$\boxed{\text{I.1}}$ Elementary Properties and Examples

- $\mathbb{F}$ will mean either $\mathbb{R}$ or $\mathbb{C}$.

Def If V is a vector space over $\mathbb{F}$, an inner product on V is a function
$$u: V \times V \to \mathbb{F}$$
such that for all $\alpha, \beta \in \mathbb{F}$ and $x, y, z \in V$ the following are satisfied

1. $u(\alpha x + \beta y, z) = \alpha u(x,z) + \beta u(y,z)$ (linearity)
2. $u(x,y) = \overline{u(y,x)}$ (conjugate symmetry)
3. $u(x,x) > 0$ whenever $x \neq 0$. (positive definite)

V together with such a function u is called an inner product space.

Notation we always denote our inner products by $\langle x,y\rangle := u(x,y)$. In physics, you are more likely to see $\langle x | y \rangle$ with linearity in the second coordinate. We will also denote the associated "norm" by
$$\|x\| := \langle x, x\rangle^{1/2}$$
which we note is well-defined by 3 above.

- You have likely encountered such objects in a linear algebra course

Prop Let V be an inner product space.

1. For $x,y,z \in V$ and $\alpha, \beta \in \mathbb{F}$, $\langle x, \alpha y + \beta z\rangle = \bar{\alpha}\langle x, y\rangle + \bar{\beta}\langle x, z\rangle$. (conjugate linearity)
2. For $x \in V$, $\langle x, y\rangle = 0$ $\forall y \in V$ iff $x = 0$.
3. For $x \in V$ and $\alpha \in \mathbb{F}$, $\|\alpha x\| = |\alpha| \, \|x\|$. In particular, $\|-x\| = \|x\|$.

Proof (1) This follows from 1 and 2 above.

(2) $(\Rightarrow)$ In particular, $\langle x, x\rangle = 0$. By 3 we must have $x = 0$.

$(\Leftarrow)$ Using 1 we have $\forall y \in V$.
$$\langle 0, y\rangle = \langle 0\cdot 0, y\rangle = 0\langle 0, y\rangle = 0$$

(3) Using the def. of the norm and 1 and 1 above we have
$$\|\alpha x\|^2 = \langle \alpha x, \alpha x\rangle = \alpha\langle x, \alpha x\rangle = \alpha\bar{\alpha}\langle x, x\rangle = |\alpha|^2 \|x\|^2.$$
Hence $\|\alpha x\| = |\alpha| \, \|x\|$. $\square$

Ex Let (X, Ω, μ) be a measure space with positive measure μ. Then
$$L^2(X, \Omega, \mu) := \{ f : X \to \mathbb{C} \ \mu\text{-measurable} : \int_X |f|^2 \, d\mu < \infty \}$$
has inner product
$$\langle f, g\rangle := \int_X f \bar{g} \, d\mu.$$
Any subspace of $L^2(X, \Omega, \mu)$ is also an inner product space with the above inner product.

In particular, if $X = \mathbb{N}$ and μ is the counting measure then

$$L^2(X,\Omega,\mu) = \ell^2(\mathbb{N}) = \{ (a_n)_{n\in\mathbb{N}} \subseteq \mathbb{C} : \sum_{n\in\mathbb{N}} |a_n|^2 < \infty \}$$

and

$$\langle (a_n)_{n\in\mathbb{N}}, (b_n)_{n\in\mathbb{N}} \rangle = \sum_{n\in\mathbb{N}} a_n \bar{b}_n .$$

Moreover, if $X = \{1,2,\ldots,d\}$ with μ the counting measure $L^2(X,\Omega,\mu) \cong \mathbb{C}^d$ with the usual inner product. □

- Hölder's inequality from real analysis says

$$\left|\int_X f\bar{g}\,d\mu\right| \le \left(\int_X |f|^2 d\mu\right)^{1/2} \left(\int_X |g|^2 d\mu\right)^{1/2}$$

This is a special case of the following theorem:

$\boxed{\text{Thm}}$ (Cauchy–Schwarz Inequality)
Let V be an inner product space. Then for all $x,y \in V$

$$|\langle x,y\rangle| \le \|x\|\,\|y\|$$

Proof For any $\alpha \in \mathbb{F}$ we have

$$0 \le \langle x-\alpha y, x-\alpha y\rangle = \langle x,x\rangle - \alpha\langle y,x\rangle - \bar{\alpha}\langle x,y\rangle + |\alpha|^2\langle y,y\rangle$$

Now, suppose $\langle x,y\rangle = |\langle x,y\rangle| e^{i\theta}$ Consider the above with $\alpha = te^{i\theta}$ where $t \in \mathbb{R}$:

$$0 \le \langle x,x\rangle - 2t\,|\langle x,y\rangle| + t^2\langle y,y\rangle$$
$$= \|y\|^2 t^2 - 2|\langle x,y\rangle| t + \|x\|^2.$$

The right-side is a quadratic polynomial in t, thus it being non-negative implies it has at most one real root. Using the quadratic formula we can infer that the discriminant is non-positive:

$$4|\langle x,y\rangle|^2 - 4\|y\|^2\|x\|^2 \le 0 \implies |\langle x,y\rangle| \le \|x\|\,\|y\|.$$ □

$\boxed{\text{Cor}}$ Let V be an inner product space.

1. $\|x+y\| \le \|x\| + \|y\|$ for all $x,y \in V$. (Triangle Inequality)
2. $\|x\| = 0$ iff $x = 0$.

Proof (1) we expand

$$\|x+y\|^2 = \langle x+y, x+y\rangle = \|x\|^2 + \langle x,y\rangle + \langle y,x\rangle + \|y\|^2$$
$$= \|x\|^2 + 2\,\mathrm{Re}\langle x,y\rangle + \|y\|^2$$

Now $\mathrm{Re}\langle x,y\rangle \le |\langle x,y\rangle| \le \|x\|\,\|y\|$, so

$$\|x+y\|^2 \le \|x\|^2 + 2\|x\|\,\|y\| + \|y\|^2 = (\|x\| + \|y\|)^2$$

Taking the square-root of each side yields the desired inequality.

(2) ($\Rightarrow$) For all $y \in X$ we have

$$|\langle x,y\rangle| \le \|x\|\,\|y\| = 0$$

Thus $x = 0$ by a previous proposition.

($\Leftarrow$) $\|0\| = (\langle 0,0\rangle)^{1/2} = (0)^{1/2} = 0.$ □

Remark Using the above Corollary and the earlier Proposition, it follows that we can make V into a metric space with metric

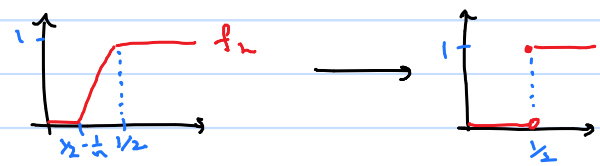
$$d(x,y) := \|x-y\|.$$

(Exercise: verify this is a metric). Consequently we are therefore in a position to do some analysis.

Def A Hilbert space is an inner product space that is complete with respect to the induced metric $d(x,y) = \|x-y\|$.

Ex 1. It follows from measure theory that $L^2(X,\Omega,\mu)$ is complete and therefore a Hilbert space. Consequently so are $\ell^2(\mathbb{N})$ and $\mathbb{C}^d$.

Suppose $(X,\Omega,\mu) = ([0,1],\mathcal{M},m)$ is the unit interval with Lebesgue measure m. Then cts functions $C([0,1],\mathbb{C}) \subseteq L^2([0,1],\mathcal{M},m)$ form a subspace and hence are inner product space. However, it is not complete and therefore not a Hilbert space. space. Indeed.



The sequence $(f_n) \subseteq C([0,1],\mathbb{C})$ converges to $f \notin C([0,1],\mathbb{C})$ with respect to the metric $\|f_n - f\| = \left(\int_{[0,1]} |f_n - f|^2 \, d\mu\right)^{1/2}$.

On the otherhand, any closed subspace of $L^2([0,1],\mathcal{M},m)$ will be a Hilbert space

2. For $A,B \in M_{m\times n}(\mathbb{F})$, define

$$\langle A,B\rangle := \mathrm{Tr}(B^*A)$$

Then $M_{m\times n}(\mathbb{F})$ is a Hilbert space with this inner product. $\square$

- Even though $C([0,1],\mathbb{C})$ is not complete with respect to the norm-metric, we can always take its completion, which in this case is $L^2([0,1],m)$. Observe that this completion is a Hilbert space whose inner product restricts to the original inner product. It turns out this is always the case:

Prop Let V be an inner product space with inner product $\langle\cdot,\cdot\rangle_V$. Then its completion with respect to the norm-metric, $\mathcal{H}$, admits an inner product $\langle\cdot,\cdot\rangle_{\mathcal{H}}$ satisfying $\langle x,y\rangle_{\mathcal{H}} = \langle x,y\rangle_V$ $\forall x,y \in V$. That is, $\mathcal{H}$ is a Hilbert space.

Proof Define $\langle (x_n)_n, (y_n)_n\rangle_{\mathcal{H}} := \lim_{n\to\infty} \langle x_n, y_n\rangle$ for Cauchy sequences $\square$

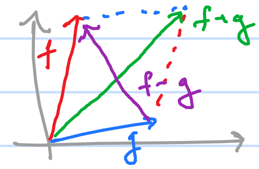
II. 2 Orthogonality

Def If $\mathcal{H}$ is a Hilbert space, we say $f, g \in \mathcal{H}$ are orthogonal if $\langle f, g\rangle = 0$, in which case we write $f \perp g$. We say subsets $A, B \subseteq \mathcal{H}$ are orthogonal if $f \perp g$ for all $f \in A$ and $g \in B$, and we write $A \perp B$.

Thm (Pythagorean Theorem)
If $f_1, \dots, f_n$ are pairwise orthogonal vectors in a Hilbert space $\mathcal{H}$, then
$$\|f_1 + f_2 + \cdots + f_n\|^2 = \|f_1\|^2 + \|f_2\|^2 + \cdots + \|f_n\|^2$$

Proof Expand
$$\|f_1 + f_2\|^2 = \|f_1\|^2 + 2\operatorname{Re}\langle f_1, f_2\rangle + \|f_2\|^2 = \|f_1\|^2 + \|f_2\|^2$$
Then use induction. $\square$

Prop (Parallelogram Law)
For any vectors f, g in a Hilbert space $\mathcal{H}$
$$\|f+g\|^2 + \|f-g\|^2 = 2(\|f\|^2 + \|g\|^2)$$



Proof Expand the left-side. $\square$

Def Let V be a vector space over $\mathbb{F}$. We say a subset $A \subseteq V$ is convex if for any $x, y \in A$, $tx + (1-t)y \in A$ for all $t \in [0,1]$. That is, the line segment from x to y is contained in A.

Ex
1. Subspaces are convex
2. Singleton sets are convex
3. Intersections of convex sets are convex
4. If V is a inner product space, then for any $f \in V$ and $r > 0$ the open ball
$$B(f, r) := \{g \in V : \|f - g\| < r\}$$
is convex, as is the closed ball
$$\overline{B}(f, r) := \{g \in V : \|f - g\| \leq r\}.$$ $\square$

Thm Let $\mathcal{H}$ be a Hilbert space with $K \subset \mathcal{H}$ closed, convex, and nonempty. For all $f \in \mathcal{H}$ there exists a unique $k_0 \in K$ satisfying
$$\|f - k_0\| = \operatorname{dist}(f, K) = \inf_{k \in K} \|f - k\|.$$

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Proof Let $d := \operatorname{dist}(f, K)$. Then there exists a sequence $(k_n)_{n \in \mathbb{N}} \subseteq K$ s.t.

$$\lim_{n\to\infty} \|f-k_n\| = d.$$

We claim this is a Cauchy sequence. Let $\varepsilon>0$. By the parallelogram law we have

$$\|k_n - k_m\|^2 = \|(f-k_m)-(f-k_n)\|^2 = 2\left(\|f-k_m\|^2+\|f-k_n\|^2\right) - \|2f-(k_m+k_n)\|^2$$
$$= 2\left(\|f-k_m\|^2+\|f-k_n\|^2\right) - 4\left\|f-\frac{k_m+k_n}{2}\right\|^2$$

Now, $\frac{k_m+k_n}{2}\in K$ be virtue of convexity and so $\left\|f-\frac{k_m+k_n}{2}\right\|^2\geq d^2$. Let $N\in\mathbb{N}$ be s.t. $\forall n\geq N$, $\|f-k_n\|^2< d^2+\varepsilon$. Then for $m,n\geq N$ combining the above estimates yields

$$\|k_n-k_m\|^2 < 4d^2+4\varepsilon-4d^2 = 4\varepsilon.$$

Thus $(k_n)_{n\in\mathbb{N}}$ is a Cauchy sequence. Let $k_0\in\mathcal{H}$ be its limit, which exists since $\mathcal{H}$ is complete. Then $k_0\in K$ since K is closed, and moreover

$$d\leq \|f-k_0\|\leq \|f-k_n\|+\|k_n-k_0\| \longrightarrow d+0.$$

Thus $\|f-k_0\|=d$.

To see that k_0 is unique, suppose $k_0'\in K$ also satisfies $\|f-k_0'\|=d$. Then by convexity we have

$$d\leq \left\|f-\frac{k_0+k_0'}{2}\right\| \leq \frac{1}{2}\|f-k_0\|+\frac{1}{2}\|f-k_0'\| = \frac{1}{2}d+\frac{1}{2}d=d.$$

so that $\left\|f-\frac{k_0+k_0'}{2}\right\|=d$. The parallelogram law then implies

$$d^2 = \left\|f-\frac{k_0+k_0'}{2}\right\|^2 = \frac{1}{2}\|f-k_0\|^2+\frac{1}{2}\|f-k_0'\|^2-\left\|\frac{k_0-k_0'}{2}\right\|^2$$
$$= d^2 - \left\|\frac{k_0-k_0'}{2}\right\|^2$$

Thus $\left\|\frac{k_0-k_0'}{2}\right\|^2=0 \Rightarrow k_0=k_0'$. $\square$

- If K is in particular a closed subspace, then we can say more:

$\boxed{\text{Thm}}$ Let $\mathcal{H}$ be a Hilbert space and $\mathcal{K}\subseteq\mathcal{H}$ a closed subspace. For $f\in\mathcal{H}$, then $f_0\in\mathcal{K}$ is the unique element of $\mathcal{K}$ satisfying

$$\|f-f_0\| = \operatorname{dist}(f,\mathcal{K})$$

iff $f-f_0\perp\mathcal{K}$.

Proof ($\Rightarrow$) Let $g\in\mathcal{K}$, then

$$\|f-(g+f_0)\|\geq \operatorname{dist}(f,\mathcal{K}) = \|f-f_0\|$$

Thus

$$\|f-f_0\|^2\leq \|f-(g+f_0)\|^2 = \|(f-f_0)-g\|^2 = \|f-f_0\|^2-2\operatorname{Re}\langle f-f_0,g\rangle+\|g\|^2$$

which implies

$$2\operatorname{Re}\langle f-f_0,g\rangle\leq \|g\|^2$$

Suppose $\langle f-f_0,g\rangle = re^{i\theta}$, we must show $r=0$. Replacing g above with $te^{i\theta}g$, $t>0$:

$$2\operatorname{Re}\langle f-g, te^{i\theta}g\rangle \leq t^2\|g\|^2$$
$$2\operatorname{Re}(te^{-i\theta}re^{i\theta})\leq$$
$$2tr\leq t^2\|g\|^2 \Longrightarrow 2r\leq t\|g\|^2.$$

Letting $t \searrow 0$ implies $r=0$. Since $g \in \mathcal{K}$ was arbitrary, we have $f - f_0 \perp \mathcal{K}$.

($\Leftarrow$) For any $g \in \mathcal{K}$ we have:

$$\|f-g\|^2 = \|(f-f_0)+(f_0-g)\|^2 = \|f-f_0\|^2 + \|f_0-g\|^2 \geq \|f-f_0\|^2.$$

Hence we must have $\|f-f_0\| = \operatorname{dist}(f,\mathcal{K})$. □

Def For $A \subseteq \mathcal{H}$ define the orthogonal complement of A by:

$$A^\perp := \{f \in \mathcal{H} : f \perp A\}$$

- One easily checks that $A^\perp$ is a closed subspace of $\mathcal{H}$ (even if A is not).
- The two previous theorems tell us that given a closed subspace $\mathcal{K}$ of $\mathcal{H}$, for every $f \in \mathcal{H}$ there is a unique $f_0 \in \mathcal{K}$ satisfying $f - f_0 \in \mathcal{K}^\perp$. Hence we can define a map

$$P : \mathcal{H} \to \mathcal{K}$$
$$f \mapsto f_0.$$

Prop Let $\mathcal{H}, \mathcal{K}$, and P be as above. Then:

1. P is a linear transformation on $\mathcal{H}$.
2. $\|Pf\| \leq \|f\|$ for all $f \in \mathcal{H}$.
3. $P^2 = P$.
4. $\operatorname{ran} P = \mathcal{K}$ and $\ker P = \mathcal{K}^\perp$.

Proof (1) Let $f_1, f_2 \in \mathcal{H}$ and $\alpha_1, \alpha_2 \in \mathbb{F}$. Then for $g \in \mathcal{K}$ we have

$$\langle \alpha_1 f_1 + \alpha_2 f_2 - (\alpha_1 Pf_1 + \alpha_2 Pf_2), g \rangle = \alpha \langle f_1 - Pf_1, g\rangle + \alpha_2 \langle f_2 - Pf_2, g\rangle = 0$$

By the previous theorem we have $\alpha_1 Pf_1 + \alpha_2 Pf_2 = P(\alpha_1 f_1 + \alpha_2 f_2)$. That is, P is linear.

(2) By the Pythagorean theorem:

$$\|f\|^2 = \|f - Pf + Pf\|^2 = \|f-Pf\|^2 + \|Pf\|^2 \geq \|Pf\|^2.$$

(3) If $g \in \mathcal{K}$, then $Pg = g$ since $\|g-g\| = 0 = \operatorname{dist}(g,\mathcal{K})$. Now for any $f \in \mathcal{H}$, $Pf \in \mathcal{K}$ and hence $P(Pf) = Pf$. That is, $P^2 = P$.

(4) Certainly $\operatorname{ran} P \subset \mathcal{K}$, and the previous part implies equality.

Suppose $f \in \ker P$. Then $f = f - 0 = f - Pf \in \mathcal{K}^\perp$. Thus $\ker P \subset \mathcal{K}^\perp$. Conversely, if $f \in \mathcal{K}^\perp$. Then for all $g \in \mathcal{K}$

$$\|f-g\|^2 = \|f\|^2 + \|-g\|^2 \geq \|f\|^2 = \|f-0\|^2.$$

Thus $0 \in \mathcal{K}$ satisfies $\|f-0\| = \operatorname{dist}(f,\mathcal{M})$ and therefore $Pf = 0$. □

Def If $\mathcal{K}$ is a closed subspace of a Hilbert space $\mathcal{H}$, the linear transformation $P : \mathcal{H} \to \mathcal{K}$ defined by $Pf \in \mathcal{K}$ and $f - Pf \in \mathcal{K}^\perp$ is called the orthogonal projection of $\mathcal{H}$ onto $\mathcal{K}$, and may be denoted $P_\mathcal{K} := P$.

Notation We write $\mathcal{K} \leq \mathcal{H}$ when $\mathcal{K} \subseteq \mathcal{H}$ is a closed subspace (to reflect that $\mathcal{K}$ is also a Hilbert space w.r.t. the restriction of the inner product).

For $A \subseteq \mathcal{H}$ the span of A is denoted.

$$\operatorname{span} A := \left\{ \sum_{i=1}^{n} \alpha_i f_i : \alpha_1, \dots, \alpha_n \in \mathbb{F},\ f_1, \dots, f_n \in A \right\}$$

We write $\overline{\operatorname{span}} A$ for the closure of the above set (w.r.t. the norm).

⚠ The textbook changes terminology at this point and assumes all subspaces are closed. Non-closed subspaces are called "linear manifolds". We will not be adopting this convention.

Exercise: If $\operatorname{span} A$ is fin. dim'l, show $\operatorname{span} A = \overline{\operatorname{span}} A$.

Cor For $A \subseteq \mathcal{H}$, $(A^{\perp})^{\perp} = \overline{\operatorname{span}} A$. In particular, if $\mathcal{K} \leq \mathcal{H}$ then $(\mathcal{K}^{\perp})^{\perp} = \mathcal{K}$.

Proof Homework 1. □

Cor Let $N \subseteq \mathcal{H}$ be a (not necessarily closed) subspace of $\mathcal{H}$. Then N is dense iff $N^{\perp} = \{0\}$.

Proof By the previous corollary,

$$(*) \qquad (N^{\perp})^{\perp} = \overline{\operatorname{span}} N = \overline{N}.$$

If N is dense, $\overline{N} = \mathcal{H}$ and thus

$$\langle f, g \rangle = 0 \qquad \forall f \in N^{\perp},\ g \in \mathcal{H}$$

In particular, $\langle f, f \rangle = 0$ for all $f \in N^{\perp}$ implying $N^{\perp} = \{0\}$.

If $N^{\perp} = \{0\}$, then $(N^{\perp})^{\perp} = (\{0\})^{\perp} = \mathcal{H}$. So $(*)$ implies $\overline{N} = \mathcal{H}$. □

I.3 A Riesz Representation Theorem

Prop Let $\mathcal{H}$ be a Hilbert space and $L: \mathcal{H} \to \mathbb{F}$ a linear functional. TFAE

1. L is uniformly continuous.
2. L is continuous at 0.
3. L is continuous at some point
4. There exists a constant $C>0$ such that $\frac{|Lf|}{\|f\|} \le C$ for all $f \in \mathcal{H}\setminus\{0\}$

Proof $(1) \Rightarrow (2) \Rightarrow (3)$ and $(4) \Rightarrow (2)$ are immediate.

$(3) \Rightarrow (1)$: suppose L is cts at $f_0 \in \mathcal{H}$. Fix $f \in \mathcal{H}$ and let $\varepsilon > 0$. Let $\delta > 0$ be such that $\|g - f_0\| < \delta$ implies $|L(g) - L(f_0)| < \varepsilon$. If $\|f - g\| < \delta$, then note that

$$\|(f-g+f_0) - f_0\| = \|f-g\| < \delta.$$

Thus by linearity

$$|L(f) - L(g)| = |L(f-g+f_0) - L(f_0)| < \varepsilon.$$

That is, L is cts at f. Since the same δ works for both f and f_0, this implies L is uniformly cts.

$(2) \Rightarrow (4)$: By cty at 0, $L^{-1}(\{\alpha \in \mathbb{F} : |\alpha| < 1\})$ contains an open ball $B(0,\delta)$. Thus $\|f\| < \delta$ implies $L(f) \in \{\alpha \in \mathbb{F} : |\alpha| < 1\}$; that is, $|L(f)| < 1$. Now, for arbitrary $f \in \mathcal{H}\setminus\{0\}$ and $0 < t < 1$ we have

$$\left\| \frac{t\delta f}{\|f\|} \right\| = t\delta < \delta.$$

Hence

$$1 > \left| L\left(\frac{t\delta f}{\|f\|}\right)\right| = \frac{t\delta}{\|f\|} |L(f)| \quad \Rightarrow \quad |L(f)| < \frac{1}{t\delta}\|f\|$$

Letting $t \nearrow 1$ we see that we can choose $c = \frac{1}{\delta}$. $\square$

Def A <u>bounded linear functional</u> on $\mathcal{H}$ is a linear functional $L: \mathcal{H} \to \mathbb{F}$ satisfying any (hence all) of the conditions in the above proposition. The <u>norm</u> of such a linear functional is the quantity

$$\|L\| := \sup_{f \in \mathcal{H}\setminus\{0\}} \frac{|L(f)|}{\|f\|}$$

Prop For a bounded linear functional L

$$\|L\| = \inf\{c > 0 : |L(f)| \le c\|f\| \ \forall f \in \mathcal{H}\} = \sup_{\|f\| \le 1} |L(f)| = \sup_{\|f\| = 1} |L(f)|$$

In particular, $|L(f)| \le \|L\|\,\|f\|$ for all $f \in \mathcal{H}$.

Proof Let $c > 0$ be s.t. $|L(f)| \le c\|f\|$ $\forall f \in \mathcal{H}$. Then $\forall f \in \mathcal{H}\setminus\{0\}$

$$c \ge \frac{|L(f)|}{\|f\|}.$$

Thus $c \ge \|L\|$. Taking the infimum over all such c yields

$$\inf\{c>0 : |L(f)| \le c\|f\| \ \forall f\in\mathcal{H}\} \stackrel{?}{=} \|L\| \ge \sup_{\|f\|\ge 1} |L(f)| \ge \sup_{\|f\|=1} |L(f)|$$

where the other inequalities are immediate. Hence it suffices to show

$$\sup_{\|f\|=1} |L(f)| \ge \inf\{c>0 : |L(f)| \le c\|f\| \ \forall f\in\mathcal{H}\}.$$

Let $c := \sup_{\|f\|=1} |L(f)|$. Then for $f\in\mathcal{H}\setminus\{0\}$ we have

$$c \ge \left|L\left(\frac{f}{\|f\|}\right)\right| = \frac{1}{\|f\|}|L(f)| \Rightarrow |L(f)| \le c\|f\|.$$

Also of course have $|L(0)| \le c\|0\|$. $\square$

$\boxed{\text{Ex}}$ Fix $f_0\in\mathcal{H}$. Then $f\mapsto\langle f, f_0\rangle$ defines a linear functional L. By the Cauchy Schwarz inequality, for all $f\in\mathcal{H}\setminus\{0\}$ we have

$$\frac{|\langle f, f_0\rangle|}{\|f\|} \le \frac{\|f\|\cdot\|f_0\|}{\|f\|} = \|f_0\|.$$

Hence $\|L\| \le \|f_0\|$. By computing $|L(f_0)| = \|f_0\|^2$, we see that $\|L\| = \|f_0\|$. $\square$

- The next theorem tells us that <u>all</u> bounded linear functionals are of the above form.

$\boxed{\text{Thm}}$ (A Riesz Representation Theorem)
If $L:\mathcal{H}\to\mathbb{F}$ is a bounded linear functional, then there exists a unique vector $f_0\in\mathcal{H}$ such that $L(f) = \langle f, f_0\rangle$ for all $f\in\mathcal{H}$. Moreover, $\|L\| = \|f_0\|$.

<u>Proof</u> Let $\mathcal{K} := \ker L$, which is a closed subspace of $\mathcal{H}$ since L is linear and continuous. If $\mathcal{K}=\mathcal{H}$, then $L\equiv 0$ and we take $f_0 = 0$. Otherwise, $\mathcal{K}\subsetneq\mathcal{H}$ and consequently $\mathcal{K}^\perp\neq\{0\}$. Let $g\in\mathcal{K}^\perp\setminus\{0\}$. Then $g\notin\mathcal{K}$ (since $\mathcal{K}\cap\mathcal{K}^\perp=\{0\}$) and so $L(g)\neq 0$. Set $g_0 := \frac{1}{L(g)}g\in\mathcal{K}^\perp$ so that $L(g_0)=1$. Now, for any $f\in\mathcal{H}$ we have

$$L(f - L(f)g_0) = L(f) - L(f)L(g_0) = 0$$

Hence $f - L(f)g_0\in\mathcal{K}$, and thus

$$0 = \langle f - L(f)g_0, g_0\rangle = \langle f, g_0\rangle - L(f)\langle g_0, g_0\rangle$$

or

$$L(f) = \frac{\langle f, g_0\rangle}{\langle g_0, g_0\rangle} = \left\langle f, \frac{1}{\|g_0\|^2}g_0\right\rangle$$

So we take $f_0 := \frac{1}{\|g_0\|^2}g_0$.

To see uniqueness, suppose f_0' also satisfies $L(f) = \langle f, f_0'\rangle$ $\forall f\in\mathcal{H}$. Then

$$\langle f_0 - f_0', f\rangle = \langle f_0, f\rangle - \langle f_0', f\rangle = L(f) - L(f) = 0$$

Thus $f_0 - f_0' \perp \mathcal{H} \Rightarrow f_0 - f_0' = 0$. The final statement about $\|L\|$ follows from the preceding example. $\square$

I.4 Orthonormal Sets of Vectors and Bases

Def A subset $A \subseteq \mathcal{H}$ in a Hilbert space is said to orthogonal if $\langle f, g \rangle = 0$ for all $f, g \in A$ s.t. $f \neq g$. We say it is orthonormal if we further require $\|f\| = 1$ for all $f \in A$. (Note that the former permits $0 \in A$, but the latter does not). An (orthonormal) basis is a maximal orthonormal set.

Ex ① $\mathbb{F}^d$ has o.n. basis $\{e_1, e_2, \ldots, e_d\}$ where $e_i = (0, \ldots, 0, \underset{\text{i-th position}}{1}, 0, \ldots, 0)^T$

② $\ell^2(\mathbb{N})$ has o.n. basis $\{e_n : n \in \mathbb{N}\}$ where $e_n(m) = \delta_{n=m}$.

③ $M_{m \times n}(\mathbb{C})$ has o.n. basis $\{E_{i,j} : 1 \leq i \leq m,\ 1 \leq j \leq n\}$ where $(E_{i,j})_{k,l} = \delta_{i=k}\,\delta_{j=l}$.

④ $L^2([0,1], m)$ has o.n. basis $\{ e^{2\pi i n t} : n \in \mathbb{Z}\}$ □

Remark This is different from the notion a "Hamel basis", which is a maximal linear independent set. The main distinction, as we will see is that while Hamel bases allow one to write every vector as a finite linear combination of basis vectors, an orthonormal basis will allow infinite linear combinations. The distinction essentially comes down to Algebra vs. Analysis (i.e. lacking vs having a notion of convergence).

Prop For an orthonormal set $A \subseteq \mathcal{H}$, there exists an orthonormal basis containing A.

Proof Apply Zorn's Lemma to the poset $\{A \subseteq \mathcal{H} : A \text{ orthonormal}\}$, which we order by inclusion. □

- The following is proven exactly as in linear algebra.

Prop (Gram-Schmidt Orthogonalization Process)
Let $\mathcal{H}$ be a Hilbert space and $\{f_n : n \in \mathbb{N}\} \subseteq \mathcal{H}$ a linearly independent subset. Then there exists an orthonormal set $\{e_n : n \in \mathbb{N}\}$ s.t. for all $n \in \mathbb{N}$

$$\operatorname{span}\{f_1, \ldots, f_n\} = \operatorname{span}\{e_1, \ldots, e_n\}.$$

- We will encounter not just infinite linear combinations, but possibly uncountable ones. We must therefore make sense of such sums.

Def Let $\mathcal{H}$ be a Hilbert space and let I be a (possibly uncountable) index set. For $\{f_i \in \mathcal{H} : i \in I\}$, we say the series

$$\sum_{i \in I} f_i$$

converges if the net $\left(\sum_{i \in F} f_i\right)_{F \in \mathcal{F}}$ converges. Here $\mathcal{F}$ is the collection of finite subsets $F \subset I$ ordered by inclusion.

⚠ For $I = \mathbb{N}$, this notion of convergence is not equivalent to saying the partial sums $(\sum_{n=1}^{N} f_i)_{N\in\mathbb{N}}$ converge. Indeed, it requires converge along all inclusions of finite sets and consequently implies the convergence of the partial sums (Exercise).
For $\mathcal{H} = \mathbb{F}$, this notion of convergence is equivalent to absolute convergence (see Exercises I.4.10-12)

$\boxed{\text{Thm}}$ Let $\mathcal{E} \subseteq \mathcal{H}$ be an orthonormal set. For $f \in \mathcal{H}$,

$$\sum_{e\in\mathcal{E}} \langle f, e\rangle e$$

always converges and satisfies

$$\Big\| \sum_{e\in\mathcal{E}} \langle f, e\rangle e \Big\|^2 = \sum_{e\in\mathcal{E}} |\langle f, e\rangle|^2 \leq \|f\|^2$$

Furthermore, $\langle f, e\rangle \neq 0$ for at most countably many $e \in \mathcal{E}$.

Proof First let $\{e_n : n\in\mathbb{N}\} \subseteq \mathcal{E}$ be an orthonormal subset, and define for each $N\in\mathbb{N}$

$$f_N := f - \sum_{n=1}^{N} \langle f, e_n\rangle e_n.$$

Then $f_N \perp e_n$ for $1 \leq n \leq N$. Thus the pythagorean theorem implies

$$\|f\|^2 = \Big\| f_N + \sum_{n=1}^{N} \langle f, e_n\rangle e_n \Big\|^2$$
$$= \|f_N\|^2 + \sum_{n=1}^{N} |\langle f, e_n\rangle|^2 \geq \sum_{n=1}^{N} |\langle f, e_n\rangle|^2$$

Letting $N \to \infty$ yields

$$(*) \qquad \sum_{n=1}^{\infty} |\langle f, e_n\rangle|^2 \leq \|f\|^2 \qquad \text{(Bessel's Inequality)}$$

Now, returning to $\mathcal{E}$, let $\mathcal{E}_k := \{e\in\mathcal{E} : |\langle f, e\rangle| > \frac{1}{k}\}$. By $(*)$, $\mathcal{E}_k$ can contain only finitely many elements. Hence

$$\bigcup_{k=1}^{\infty} \mathcal{E}_k = \{e\in\mathcal{E} : |\langle f, e\rangle| > 0\} = \{e\in\mathcal{E} : \langle f, e\rangle \neq 0\}$$

is countable.

Next, let $\varepsilon > 0$ and enumerate $\{e\in\mathcal{E} : \langle f, e\rangle \neq 0\} = \{e_n : n\in\mathbb{N}\}$. From $(*)$, we can find $N_0\in\mathbb{N}$ s.t. $\forall N \geq N_0$

$$\sum_{n=N+1}^{\infty} |\langle f, e_n\rangle|^2 < \varepsilon.$$

Let $F, G \subset \mathcal{E}$ be finite subsets containing $\{e_1, \ldots, e_{N_0}\}$. Then

$$\Big\| \sum_{e\in F} \langle f, e\rangle e - \sum_{e\in G} \langle f, e\rangle e \Big\|^2 = \sum_{e\in F\Delta G} |\langle f, e\rangle|^2$$

Every $e \in F\Delta G$ satisfies either $\langle f, e\rangle = 0$ or $e = e_n$ for $n \geq N+1$. Thus the above is bounded by ε. If $\mathcal{F}$ is the collection of finite subsets of $\mathcal{E}$, then the above implies $(\sum_{e\in F} \langle f, e\rangle e)_{F\in\mathcal{F}}$ is a Cauchy net. Since $\mathcal{H}$ is complete it necessarily converges.

Finally, for any $F\in\mathcal{F}$ we have by $(*)$

$$\Big\| \sum_{e\in F} \langle f, e\rangle e \Big\|^2 = \sum_{e\in F} |\langle f, e\rangle|^2 \leq \|f\|^2.$$

Taking the limit $F \to \infty$ in $\mathcal{F}$ yields the claimed inequality. □

Prop Let $\mathcal{E}$ be an orthonormal set in $\mathcal{H}$. If $\mathcal{K} := \overline{\text{span}}\, \mathcal{E}$, then the orthogonal projection onto $\mathcal{K}$ is given by:

$$P_{\mathcal{K}} f = \sum_{e \in \mathcal{E}} \langle f, e \rangle e$$

Proof Set Qf equal to the right-side above. Clearly $Qf \in \mathcal{K}$, and also for each

$$\langle Qf, e \rangle = \langle f, e \rangle \quad \forall e \in \mathcal{E}.$$

Thus $\langle f - Qf, e \rangle = 0$ and therefore

$$f - Qf \in (\text{span}\, \mathcal{E})^{\perp} = (\overline{\text{span}}\, \mathcal{E})^{\perp} = \mathcal{K}^{\perp}$$

$\square$

- As you might expect, if $\mathcal{E}$ is an orthonormal basis, then the inequality in the above theorem is an equality. In fact it is equivalent to this.

Thm Let $\mathcal{E} \subset \mathcal{H}$ be an orthonormal set. The following are equivalent:

1. $\mathcal{E}$ is an orthonormal basis
2. For $f \in \mathcal{H}$, if $f \perp \mathcal{E}$ then $f = 0$.
3. $\overline{\text{span}}\, \mathcal{E} = \mathcal{H}$.
4. For $f \in \mathcal{H}$, $\quad f = \sum_{e \in \mathcal{E}} \langle f, e \rangle e$.
5. For $f, g \in \mathcal{H}$, $\langle f, g \rangle = \sum_{e \in \mathcal{E}} \langle f, e \rangle \langle e, g \rangle$.
6. For $f \in \mathcal{H}$, $\|f\|^2 = \sum_{e \in \mathcal{E}} |\langle f, e \rangle|^2$. (Parseval's Identity)

Proof $(1 \Rightarrow 2)$ Suppose $f \perp \mathcal{E}$. If $f \neq 0$, then $\mathcal{E} \cup \{f/\|f\|\}$ contradicts $\mathcal{E}$ being a maximal orthonormal set.

$(2 \Rightarrow 3)$ $\overline{\text{span}}\, \mathcal{E} = (\mathcal{E}^{\perp})^{\perp} = (\{0\})^{\perp} = \mathcal{H}$.

$(3 \Rightarrow 4)$ Set $g := \sum_{e \in \mathcal{E}} \langle f, e \rangle e$, which exists by the previous theorem.

Then $\forall e' \in \mathcal{E}$

$$\langle f - g, e' \rangle = \langle f, e' \rangle - \langle g, e' \rangle = \langle f, e' \rangle - \langle \sum_{e \in \mathcal{E}} \langle f, e \rangle e, e' \rangle$$

$$\overset{\text{Cauchy-Schwarz}}{=} \langle f, e' \rangle - \sum_{e \in \mathcal{E}} \langle f, e \rangle \langle e, e' \rangle = \langle f, e' \rangle - \langle f, e' \rangle = 0.$$

Thus $f - g \in \mathcal{E}^{\perp}$. But

$$\mathcal{E}^{\perp} = ((\mathcal{E}^{\perp})^{\perp})^{\perp} = (\overline{\text{span}}\, \mathcal{E})^{\perp} = \mathcal{H}^{\perp} = \{0\}.$$

So $f - g = 0$.

$(4 \Rightarrow 5)$ $\quad \langle f, g \rangle = \langle \sum_{e \in \mathcal{E}} \langle f, e \rangle e, g \rangle \overset{C-S}{=} \sum_{e \in \mathcal{E}} \langle f, e \rangle \langle e, g \rangle$

$(5 \Rightarrow 6)$ This follows from $\|f\|^2 = \langle f, f \rangle$.

$(6 \Rightarrow 1)$ Suppose $\mathcal{E}$ is ~~not~~ an orthonormal basis. Then $\exists f \perp \mathcal{E}$, $f \neq 0$. But 6 implies

$$\|f\|^2 = \sum_{e \in \mathcal{E}} |\langle f, e \rangle|^2 = \sum_{e \in \mathcal{E}} 0 = 0.$$

a contradiction. $\square$

Prop If $\mathcal{H}$ is a Hilbert space, then any two orthonormal bases have the same cardinality.

Proof Let $\mathcal{E}$ and $\mathcal{F}$ be two orthonormal bases for $\mathcal{H}$. Let $\varepsilon = |\mathcal{E}|$ and $\eta = |\mathcal{F}|$. If either ε or η is finite, we appeal to linear algebra. So suppose both are infinite. For $e \in \mathcal{E}$, define

$$\mathcal{F}_e := \{f \in \mathcal{F} : \langle e, f \rangle \neq 0\}$$

Then $\mathcal{F}_e$ is countable. By (2) in the previous theorem

$$\mathcal{F} = \bigcup_{e \in \mathcal{E}} \mathcal{F}_e .$$

Hence $\eta \leq \varepsilon \cdot \aleph_0 = \varepsilon$. Similarly $\varepsilon \leq \eta$ and so $\varepsilon = \eta$. $\square$

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Def The <u>dimension</u> of a Hilbert space $\mathcal{H}$ is the cardinality of a basis, and is denoted by $\dim \mathcal{H}$.

- Recall that we say a metric space (X, d) is separable if it has a countable dense subset D. Note this implies that for any pairwise disjoint collection of open balls $\{B_i : i \in I\}$, I is countable. Indeed, $B_i \cap D \neq \phi$ $\forall i \in I$, pick $x_i \in B_i \cap D$ for each $i \in I$. Then $x_i \neq x_j$ for $i \neq j$ so that $|\{x_i : i \in I\}| = |I|$. But $\{x_i : i \in I\} \subset D$, and so $|I| \leq \aleph_0$

Prop If $\mathcal{H}$ is an infinite dimensional Hilbert space, then $\mathcal{H}$ is separable iff $\dim \aleph_0$.

Proof ($\Longrightarrow$) Let $\mathcal{E}$ be an orthonormal basis for $\mathcal{H}$. Observe for $e_1, e_2 \in \mathcal{E}$

$$\|e_1 - e_2\|^2 = \|e_1\|^2 + \|-e_2\|^2 = 2$$

Thus $\{B(e, \sqrt{2}/2) : e \in \mathcal{E}\}$ is a collection of pairwise disjoint collection of open balls. So by the above, $\mathcal{E}$ is ctble.

($\Longleftarrow$) Let $\mathcal{E}$ be an orthonormal basis for $\mathcal{H}$, which we enumerate as $\mathcal{E} = \{e_n : n \in \mathbb{N}\}$. Then

$$D = \left\{ \sum_{i=1}^{n} \alpha_i e_i : n \in \mathbb{N}, \alpha_i \in \mathbb{Q} \right\}$$

is countable (exercise) and dense by (4) in the above theorem. $\square$

Ex (1) $\ell^2(\mathbb{N})$ has basis $e_n = (0, \ldots, 0, \underset{n^{\text{th}}}{1}, 0, \ldots)$, $n \in \mathbb{N}$, and so is separable

(2) In $L^2(\mathbb{R}, m)$, consider

$$D = \left\{ \sum_{j=1}^{n} (\alpha_j + i\beta_j) 1_{(a_j, b_j)} : n \in \mathbb{N} \text{ and } \alpha_j, \beta_j, a_j, b_j \in \mathbb{Q} \text{ for } j = 1, \ldots, n \right\}.$$

This is a countable dense set (Exercise), and hence $\dim L^2(\mathbb{R}, m) = \aleph_0$.

Similarly $\dim L^2([0,1], m) = \aleph_0$. $\square$

I.5 Isomorphic Hilbert Spaces and the Fourier Transform for the Circle

- We must decide on a notion of isomorphism between Hilbert spaces. It should be a map preserving the essential structure of a Hilbert space:
  - vector space
  - inner product
  - complete metric space

Def If $\mathcal{H}$ and $\mathcal{K}$ are Hilbert spaces, an isomorphism between $\mathcal{H}$ and $\mathcal{K}$ is a surjective linear map $U: \mathcal{H} \to \mathcal{K}$ such that

$$\langle Uf, Ug \rangle = \langle f, g \rangle \qquad \forall f, g \in \mathcal{H}.$$

In this case we say $\mathcal{H}$ and $\mathcal{K}$ are isomorphic.

- Note that injectivity follows from $\|Uf - Ug\|^2 = \|U(f-g)\|^2 = \|f-g\|^2$. Also completeness is also preserved.
- Recall that a distance preserving map between metric spaces is called an isometry.

Prop A linear map $V: \mathcal{H} \to \mathcal{K}$ between Hilbert spaces is an isometry iff $\langle Vf, Vg \rangle = \langle f, g \rangle \quad \forall f, g \in \mathcal{H}$.

Proof ($\Rightarrow$) Let $f, g \in \mathcal{H}$ and $\alpha \in \mathbb{F}$. Then

$$(*) \qquad \|Vf - \alpha Vg\| = \|Vf - V(\alpha g)\| = \|f - \alpha g\|.$$

Squaring the left and right sides and expanding yields:

$$\|Vf\|^2 - 2\operatorname{Re} \bar{\alpha} \langle Vf, Vg \rangle + |\alpha|^2 \|Vg\|^2 = \|f\|^2 - 2\operatorname{Re} \bar{\alpha} \langle f, g \rangle + \|g\|^2.$$

Observe that $(*)$ applied to $g = 0$ yields $\|Vf\| = \|f\|$. Similarly $\|Vg\| = \|g\|$. So we obtain

$$\operatorname{Re} \bar{\alpha} \langle Vf, Vg \rangle = \operatorname{Re} \bar{\alpha} \langle f, g \rangle$$

Taking $\alpha = 1$ (then $\alpha = i$ if $\mathbb{F} = \mathbb{C}$) yields $\langle Vf, Vg \rangle = \langle f, g \rangle$.

($\Leftarrow$) This follows from

$$\|Vf - Vg\|^2 = \langle V(f-g), V(f-g) \rangle = \langle f-g, f-g \rangle = \|f-g\|^2. \qquad \square$$

- Thus for linear $U: \mathcal{H} \to \mathcal{K}$, U is an isomorphism iff it is a surjective isometry.

Ex Define $S: \ell^2(\mathbb{N}) \to \ell^2(\mathbb{N})$ by

$$S(x_1, x_2, x_3, \dots) = (0, x_1, x_2, x_3, \dots).$$

Then clearly S preserves the inner product and so is an isometry by the proposition. However, S is <u>not</u> surjective and so is <u>not</u> an isomorphism. $\square$

Thm Hilbert spaces $\mathcal{H}$ and $\mathcal{K}$ are isomorphic iff $\dim \mathcal{H} = \dim \mathcal{K}$.

Proof ($\Rightarrow$) Let $U: \mathcal{H} \to \mathcal{K}$ be an isomorphism. If $\mathcal{E}$ is an orthonormal basis for $\mathcal{H}$, then $\{Ue : e \in \mathcal{E}\}$ is an orthonormal basis for $\mathcal{K}$. Indeed, $\langle Ue, Ue' \rangle = \langle e, e' \rangle$ implies it is orthonormal. For $f \in \mathcal{K}$, $f \perp \{Ue : e \in \mathcal{E}\} \Rightarrow U^{-1}(f) \perp \mathcal{E} \Rightarrow U^{-1}(f) = 0 \Rightarrow f = 0$. Thus it is an orthonormal basis by a theorem from section I.4.

($\Leftarrow$) Let $\mathcal{E}$ and $\mathcal{F}$ be bases for $\mathcal{H}$ and $\mathcal{K}$, respectively. Since $\dim \mathcal{H} = \dim \mathcal{K}$, there is a bijection $\sigma: \mathcal{E} \to \mathcal{F}$. Define $U: \mathcal{H} \to \mathcal{K}$ as follows:

$$U(f) := \sum_{e \in \mathcal{E}} \langle f, e \rangle \sigma(e). \qquad f \in \mathcal{H}$$

Note that this series converges because for finite $F \subseteq \mathcal{E}$

$$\Big\| \sum_{e \in F} \langle f, e \rangle \sigma(e) \Big\|^2 = \sum_{e \in F} |\langle f, e \rangle|^2 = \Big\| \sum_{e \in F} \langle f, e \rangle e \Big\|^2$$

and $\sum_{e \in \mathcal{E}} \langle f, e \rangle e = f$ converges. It is also easy to see that U is linear and for $g \in \mathcal{H}$ we have

$$\begin{aligned}
\langle f, g \rangle &= \sum_{e \in \mathcal{E}} \langle f, e \rangle \langle e, g \rangle \\
&= \sum_{e \in \mathcal{E}} \langle \langle f, e \rangle \sigma(e), \langle g, e \rangle \sigma(e) \rangle \\
&= \sum_{e \in \mathcal{E}} \sum_{e' \in \mathcal{E}} \langle \langle f, e \rangle \sigma(e), \langle g, e' \rangle \sigma(e') \rangle \\
&= \Big\langle \sum_{e \in \mathcal{E}} \langle f, e \rangle \sigma(e), \sum_{e' \in \mathcal{E}} \langle g, e' \rangle \sigma(e') \Big\rangle = \langle Uf, Ug \rangle.
\end{aligned}$$

Thus U is an isometry. It remains to show it is surjective. Since U^{-1} sends Cauchy sequences to Cauchy sequences, $\operatorname{ran} U$ is complete and therefore closed. So U being surjective is equivalent to $\mathcal{K} = \operatorname{ran} U = \overline{\operatorname{ran} U} = (\operatorname{ran} U^{\perp})^{\perp}$. But $\mathcal{F} \subseteq \operatorname{ran} U$ and so $\operatorname{ran} U^{\perp} = \{0\} \Rightarrow (\operatorname{ran} U^{\perp})^{\perp} = \mathcal{K}$. $\square$

- We saw in Section I.4 that for infinite dimensional Hilbert spaces, separability is equivalent to having countable dimension. We consequently obtain:

Cor All separable infinite dimensional Hilbert spaces are isomorphic.

also given by Fourier Analysis.

Ex $L^2(\mathbb{R}, m) \cong L^2([0,1], m) \cong \ell^2(\mathbb{N})$ $\square$

Exercise Read the results on the Fourier transform of the circle.

I.6 Operations on Hilbert spaces

Direct Sums

- Let $\mathcal{H}$ and $\mathcal{K}$ be Hilbert spaces. Then
$$\mathcal{H} \oplus \mathcal{K} := \{(h,k) : h \in \mathcal{H},\ k \in \mathcal{K}\}$$
is a vector space with operations
$$\alpha(h_1,k_1) + (h_2,k_2) = (\alpha h_1 + h_2,\ \alpha k_1 + k_2) \qquad \alpha \in \mathbb{F},\ h_1,h_2 \in \mathcal{H},\ k_1,k_2 \in \mathcal{K}$$
Moreover,
$$\langle (h_1,k_1), (h_2,k_2) \rangle := \langle h_1,h_2 \rangle + \langle k_1,k_2 \rangle$$
is an inner product which makes $\mathcal{H} \oplus \mathcal{K}$ into a Hilbert space (Exercise: check this).

- Observe that
$$\|(h,k)\| = \langle (h,k),(h,k) \rangle^{1/2} = \left(\|h\|^2 + \|k\|^2\right)^{1/2}$$

- If $\mathcal{E}$ and $\mathcal{F}$ are orthonormal bases for $\mathcal{H}$ and $\mathcal{K}$, respectively, then
$$\{(e,0) : e \in \mathcal{E}\} \cup \{(0,f) : f \in \mathcal{F}\}$$
is an orthonormal basis for $\mathcal{H} \oplus \mathcal{K}$. Consequently
$$\dim(\mathcal{H} \oplus \mathcal{K}) = \dim \mathcal{H} + \dim \mathcal{K}$$

- Iterating this construction allows us to consider
$$\mathcal{H}_1 \oplus \mathcal{H}_2 \oplus \cdots \oplus \mathcal{H}_n,$$
but to pass to infinite direct sums we need a bit more care.

Prop Let $\mathcal{H}_n$, $n \in \mathbb{N}$, be Hilbert spaces and define
$$\mathcal{H} := \{(h_n)_{n \in \mathbb{N}} : h_n \in \mathcal{H}_n \text{ and } \sum_{n \in \mathbb{N}} \|h_n\|^2 < \infty\}$$
For $(h_n),(g_n) \in \mathcal{H}$ define
$$\langle (h_n),(g_n) \rangle := \sum_{n \in \mathbb{N}} \langle h_n, g_n \rangle$$
Then this is an inner product whose induced norm is
$$\|(h_n)\| = \left(\sum_{n \in \mathbb{N}} \|h_n\|^2\right)^{1/2}.$$
Moreover $\mathcal{H}$ is a Hilbert space when equipped with this inner product.

Proof Observe that by the Cauchy-Schwarz inequality
$$\sum_{n \in \mathbb{N}} |\langle h_n, g_n \rangle| \le \sum_{n \in \mathbb{N}} \|h_n\|\,\|g_n\| \underset{\text{C-S in } \ell^2(\mathbb{N})}{\le} \left(\sum_{n \in \mathbb{N}} \|h_n\|^2\right)^{1/2} \left(\sum_{n \in \mathbb{N}} \|g_n\|^2\right)^{1/2} < \infty$$

Thus the series defining the inner product converges absolutely and is therefore well-defined. Checking the remaining details is left as an exercise. $\square$

• $\mathcal{H}$ in the proposition is denoted

$$\mathcal{H} = \mathcal{H}_1 \oplus \mathcal{H}_2 \oplus \cdots = \bigoplus_{n \in \mathbb{N}} \mathcal{H}_n$$

$\mathbb{N}$ can be replaced with an arbitrary indexing set I:

$$\bigoplus_{i \in I} \mathcal{H}_i = \{ (h_i)_{i \in I} : h_i \in \mathcal{H}_i \text{ and } \underbrace{\sum_{i \in I} \|h_i\|^2}_{\text{converges as a net}} < \infty \}$$

• If $\mathcal{H}_i = \mathcal{H}$ $\forall i \in I$ then we also write

$$\ell^2(I, \mathcal{H}) := \bigoplus_{i \in I} \mathcal{H}$$

In particular, $\ell^2(\mathbb{N}, \mathbb{C}) = \ell^2(\mathbb{N})$.

Tensor Products

• Let $\mathcal{H}$ and $\mathcal{K}$ be Hilbert spaces. Let $\mathcal{H} \odot \mathcal{K}$ denote linear combinations of symbols $h \otimes k$, with $h \in \mathcal{H}$ and $k \in \mathcal{K}$, subject to the relations

$$(h_1 + h_2) \otimes k = h_1 \otimes k + h_2 \otimes k \qquad h_1, h_2 \in \mathcal{H},\ k \in \mathcal{K}$$
$$h \otimes (k_1 + k_2) = h \otimes k_1 + h \otimes k_2 \qquad h \in \mathcal{H},\ k_1, k_2 \in \mathcal{K}$$
$$(\alpha h) \otimes k = h \otimes (\alpha k) = \alpha (h \otimes k) \qquad \alpha \in \mathbb{F},\ h \in \mathcal{H},\ k \in \mathcal{K}$$

Then

$$\langle h_1 \otimes k_1, h_2 \otimes k_2 \rangle := \langle h_1, h_2 \rangle \langle k_1, k_2 \rangle$$

is an inner product on $\mathcal{H} \odot \mathcal{K}$. We denote its completion (a Hilbert space) by $\mathcal{H} \otimes \mathcal{K}$.

• If $\mathcal{E}$ and $\mathcal{F}$ are orthonormal bases for $\mathcal{H}$ and $\mathcal{K}$, respectively, then

$$\{ e \otimes f : e \in \mathcal{E}, f \in \mathcal{F} \}$$

is an orthonormal basis for $\mathcal{H} \otimes \mathcal{K}$. Consequently,

$$\dim(\mathcal{H} \otimes \mathcal{K}) = \dim \mathcal{H} \cdot \dim \mathcal{K}$$

$\boxed{\text{Ex}}$ Let $\mathcal{H}$ be a Hilbert space over $\mathbb{C}$. Recall that $M_{m \times n}(\mathbb{C})$ has an orthonormal basis $\{ E_{i,j} : 1 \le i \le m, 1 \le j \le n \}$ where

$$E_{i,j} = \begin{array}{c} \\ i \end{array} \begin{pmatrix} 0 & \overset{j}{} & 0 \\ & 1 & \\ 0 & & 0 \end{pmatrix}$$

Then every element of $\mathcal{H} \otimes M_{m \times n}(\mathbb{C})$ is of the form

$$\sum_{i=1}^{m} \sum_{j=1}^{n} f_{i,j} \otimes E_{i,j} \qquad f_{i,j} \in \mathcal{H}$$

($\boxed{\text{Exercise}}$: verify this.) We can visualize this by

$$f \otimes E_{ij} = \begin{pmatrix} 0 & & 0 \\ & f & \\ 0 & & 0 \end{pmatrix}$$

so that

$$\sum_{i=1}^{m} \sum_{j=1}^{n} f_{i,j} \otimes E_{i,j} = \begin{pmatrix} f_{1,1} & f_{1,2} & \cdots & f_{1,n} \\ f_{2,1} & f_{2,2} & \cdots & f_{2,n} \\ \vdots & \vdots & & \vdots \\ f_{m,1} & f_{m,2} & \cdots & f_{m,n} \end{pmatrix} \in M_{m \times n}(\mathcal{H})$$

Exercise: Check that this respects the inner product. $\square$

- A fact that physicists often lament is that not all elements of $\mathcal{H} \otimes \mathcal{K}$ are "elementary" tensor products: $h \otimes k$. However, in the context of the above example this would be like only considering matrices with a single non-zero entry.

Exercise Prove that

$$U : \ell^2(\mathbb{N}, \mathcal{H}) \longrightarrow \ell^2(\mathbb{N}) \otimes \mathcal{H}$$
$$(h_n)_{n \in \mathbb{N}} \longmapsto \sum_{n \in \mathbb{N}} e_n \otimes h_n$$

is an isomorphism.

Conjugate Hilbert Space

- Given a Hilbert space $\mathcal{H}$, let $\overline{\mathcal{H}}$ be the set consisting of symbols $\overline{f}$ for $f \in \mathcal{H}$. We make this into a vector space via

$$\alpha \overline{f} + \overline{g} := \overline{\overline{\alpha} f + g} \qquad \alpha \in \mathbb{F},\ f, g \in \mathcal{H}$$

$\overline{\mathcal{H}}$ is a Hilbert space with inner product

$$\langle \overline{f}, \overline{g} \rangle := \langle g, f \rangle$$

Exercise For each $\overline{f} \in \overline{\mathcal{H}}$, define a map $L_{\overline{f}} : \mathcal{H} \longrightarrow \mathbb{F}$ by

$$L_{\overline{f}}(g) = \langle g, f \rangle.$$

Show that $\overline{f} \mapsto L_{\overline{f}}$ is a bijective, bounded, linear transformation