis, and yet we can characterize those A that only fail to be unital.

Cor Let X be a compact Hausdorff space. If $A\subseteq C(X)$ is a closed $*$-subalgebra that separates points in X, then either $A=C(X)$ or there exists $x_0\in X$ such that

$$A = \{f\in C(X) : f(x_0)=0\}.$$

Proof First note that $A+\mathbb{F}$ is a closed unital $*$-subalgebra which separates points in X (exercise). Consequently, $A+\mathbb{F}=C(X)$ by the Stone–Weierstrass theorem. Suppose $A\neq C(X)$. Then by a theorem from Section I.2 we have

$$A^\perp \cong (C(X)/A)^* \cong (\mathbb{F})^* \cong \mathbb{F}.$$

Let $\mu \in A^\perp$ with $\|\mu\| = 1$. As we saw in the proof of the Stone-Weierstrass theorem, $f\mu \in A^\perp$ for all $f \in A$. Consequently $f\mu = \alpha\mu$ for some $\alpha \in \mathbb{F}$. Thus $f|_{\mathrm{supp}(\mu)}$ is constant for all $f \in A$. Since A separates points, we must have $\mathrm{supp}(\mu) = \{x_0\}$ for some $x_0 \in X$. Thus $A^\perp = \mathrm{span}\{\delta_{x_0}\}$ and thus

$$A = {}^\perp(A^\perp) = {}^\perp(\mathrm{span}\{\delta_{x_0}\}) = \{f \in C(X) : f(x_0) = 0\}. \qquad \square$$

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Cor Let X be a locally compact Hausdorff space. If $A \subseteq C_0(X)$ is a closed $*$-subalgebra that separates points in X and for all $x \in X$ there exists $f \in A$ with $f(x) \neq 0$, then $A = C_0(X)$.

Proof Let $X_\infty = X \sqcup \{\infty\}$ be the one point compactification of X. Then every $f \in C_0(X)$ extends continuously to X_∞ by setting $f(\infty) = 0$; that is

$$C_0(X) = \{f \in C(X_\infty) : f(\infty) = 0\}.$$

Now, A still separates points in X_∞ since, by assumption, $\forall x \in X$ $\exists f \in A$ with $f(x) \neq 0 = f(\infty)$. Thus the previous corollary implies (since A is non-unital in $C(X_\infty)$) that there exists $x_0 \in X_\infty$ such that $A = \{f \in C(X_\infty) : f(x_0) = 0\}$. Our assumption implies that we must have $x_0 = \infty$, and so $A = C_0(X)$. $\square$

- For a locally compact Hausdorff space, denote

$$P(X) := \{\mu \in M(X) : \mu \text{ is positive}, \ \mu(X) = 1\}$$

i.e. the regular probability measures on X. Note that $P(X)$ is convex.

Thm Let X be a compact Hausdorff space. Then

$$\mathrm{ext}\,(M(X))_1 = \{\alpha\delta_x : |\alpha| = 1 \text{ and } x \in X\}$$

$$\mathrm{ext}\,P(X) = \{\delta_x : x \in X\}$$

Proof Fix $x \in X$ and $\alpha \in \mathbb{F}$. Suppose $\alpha\delta_x = \frac{1}{2}\mu_1 + \frac{1}{2}\mu_2$ for some $\mu_1, \mu_2 \in (M(X))_1$. Then

$$1 = |\alpha\delta_x(\{x\})| \leq \tfrac{1}{2}|\mu_1(\{x\})| + \tfrac{1}{2}|\mu_2(\{x\})| \leq \tfrac{1}{2}\|\mu_1\| + \tfrac{1}{2}\|\mu_2\| \leq 1$$

So it must be that $|\mu_1(\{x\})| + |\mu_2(\{x\})| = 2$, and so $|\mu_1(\{x\})| = |\mu_2(\{x\})| = 1$ since $\|\mu_1\|, \|\mu_2\| \leq 1$. Note that $|\mu_j|(\{x\}) = |\mu_j(\{x\})|$, so $|\mu_j|(X \setminus \{x\}) = 0$ for $j = 1, 2$. So we must have $\mu_j = \alpha_j\delta_x$ with $|\alpha_j| = 1$ for $j = 1, 2$. Then $\alpha = \frac{1}{2}\alpha_1 + \frac{1}{2}\alpha_2$, but $|\alpha| = 1$ implies $\alpha \in \mathrm{ext}\, S^1$. So $\alpha_1 = \alpha_2 = \alpha$ and $\mu_1 = \mu_2 = \alpha\delta_x$. Thus $\alpha\delta_x \in \mathrm{ext}\,(M)_1$. The proof of $\delta_x \in \mathrm{ext}\,P(X)$ is similar (easier even).

We next argue the reverse inclusions. Let $\mu \in \mathrm{ext}\,(M(X))_1$. It suffices to show $\mathrm{supp}(\mu)$ is a singleton. Note that $\mu \neq 0$ implies $\exists x_0 \in \mathrm{supp}(\mu)$. Suppose, towards a contradiction, that $\exists x \in \mathrm{supp}(\mu) \setminus \{x_0\}$. Let U, V be open neighborhoods of x_0 and x, respectively, with $\bar{U}\cap\bar{V}=\phi$. Urysohn's lemma implies $\exists\, f: X\to[0,1]$ continuous with $f|_{\bar{U}}\equiv 1$ and $f|_{\bar{V}}\equiv 0$. Set

$$\alpha := \|f\mu\| \geq \int_U |f|\, d|\mu| = |\mu|(U) > 0,$$

where the last inequality follows from $x_0\in \operatorname{supp}(\mu)$. Also $1-\alpha = \|(1-f)\cdot\mu\|$ and

$$1-\alpha = \int_X (1-f)\, d|\mu| \geq \int_V (1-f)\, d|\mu| = |\mu|(V) > 0,$$

where the last inequality follows from $x\in \operatorname{supp}(\mu)$. Thus $0<\alpha<1$ and

$$\mu = \alpha\,\frac{f\cdot\mu}{\|f\cdot\mu\|} + (1-\alpha)\,\frac{(1-f)\cdot\mu}{\|(1-f)\cdot\mu\|}$$

Since μ is an extreme point, we must have $\frac{1}{\|f\cdot\mu\|} f\cdot\mu = \mu$. Thus $f = \|f\cdot\mu\| = \alpha$ μ-a.e., but $f|_U\equiv 1 > \alpha$, a contradiction. Thus $|\operatorname{supp}(\mu)|=1$, and $\operatorname{ext}(M(X))_1$ is the claimed set.

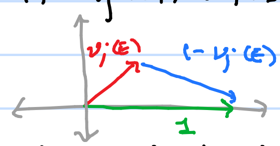
Finally, let $\mu\in \operatorname{ext} P(X)$. We'll show $\mu \in \operatorname{ext}(M(X))_1$, in which case $\mu=\alpha\cdot\delta_x$ for some $|\alpha|=1$ and $x\in X$ by the above argument. Since μ is a probability measure, then necessarily $\alpha=1$ and we are done. So suppose $\mu=\frac{1}{2}\nu_1+\frac{1}{2}\nu_2$ for $\nu_1,\nu_2\in(M(X))_1$. Note that

$$1=\|\mu\|\leq \tfrac{1}{2}\|\nu_1\|+\tfrac{1}{2}\|\nu_2\|\leq 1$$

so $\|\nu_1\|+\|\nu_2\|=2$, but $\|\nu_1\|, \|\nu_2\|\leq 1$ implies $\|\nu_1\|=\|\nu_2\|=1$. Also

$$1=\mu(X)=\tfrac{1}{2}\nu_1(X)+\tfrac{1}{2}\nu_2(X).$$

Since $|\nu_1(X)|, |\nu_2(X)|\leq 1$ and $1\in \operatorname{ext}\{z\in\mathbb{C} : |z|\leq 1\}$, we must have $\nu_1(X)=\nu_2(X)=1$. Consequently, $\nu_j(X)=\|\nu_j\|$ for $j=1,2$. For any Borel measurable $E\subseteq X$, if $\nu_j(E)\notin[0,1]$ then



$$1 < |\nu_j(E)| + |1-\nu_j(E)| = |\nu_j(E)| + |\nu_j(E^c)| \leq |\nu_j|(X) = 1$$

a contradiction. Thus $\nu_j\in P(X)$ for $j=1,2$. Since $\mu\in \operatorname{ext} P(X)$, we have $\nu_1=\nu_2=\mu$, and thus $\mu\in\operatorname{ext}(M(X))_1$. $\square$

V.12 The Krein-Smulian Theorem

- Let X be a Banach space and $A \subseteq X$ convex. If A is weakly closed, then $\forall r>0$

$$(A)_r := \{x \in A : \|x\| \leq r\} = A \cap B(0,r)$$

is weakly closed as the intersection of two weakly closed sets.
Conversely, if $(A)_r$ is weakly closed for all $r>0$, then A is weakly closed. Indeed, first note that A being convex implies it suffices to show A is normed closed. So if $(x_n)_{n\in\mathbb{N}} \subseteq A$ converges to some $x_0 \in X$, then the sequence being bounded implies $\exists r>0$ s.t. $(x_n)_{n\in\mathbb{N}} \subseteq (A)_r$. Since $(A)_r$ is convex and weakly closed, it is norm closed and so $x_0 \in (A)_r \subseteq A$.

- Now consider $A \subseteq X^*$ convex. If A is weak* closed, then we still have $(A)_r$ is weak* closed for all $r>0$, since $(X^*)_r$ is weak* closed: suppose $(x_i^*)_{i\in I} \in (X^*)_r$ is a net converging to some $x_0^* \in X^*$. Then for all $x \in (X)_1$ we have

$$|x_0^*(x)| = \lim_{i\to\infty} |x_i^*(x)| \leq \limsup_{i\to\infty} \|x_i^*\| \leq r$$

and so $\|x_0^*\| \leq r$.
However, we can argue the converse as we did above because convex norm closed sets in X^* are <u>not</u> necessarily weak*-closed.
(I claimed otherwise in lecture on 4/1, but I was wrong.)

Ex For $\phi \in X^{**} \setminus \hat{X}$, set $A := \ker\phi$. Then A is convex (it's a hyperplane), and norm-closed since $\phi : X^* \to \mathbb{F}$ is continuous with respect to the norm topology on X^*. However, it is <u>not</u> weak*-closed. Indeed, the weak*-closed hyperplanes in X^* correspond to $f: X^* \to \mathbb{F}$ which are weak* continuous. But $(X^*, \sigma(X^*,X))^* = X$, so $f = \hat{x}$ for some $x \in X$. Since $\phi \notin \hat{X}$, A cannot be weak* closed. $\square$

- Despite these obstacles, the converse is still true:

Thm (The Krein-Smulian Theorem)
Let X be a Banach space and $A \subseteq X^*$ a convex set. If

$$(A)_r = \{x^* \in A : \|x^*\| \leq r\}$$

is weak* closed for all $r>0$, then A is weak* closed.

- Before proving this theorem, we establish some lemmas.

Lemma 1 Let X be a Banach space. For $r>0$, let $\mathcal{F}_r$ be the collection of all finite subsets of $(X)_{1/r}$. Then

$$\bigcap_{F \in \mathcal{F}_r} F^{\diamond} = (X^*)_r$$

Proof Let E be the set on the left. For $x^* \in E$, we have for any $x \in X$ with $\|x\|=1$

$$|(x, x^*)| = r \cdot |(\tfrac{1}{r}x, x^*)| \leq r \cdot 1 = r$$

since $\{\frac{1}{r}x\} \in \mathcal{F}_r$. Thus $x^* \in (X^*)_r$. Conversely, if $x^* \in (X^*)_r$ and $F \in \mathcal{F}_r$, then for any $x \in F$

$$|(x, x^*)| \leq \|x\| \cdot \|x^*\| \leq \tfrac{1}{r} \cdot r = 1,$$

so that $x^* \in F^{\diamond}$. $\square$

Lemma 2 Let X and $A \subseteq X^*$ be as in the statement of the Krein-Smulian theorem. If $(A)_1 = \emptyset$, then there exists $x \in X$ such that

$$\operatorname{Re}(x, x^*) \geq 1 \qquad \forall x^* \in A.$$

Proof We begin by inductively building a sequence of finite subsets $F_0, F_1, \ldots \subseteq X$ satisfying

(i.n) $n \cdot F_n \subseteq (X)_1$,

(ii.n) $(A)_{n+1} \cap \bigcap_{j=0}^{n} F_j^{\diamond} = \emptyset$.

Set $F_0 = \{0\}$. Then $0 \cdot F_0 = \{0\} \subseteq (X)_1$, and $(A)_1 = \emptyset$ by assumption. Suppose $F_0, F_1, \ldots, F_{n-1}$ have been constructed. Set

$$Q := (A)_{n+1} \cap \bigcap_{j=0}^{n-1} F_j^{\diamond}$$

Note that Q is weak* closed and bounded, hence weak* compact by the Banach-Alaoglu theorem. Lemma 1 implies

$$Q \cap \bigcap_{\substack{F \subseteq (X)_{1/n} \\ \text{finite}}} F^{\diamond} = Q \cap (X^*)_n$$

The latter set is empty by (ii.n-1) in our induction hypothesis. Since Q is weak* compact, the finite intersection property implies we cannot have $Q \cap F^{\diamond} \neq \emptyset$ for all $F \subseteq (X)_{1/n}$ finite. Thus we choose $F_n \subseteq (X)_{1/n}$ finite such that $Q \cap F_n^{\diamond} = \emptyset$, which is (ii.n). We also have $nF_n \subseteq n(X)_{1/n} = (X)_1$, (i.n). Thus induction yields the sequence $F_0, F_1, \ldots \subseteq X$.

Observe that

$$A \cap \bigcap_{n=1}^{\infty} F_n^{\diamond} \subseteq \bigcup_{n=1}^{\infty} (A)_n \cap \bigcap_{j=1}^{n} F_j^{\diamond} = \emptyset$$

Enumerate $\bigcup_{n=1}^{\infty} F_n$ as a sequence $(x_n)_{n \in \mathbb{N}}$. Then $F_n \subseteq (X)_{1/n}$ implies

$$\lim_{n \to \infty} \|x_n\| = 0.$$

Consequently, we can define a map

$$T: X^* \to c_0(\mathbb{N})$$
$$x^* \mapsto (x^*(x_n))_{n\in\mathbb{N}}$$

Clearly T is linear, so $T(A) \subseteq c_0(\mathbb{N})$ is convex. Also, for $x^* \in A$ we must have

$$\|T(x^*)\| = \sup_{n\in\mathbb{N}} |x^*(x_n)| \geq 1$$

since $x^* \notin \bigcap_{n=1}^{\infty} F_n^{\circ}$. Thus $T(A) \cap (c_0(\mathbb{N}))_1 = \phi$. In particular, $T(A)$ and $(c_0(\mathbb{N}))_1^{\circ}$ are disjoint convex sets with the latter open, so they are separated. That is, there exists $f \in \ell^1(\mathbb{N}) = c_0(\mathbb{N})$ and $\alpha \in \mathbb{R}$ such that

$$\underset{\text{Re}\sum_{n=1}^{\infty}\phi(n)f(n)}{\underset{\shortparallel}{\text{Re}(\phi, f)}} < \alpha \leq \underset{\text{Re}\sum_{n=1}^{\infty} x^*(x_n)f(n)}{\underset{\shortparallel}{\text{Re}(T(x^*), f)}} \qquad \forall \phi \in (c_0(\mathbb{N}))_1^{\circ},\ x^* \in A.$$

By dividing by $\|f\|_1$ and replacing α with $\alpha/\|f\|$ we see that we can assume $\|f\|_1 = 1$. Since we can always get $|(\phi, f)| = \text{Re}(e^{i\theta}\phi, f)$ for an appropriate θ, we see that

$$1 = \|f\|_1 = \sup_{\substack{\phi \in c_0(\mathbb{N}) \\ \|\phi\|_\infty < 1}} |(\phi, f)| \leq \text{Re}\sum_{n=1}^{\infty} x^*(x_n) f(n) \qquad \forall x^* \in A$$

Banach space

Now, $f \in \ell^1(\mathbb{N})$ implies $x := \sum_{n=1}^{\infty} x_n f(n) \in X$. The above implies $\text{Re}(x, x^*) \geq 1$ for all $x^* \in A$. $\square$

Proof (Krein–Smulian)

Obviously $A \subset \overline{A}^{wk*}$. To show the converse, we will fix $x_0^* \in X^* \setminus A$ and show $x_0^* \notin \overline{A}^{wk*}$. Note that each $(A)_r$ is norm-closed by virtue of being weak* closed. It follows that A is norm-closed (since any norm-convergent sequence is bounded). Consequently $\exists\, r > 0$ s.t. $\overline{B}(x_0^*, r) \cap A = \phi$. It follows that $(A - x_0^*)_r = \phi$, and that $\frac{1}{r}(A - x_0^*)$ satisfies the hypotheses of Lemma 2. Thus there is some $x \in X$ such that

$$\text{Re}(x, x^*) \geq 1 \qquad \forall x^* \in \tfrac{1}{r}(A - x_0^*)$$

We therefore cannot have $0 \in \overline{\frac{1}{r}(A - x_0^*)}^{wk*}$ which implies $x_0^* \notin \overline{A}^{wk*}$. $\square$

- If $A \subseteq X^*$ is invariant under scalar multiplication (e.g. A is a subspace) then $(A)_r = r\cdot(A)_1$ for all $r > 0$. Since multiplication by $r > 0$ is a homeomorphism, $(A)_r$ is then weak* closed iff $(A)_1$ is. This yields the following corollary.

Cor Let X be a Banach space and $Y \subseteq X^*$ a subspace. Then Y is weak* closed if and only if $(Y)_1$ is weak* closed.

Cor Let X be a separable Banach space. If $A \subseteq X^*$ is convex and weak* sequentially closed, then A is weak* closed.

Proof Since X is separable, $(X^*)_1$ is weak* metrizable. In fact, $(X^*)_r$ is weak* metrizable $\forall r > 0$. Then $(A)_r$ is weak* closed for each $r > 0$ since it is weak* sequentially closed, and so A is weak* closed by Krein–Smulian. $\square$

Cor Let X be a separable Banach space. A linear functional $\phi : X^* \to \mathbb{F}$ is weak* continuous if and only if ϕ is weak* sequentially continuous.

Proof We know ϕ is weak* continuous iff $\ker \phi$ is weak* closed. By the previous corollary, this is equivalent to being weak* sequentially closed, and further equivalent to ϕ being weak* sequentially continuous. $\square$

Thm Let X be a separable Banach space. For a subspace $Y \subseteq X^*$, the following are equivalent:

1. Y is weak* sequentially dense in X^*.
2. There is $c > 0$ such that for all $x \in X$
$$\|x\| \leq \sup_{x^* \in (Y)_c} |\langle x, x^* \rangle|$$
3. There is $c > 0$ such that if $x^* \in (X^*)_1$ then there is a sequence $(x_n^*)_{n \in \mathbb{N}} \subseteq (Y)_c$ converging weak* to x^*.

Proof $(1) \Rightarrow (3)$: For each $n \in \mathbb{N}$, define $Z_n := \overline{(Y)_n}^{wk*}$. For $x^* \in X^*$, there exists $(x_n^*)_{n \in \mathbb{N}} \subseteq Y$ converging weak* to x^*. Since it is a sequence, the Banach–Steinhaus theorem implies $\exists N \in \mathbb{N}$ such that $x^* \in Z_N$. Thus
$$X^* = \bigcup_{n=1}^{\infty} Z_n.$$
Note that each Z_n is norm closed by virtue of being weak* closed. Consequently, the Baire–Category Theorem implies one of the Z_n has non-empty norm-interior. Hence $\exists x_0^* \in Z_n$ and $r > 0$ such that $\overline{B}(x_0^*, r) \subseteq Z_n$. Since $Z_n \subseteq (X^*)_n$ is weak* metrizable, for $x_0^* \in Z_n = \overline{(Y)_n}^{wk*}$ we can find $(y_k^*)_{k \in \mathbb{N}} \subseteq (Y)_n$ converging weak* to x_0^*. Now, for $x^* \in (X^*)_1$ we have $x_0^* + r \cdot x^* \in \overline{B}(x_0^*, r) \subseteq Z_n$. So as above we can find a sequence $(z_k^*)_{k \in \mathbb{N}} \subseteq (Y)_n$ converging weak* to $x_0^* + r \cdot x^*$. Then for $c := \frac{2n}{r}$ we have $x_k^* := \frac{1}{r}(z_k^* - y_k^*) \in (Y)_c$ and $x_k^* \to x^*$ weak*.

$(3) \Rightarrow (1)$: This is immediate.

$(2) \Rightarrow (3)$: Note that (2) says ${}^\diamond(Y)_c \subseteq (X)_1$. Thus
$$(X^*)_1 = (X)_1^\diamond \subseteq {}^\diamond(Y)_c^\diamond = \overline{(Y)_c}^{wk*}$$
where the last equality holds by the Bipolar theorem. Since X is separable, bounded sets are weak* metrizable. Hence $(X^*)_1 \subseteq \overline{(Y)_c}^{wk*}$ implies (3).

$(3) \Rightarrow (2)$: For $x \in X$, $\exists x^* \in (X^*)_1$ with $\langle x, x^* \rangle = \|x\|$ by Hahn–Banach. Then (3) implies $\exists (x_n^*)_{n \in \mathbb{N}} \subseteq (Y)_c$ with $\langle x, x_k^* \rangle \to \|x\|$, which gives (2). $\square$