IV.1 Elementary Properties and Examples

- In this chapter we continue to strip away the structure on the vector spaces we consider. We will now abandon the norm, which means the vector spaces will ~~not~~ not be metric spaces. However, we still want them to be topological spaces.

Def A topological vector space (TVS) is a vector space X equipped with a topology such that the maps

$$X \times X \ni (x,y) \mapsto x+y \in X$$
$$\mathbb{F} \times X \ni (\alpha, x) \mapsto \alpha x \in X$$

are continuous.

Ex (1) Let X be a normed space. Then this is a TVS when equipped with the metric space topology induced by its norm.

(2) Let X be a vector space and let $\mathcal{P}$ be a family of seminorms. One can consider the topology $\mathcal{T}$ generated by the sets

$$\{x \in X : p(x-x_0) < \varepsilon\} \qquad p \in \mathcal{P},\ x_0 \in X,\ \varepsilon > 0$$

This is the "coarsest" topology s.t. the seminorms $p \in \mathcal{P}$ are all continuous. Using the definition of a seminorm, one can show X is a TVS with this topology. A subset $U \subseteq X$ is open iff $\forall x_0 \in U$ $\exists p_1, \dots, p_n \in \mathcal{P}$, $\varepsilon_1, \dots, \varepsilon_n > 0$ s.t.

$$\bigcap_{j=1}^{n} \{x \in X : p_j(x-x_0) < \varepsilon_j\} \subseteq U$$ $\square$

- We will typically want to work with Hausdorff TVS's. One way to ensure this is:

Def A locally convex space (LCS) is a TVS whose topology is defined by a family of seminorms $\mathcal{P}$ (as in (2)) satisfying

$$\bigcap_{p \in \mathcal{P}} \{x \in X : p(x) = 0\} = \{0\}$$

Prop If X is a LCS, then X is Hausdorff.

~~**Proof**~~ Let $x, y \in X$ be distinct. Then $x - y \neq 0$, and so $\exists p \in \mathcal{P}$ s.t. $p(x-y) \neq 0$. Let $\varepsilon \in (0, p(x-y))$ and define

$$U := \{z \in X : p(x-z) < \tfrac{1}{2}\varepsilon\}$$
$$V := \{z \in X : p(y-z) < \tfrac{1}{2}\varepsilon\}$$

Then U and V are open, $x \in U$, $y \in V$, and $U \cap V = \emptyset$. $\square$

EX 1. Let X be a loc. cpt. Hausdorff space, and let $C(X)$ denote the set of continuous functions on X. For $K \subset X$ compact define a seminorm

$$p_K(f) = \sup_{x \in K} |f(x)|.$$

Then the family $\mathcal{P} = \{p_K : K \subseteq X \text{ compact}\}$ makes $C(X)$ into an LCS. Note that this is the topology of uniform convergence on compact subsets.

2. Let X be a normed space. For $f \in X^*$ define a seminorm on X by

$$p_f(x) = |f(x)| \qquad x \in X.$$

Then $\mathcal{P} := \{p_f : f \in X^*\}$ make X into a LCS and the induced topology on X is called the weak topology.

3. Let X be a normed space. For $x \in X$ define a seminorm on X^* by

$$p_x(f) = |f(x)| \qquad f \in X^*.$$

Then $\mathcal{P} := \{p_x : x \in X\}$ makes X^* into a LCS and the induced topology on X^* is called the weak* topology. $\square$

Prop Let X be a TVS and p a seminorm on X. The following are equivalent

1. p is continuous
2. $\{x \in X : p(x) < 1\}$ is open
3. $0 \in \{x \in X : p(x) < 1\}^\circ$
4. $0 \in \{x \in X : p(x) \leq 1\}^\circ$
5. p is continuous at 0
6. There exists a continuous seminorm q on X such that $p \leq q$.

Proof $(1) \Rightarrow (2) \Rightarrow (3) \Rightarrow (4)$ are clear.

$(4) \Rightarrow (5)$: Let $(x_i)_{i \in I} \subseteq X$ be a net converging to 0. We must show $\lim_i p(x_i) = 0$. Let $\varepsilon > 0$, and note that $x \mapsto \varepsilon x$ is a homeomorphism on X. Thus $0 \in \{x \in X : p(x) \leq \varepsilon\}^\circ$. It follows that there is $i_0 \in I$ such that $x_i \in \{x \in X : p(x) \leq \varepsilon\}^\circ$ for all $i \geq i_0$. In particular, $p(x_i) \leq \varepsilon \ \forall i \geq i_0$. Thus $\lim p(x_i) = 0$.

$(5) \Rightarrow (1)$: Fix $x_0 \in X$ and suppose $(x_i)_{i \in I} \subseteq X$ converges to x_0. By the reverse triangle inequality

$$|p(x_i) - p(x_0)| \leq p(x_i - x_0)$$

Since $(x_i - x_0)_{i \in I}$ converges to zero, the above tends to zero. Thus p is continuous at x_0.

$(1) \Rightarrow (6)$: Take $q = p$.

$(6) \Rightarrow (5)$: Suppose $x_i \to 0$. Then

$$0 \leq p(x_i) \leq q(x_i) \longrightarrow 0$$

so p is continuous at zero. $\square$

- Some operations naturally produce new seminorms.

Prop Let X be a TVS

1. If $p_1, \dots, p_n$ are continuous seminorms on X, then $p_1 + \cdots + p_n$ is a continuous seminorm.
2. If $\{p_i\}_{i\in I}$ is a family of continuous seminorms such that there exists a continuous seminorm q with $p_i \le q$ $\forall i \in I$, then $\sup_{i\in I} p_i$ defines a continuous seminorm.

Proof Exercise. $\square$

- Suppose P is a family of seminorms making a vector space X into a LCS. In light of the above proposition, it is often convenient to enlarge P so as to be closed under finite sums and supremums. It may also be convenient to assume P contains all continuous seminorms on X. Neither of these enlargements changes the topology on X.

Notation Let X be a vector space. For $x, y \in X$ we write

$$[x,y] := \{(1-t)x + ty \ : \ 0 \le t \le 1\}$$

- Recall that $C \subseteq X$ is convex iff for all $x, y \in C$ one has $[x,y] \subseteq C$. This can be strengthened as in the next proposition.

Prop Let X be a vector space.

1. $C \subseteq X$ is convex if and only if whenever $x_1, \dots, x_n \in C$ and $t_1, \dots, t_n \in [0,1]$ satisfy $\sum t_i = 1$ then one has $\sum t_i x_i \in C$.
2. If $\{C_i\}_{i\in I}$ is a collection of convex sets, then $\bigcap_{i\in I} C_i$ is convex.

Proof Exercise. $\square$

Def For $A \subseteq X$, the convex hull of A, denoted $\mathrm{co}(A)$, is the intersection of all convex sets containing A. If X is a TVS, then the closed convex hull of A, denoted $\overline{\mathrm{co}}(A)$, is the intersection of all closed convex subsets of X containing A.

- Note that X is always convex, so $\mathrm{co}(A)$ and $\overline{\mathrm{co}}(A)$ always exist. Moreover, they are both convex by the previous proposition. Also $\overline{\mathrm{co}}(A)$ is closed.

EX

1. For X a normed space, $B(0,1)$ and $\overline{B}(0,1)$ are convex. If $Y \subseteq X$ is a subspace then Y is convex and $\overline{\mathrm{co}}(Y) = \overline{Y}$.
2. Let $f \in X^*$. Then $\{x : |f(x)| \le 1\}$, $\{x : \mathrm{Re} f(x) \le 1\}$, $\{x : \mathrm{Re} f(x) > 1\}$ are all convex.
3. For $T : X \to Y$ linear and $C \subseteq Y$ convex, $T^{-1}(C) \subseteq X$ is convex. $\square$

Prop Let X be a TVS and $C \subseteq X$ convex. Then:

1. $\bar{C}$ is convex.
2. If $x \in C^\circ$ and $y \in \bar{C}$ then
$$[x,y) := \{(1-t)x + ty : 0 \le t < 1\} \subseteq C^\circ$$

Proof (1): Let $x \in C$ and $y \in \bar{C}$. Then there exists a net $(y_i)_{i\in I} \subseteq C$ converging to y. Then for $0 \le t \le 1$, $C \ni (1-t)x + ty_i \longrightarrow (1-t)x + ty$. So $[x,y] \in \bar{C}$.

Now assume $x, y \in \bar{C}$. Let $(x_i)_{i \in I} \subseteq C$ be a net converging to x. Then for all $0 \le t \le 1$, the above implies
$$\bar{C} \ni (1-t)x_i + ty \longrightarrow (1-t)x + ty$$
So that $[x,y] \subseteq \bar{C}$ and $\bar{C}$ is convex.

(2): Let $x \in C^\circ$ and $y \in \bar{C}$. Fix $0 < t < 1$, and set $z := (1-t)x + ty$. Since $x \in C^\circ$, $V := C^\circ - x$ is an open neighborhood of 0. Likewise, $\frac{1-t}{t}V$ is an open neighborhood of 0. Let $w \in C$ be such that $y - w \in \frac{1-t}{t}V$. This implies $(1-t)V + t(w-y)$ is also an open neighborhood of 0. Thus the following is an open neighborhood of z
$$(1-t)V + t(w-y) + z = (1-t)(C^\circ - x) + t(w-y) + (1-t)x + ty = (1-t)C^\circ + tw \subseteq C$$
Thus $z \in C^\circ$. $\square$

Cor If $A \subseteq X$, then $\overline{\text{co}}(A) = \overline{\text{co}(A)}$.

Def A subset $A \subseteq X$ is called balanced if $x \in A$ implies $\alpha x \in A$ for all $|\alpha| \le 1$. It is called absorbing if for all $x \in X$ there exists $t_0 > 0$ such that $tx \in A$ for all $0 \le t < t_0$. We say A is absorbing at a if $A - a$ is absorbing.

Ex If X is a vector space and p is a seminorm on X, then
$$V := \{x : p(x) < 1\}$$
is convex balanced and absorbing at each of its points. $\square$

Prop Let X be a vector space over $\mathbb{F}$. If $V \subseteq X$ is nonempty, convex, balanced, and absorbing at each of its points, then there is a unique seminorm p on X such that $V = \{x \in X : p(x) < 1\}$.

Proof First note that uniqueness follows from a lemma in Section III.1. Now define
$$p(x) := \inf\{t : t \ge 0 \text{ and } x \in tV\}.$$
Since $x \in tV \iff \frac{1}{t}x \in V$ and V is absorbing, $p(x) < \infty$. Since V is balanced, $0 \in V$ and so $p(0) = 0$. Let $x \in X$ and $\alpha \in \mathbb{F} \setminus \{0\}$. Using again that V is balanced we have:
$$p(\alpha x) = \inf\{t \ge 0 : \alpha x \in tV\} = \inf\{t \ge 0 : x \in t(\tfrac{1}{\alpha}V)\}$$
$$= \inf\{t \ge 0 : x \in t(\tfrac{1}{|\alpha|}V)\} = |\alpha| \inf\{\tfrac{t}{|\alpha|} : x \in \tfrac{t}{|\alpha|}V\} = |\alpha|\, p(x).$$
Next we show p satisfies the triangle inequality. First observe that for $x, y \in V$ and

$\alpha, \beta \geq 0$ we have by the convexity of V

$$\alpha x + \beta y = (\alpha+\beta)\left(\frac{\alpha}{\alpha+\beta} x + \frac{\beta}{\alpha+\beta} y \right) \in (\alpha+\beta) V$$

That is,

$$\alpha V + \beta V = (\alpha+\beta) V.$$

Now, suppose $p(x) = \alpha$ and $p(y) = \beta$. Then for any $\delta > 0$ we have $x \in (\alpha+\delta)V$ and $y \in (\beta+\delta)V$. Thus $x+y \in (\alpha+\beta+2\delta)V$. So $p(x+y) \leq \alpha+\beta+2\delta$. Letting $\delta \to 0$ yields

$$p(x+y) \leq \alpha+\beta = p(x)+p(y).$$

Thus p is a seminorm. It remains to show $V = \{x \in X : p(x) < 1\}$. Suppose $p(x) = \alpha < 1$. If $\alpha < \beta < 1$, then $x \in \beta V \subseteq V$ since V is balanced. Thus $\{p(x) < 1\} \subset V$. Conversely, if $x \in V$, then $p(x) \leq 1$. Since V is absorbing at x, $\exists\, t_0 > 0$ such that $tx \in V - x$ for all $0 \leq t < t_0$. That is, $(1+t)x \in V$ or $x \in \frac{1}{1+t} V$. So $p(x) \leq \frac{1}{1+t} < 1$. $\square$

Def The seminorm p defined in the previous proposition is called the <u>Minkowski function</u> of V or the <u>gauge</u> of V.

implies absorbing at all of its points

Prop Let X be a TVS and let $\mathcal{U}$ be the collection of all open convex balanced subsets of X. Then X is a LCS if and only if $\mathcal{U}$ is a basis for the neighborhood system at 0.

IV.2 Metrizable and Normable Locally Convex Spaces.

Prop Let $(p_n)_{n\in\mathbb{N}}$ be a sequence of seminorms on X satisfying
$$\bigcap_{n=1}^{\infty} \{x\in X : p_n(x)=0\} = \{0\}.$$
For $x,y\in X$ define
$$d(x,y) := \sum_{n=1}^{\infty} 2^{-n} \frac{p_n(x-y)}{1+p_n(x-y)}$$
Then d is a metric on X and the topology on X determined by d is the same as the topology determined by $\{p_n : n\in\mathbb{N}\}$. Thus an LCS is metrizable if and only if its topology is determined by a countable family of seminorms.

Proof Checking d is a metric, and that d and $\{p_n : n\in\mathbb{N}\}$ induce the same topology is on Homework 5.

If X is an LCS whose topology is determined by a countable family of seminorms, then it metrizable with precisely the metric d defined above.

Conversely, suppose X is metrizable with metric d and its topology is determined by a family of seminorms $\mathcal{P}$. Define an open set
$$U_n := \{x\in X : d(x,0) < \tfrac{1}{n}\}.$$
As an LCS, $\exists$ continuous seminorms $q_1,\dots,q_k$ and $\varepsilon_1,\dots,\varepsilon_k>0$ such that
$$\bigcap_{j=1}^{k} \{x\in X : q_j(x) < \varepsilon_j\} \subseteq U_n$$

Define $p_n := \frac{1}{\varepsilon_1} q_1 + \dots + \frac{1}{\varepsilon_k} q_k$. Then $p_n(x)<1$ implies $x\in U_n$, and p_n is continuous.
Now, if $(x_i)_{i\in I}$ is a net converging to 0 in X, then $p_n(x_i)\to 0$. Conversely, if $(x_i)_{i\in I}$ is a net such that $p_n(x_i)\to 0$ $\forall n\in\mathbb{N}$, then $\exists\, i_0\in I$ such that $p_n(x_i)<1$ whenever $i\geq i_0$. This implies $x_i\in U_n$ $\forall i\geq i_0$, or $d(x_i,0)<\frac{1}{n}$.
Consequently, $x_i\to 0$ with respect to d. Thus the topology determined by $\{p_n : n\in\mathbb{N}\}$ is the same as the topology determined by d. $\square$

EX Let X be a locally compact metric space, and let $C(X)$ be a LCS with topology determined by seminorms
$$p_K(f) = \sup_{x\in K} |f(x)|$$
for $K\subseteq X$ compact. Then $C(X)$ is metrizable iff X is σ-compact.
E.g. $C(\mathbb{R}^d)$ is metrizable since $\mathbb{R}^d = \bigcup_{n=1}^{\infty} [-n,n]^d$ $\square$

Def ① A metric d on a vector space X is translation invariant if
$$d(x+z,y+z) = d(x,y) \qquad \forall x,y,z\in X.$$
② A Fréchet space is a TVS X whose topology is determined by a translation invariant metric and such that (X,d) is complete. e.g. Banach space

Def If X is a TVS, we say $B \subseteq X$ is *bounded* if for every open neighborhood $\mathcal{U}$ of 0 there exists $\varepsilon > 0$ such that $\varepsilon B \subseteq \mathcal{U}$.

Ex ① If X is a normed space, $B \subseteq X$ is bounded in the above sense iff $B \subseteq B(0, r)$ for some $r > 0$.

② If p is only a seminorm on a vector space X, then
$$B := \{x \in X : p(x) < 1\}$$
need not be bounded. Indeed, consider
$$B = \{f \in C(\mathbb{R}) : \sup_{0 \leq t \leq 1} |f(t)| \leq 1\}$$
Since $C(\mathbb{R})$ is metrizable, this is bounded iff $B \subseteq \{f \in C(\mathbb{R}) : d(f, 0) \leq r\}$ for some $r > 0$. But the functions
$$f_n(t) := \begin{cases} 0 & \text{if } t \leq 1 \\ n(t-1) & \text{if } t > 1 \end{cases} \in B$$
preclude this. □

Prop An LCS X is normable if and only if it has a non-empty bounded open set.

Proof If X has a norm, then $B(0,1)$ is such a set.

Conversely let $\mathcal{U} \subseteq X$ be a non-empty bounded open set. By replacing $\mathcal{U}$ with $\mathcal{U} - x_0$ for any $x_0 \in \mathcal{U}$, we may assume $0 \in \mathcal{U}$. Since X is an LCS, the same argument as in the previous proposition implies $\exists$ a continuous seminorm p such that
$$V := \{x \in X : p(x) < 1\} \subseteq \mathcal{U}.$$
We claim that p is a norm and determines the topology on X. Indeed, let $x \in X \setminus \{0\}$. Since X is Hausdorff (as an LCS), $\exists$ open sets W_0, W_x such that $0 \in W_0 \not\ni x$, $x \in W_x \not\ni 0$. Let $\varepsilon > 0$ be such that $W_0 \supseteq \varepsilon \mathcal{U} \supseteq \varepsilon V = \{x \in X : p(x) < \varepsilon\}$. Since $x \notin W_0$, we must have $p(x) \geq \varepsilon$ and so $p(x) \neq 0$. Thus p is a norm.

Now, p is continuous on X. So the topology it induces is a priori coarser. We must show any other continuous seminorm q on X is still continuous with respect to this coarser topology. By a proposition from Section IV.1, it suffices to show $\exists\, \alpha > 0$ such that $q \leq \alpha p$. Since q is continuous, $\{x \in X : q(x) < 1\}$ is open. So $\exists\, \varepsilon > 0$ such that
$$\{x \in X : q(x) < 1\} \supset \varepsilon \mathcal{U} \supset \varepsilon V.$$
Thus $p(x) < \varepsilon$ implies $q(x) < 1$. By a Lemma from Section III.1, this implies $q(x) \leq \varepsilon^{-1} p(x)$. □

IV.3 Geometric Consequences of the Hahn-Banach Theorem

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- As usual, we cite the proofs from earlier chapters for the following:

Thm Let X be a TVS. For a linear functional $f: X \to \mathbb{F}$, the following are equivalent:

1. f is continuous.
2. f is continuous at 0.
3. f is continuous at some point.
4. $\ker(f)$ is closed
5. $x \mapsto |f(x)|$ is a seminorm.

If X is an LCS with topology defined by seminorms $\mathcal{P}$, then the above are further equivalent to:

6. There are $p_1, \ldots, p_n \in \mathcal{P}$ and $\alpha_1, \ldots, \alpha_n > 0$ such that
$$|f(x)| \leq \sum_{j=1}^{n} \alpha_j p_j(x) \qquad \forall x \in X.$$

- Recall that in Section IV.1 we characterized non-empty, convex, balanced sets that are absorbing at each of its points as the unit ball with respect to a unique seminorm. If we remove "balanced" we obtain the following characterization (via a similar proof):

Prop Let X be a TVS. If $0 \in G \subseteq X$ is open and convex, then
$$q(x) := \inf\{t \geq 0 : x \in tG\}$$
is a non-negative, continuous, sublinear functional and $G = \{x \in X : q(x) < 1\}$.

- The above says the is a correspondence between open convex neighborhoods of 0 and subliner fuctionals. Recall there is also a correspondence linear functionals and hyperplanes. These correspondences yield geometric results:

(Geometric Hahn-Banach Theorem)

Thm Let X be a TVS. If $G \subseteq X$ is non-empty, open, convex, and does not contain 0, then there exists a closed hyperplane $Y \subseteq X$ such that $Y \cap G = \emptyset$.

Proof First suppose X is over $\mathbb{R}$. Fix $x_0 \in G$ and set $H := x_0 - G$. Then H is an open convex neighborhood of 0. The previous Proposition implies $H = \{q(x) < 1\}$ for some continuous nonnegative sublinear functional $q: X \to \mathbb{R}$. Note that $0 \notin G \Rightarrow x_0 \notin H$ and so $q(x_0) \geq 1$. Define $Y_0 := \operatorname{span}\{x_0\}$ and $f: Y_0 \to \mathbb{R}$ by $f(\alpha x_0) = \alpha q(x_0)$. Note that for $\alpha \geq 0$
$$f(\alpha x_0) = \alpha q(x_0) = q(\alpha x_0)$$
and for $\alpha < 0$
$$f(\alpha x_0) = \alpha q(x_0) < \alpha < 0 \leq q(\alpha x_0)$$
Thus $f \leq q$ on Y_0. By the Hahn-Banach theorem $\exists F: X \to \mathbb{R}$ such that $F|_{Y_0} = f$ and $F \leq q$. Note that $F \leq q$ implies F is continuous at zero (Exercise). Thus $Y := \ker F$ is a closed

hyperplane. For $x \in G$, $x_0 - x \in H$ so that

$$F(x_0) - F(x) = F(x_0 - x) \le q(x_0 - x) < 1$$

Thus

$$F(x) > F(x_0) - 1 = f(x_0) - 1 = q(x_0) - 1 \ge 0$$

Hence $x \notin \ker F$, which implies $Y \cap G = \phi$.

Now suppose X is over $\mathbb{C}$. Viewing X as an $\mathbb{R}$-linear space the previous part yields a continuous $\mathbb{R}$-linear $F: X \to \mathbb{R}$ such that $\ker(F) \cap G = \phi$. If $\tilde{F}(x) := F(x) - iF(ix)$, then $\tilde{F}$ is continuous, $\mathbb{C}$-linear and has $\mathrm{Re}\tilde{F} = F$. Consequently, $x \in \ker \tilde{F}$ iff $F(x) = 0 = F(ix)$. Thus $Y := \ker \tilde{F} = \ker F \cap (i \ker F)$. So Y is a closed hyperplane with $Y \cap G = \phi$. $\square$

- Note that $\mathbb{R}$-linear functionals $f: X \to \mathbb{R}$ have the nice feature that

$$X = \ker(f) \sqcup \{x \in X : f(x) > 0\} \sqcup \{x \in X : f(x) < 0\}.$$

$\boxed{\text{Def}}$ Let X be a real TVS. We say $A, B \subseteq X$ are <u>strictly separated</u> if there exists a continuous linear functional $f: X \to \mathbb{R}$ and $\alpha \in \mathbb{R}$ such that

$$A \subseteq f^{-1}((\alpha, \infty)) \quad \text{and} \quad B \subseteq f^{-1}((-\infty, \alpha)).$$

We say they are <u>separated</u> if ← different from Conway def.

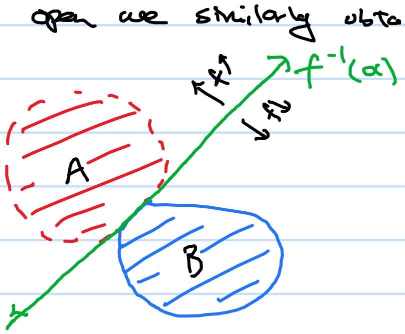
$$A \subseteq f^{-1}((\alpha, \infty)) \quad \text{and} \quad B \subseteq f^{-1}((-\infty, \alpha]).$$

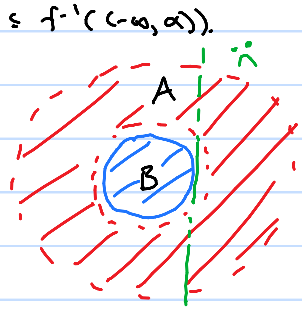
$\boxed{\text{Thm}}$ Let X be a real TVS. If $A, B \subseteq X$ are disjoint and convex with A open, then A and B are separated. If B is also open, then they are strictly separated.

<u>Proof</u> Set $G := A - B = \{a - b : a \in A, b \in B\}$. Then G is convex and open since $G = \bigcup_{b \in B} A - b$. Since $A \cap B = \phi$, $0 \notin G$. Thus there is a closed hyperplane $Y \subseteq X$ such that $Y \cap G = \phi$. Let $f: X \to \mathbb{R}$ be such that $Y = \ker f$. Then $f(G)$ is a convex subset of $\mathbb{R}$ with $0 \notin f(G)$. Consequently $f(G) \subseteq (0, \infty)$ or $f(G) \subseteq (-\infty, 0)$. Without loss of generality, assume the former. This implies $0 < f(a - b) = f(a) - f(b)$ for all $a \in A, b \in B$. Thus $\exists \alpha \in \mathbb{R}$ such that

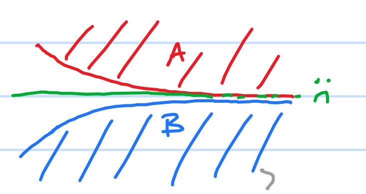
$$\sup_{b \in B} f(b) \le \alpha \le \inf_{a \in A} f(a).$$

So $B \subseteq f^{-1}((-\infty, \alpha])$ and $A \subseteq f^{-1}([\alpha, \infty))$. Since A is open, $A \subseteq f^{-1}([\alpha, \infty))^\circ = f^{-1}((\alpha, \infty))$. If B is open we similarly obtain $B \subseteq f^{-1}((-\infty, \alpha))$. $\square$





convexity is necessary



A (or B) open necessary

Lemma Let X be a TVS. If $K \subseteq X$ is compact and $V \supseteq K$ is open, then there is an open neighborhood $\mathcal{U}$ of 0 such that $K + \mathcal{U} \subseteq V$.

Proof Let $\mathscr{U}$ be the collection of all open neighborhoods of 0. Suppose, towards a contradiction, that $K + \mathcal{U} \not\subseteq V$ for all $\mathcal{U} \in \mathscr{U}$. Let $x_\mathcal{U} \in K$ and $y_\mathcal{U} \in \mathcal{U}$ be such that $x_\mathcal{U} + y_\mathcal{U} \in V^c$. Put an ordering on $\mathscr{U}$ be reverse inclusion: $\mathcal{U}_1 \leq \mathcal{U}_2 \iff \mathcal{U}_1 \supseteq \mathcal{U}_2$. This makes $(x_\mathcal{U})_{\mathcal{U} \in \mathscr{U}}$ and $(y_\mathcal{U})_{\mathcal{U} \in \mathscr{U}}$ into nets. Note that $y_\mathcal{U} \in \mathcal{U}$ implies $y_\mathcal{U} \to 0$. Since K is compact, there is a subnet converging to some $x \in K$. The corresponding subnet of $(y_\mathcal{U})_{\mathcal{U} \in \mathscr{U}}$ still converges to zero, so we have $x = x + 0 \in \overline{V^c} = V^c$. But this contradicts $K \subseteq V$. $\square$

Thm Let X be a real LCS. If $A, B \subseteq X$ are disjoint, closed, and convex with B compact, then A and B are strictly separated.

Proof $B \subseteq A^c$, so by the Lemma $\exists\, \mathcal{U} \ni 0$ open such that $B + \mathcal{U} \subseteq A^c$. Using local convexity, there exists a continuous seminorm p on X such that

$$\{x \in X : p(x) < 1\} \subseteq \mathcal{U}.$$

Set $\mathcal{U}_0 := \{x \in X : p(x) < \frac{1}{2}\}$. Then $(B + \mathcal{U}) \cap (A + \mathcal{U}) = \phi$. Since $B + \mathcal{U}$ and $A + \mathcal{U}$ are open convex subsets containing A and B, the previous theorem finishes the proof. $\square$

Rem The previous two theorems are sometimes called the Hahn-Banach separation theorems. There is also a version for complex LCS's (see below).

Cor If X is a real LCS, $A \subseteq X$ is closed and convex, and $x \notin A$, then $\{x\}$ is strictly separated from A.

Cor If X is a real LCS and $A \subseteq X$, then

$$\overline{co}(A) = \bigcap \{ f^{-1}((-\infty, \alpha]) : f : X \to \mathbb{R} \text{ cts}, A \subseteq f^{-1}((-\infty, \alpha]) \text{ for some } \alpha \in \mathbb{R}\}$$

Proof The sets in the intersection are all closed convex sets containing A, so the inclusion $\subseteq$ holds. If $x_0 \notin \overline{co}(A)$, then the previous Corollary implies there is a continuous $f : X \to \mathbb{R}$ such that $f(x_0) > \alpha$ and $f(x) < \alpha$ for all $x \in \overline{co}(A)$. This shows x_0 is not in the intersection. $\square$

- The following is the generalization of $\overline{span}(A) = \bigcap \{ \ker f : f \in X^*, A \subseteq \ker f\}$ for normed spaces.

Cor If X is a real LCS and $A \subseteq X$, then $\overline{span}(A)$ is the intersection of all all closed hyperplanes containing A.

Thm Let X be a complex LCS. If $A, B \subseteq X$ are disjoint, closed, and convex with B compact, then there exists a continuous linear functional $f: X \to \mathbb{C}$, $\alpha \in \mathbb{R}$, and $\varepsilon > 0$ such that

$$\operatorname{Re} f(a) \leq \alpha < \alpha + \varepsilon \leq \operatorname{Re} f(b) \qquad \forall a \in A,\ b \in B.$$

Proof Viewing X as a real LCS, $f_0: X \to \mathbb{R}$ be the linear functional strictly separating A and B. Replacing f_0 with $-f_0$ if necessary, we have

$$A \subseteq f_0^{-1}((-\infty, \alpha]) \quad \text{and} \quad B \subseteq f_0^{-1}((\alpha, \infty))$$

for some $\alpha \in \mathbb{R}$. Since B is compact, $f_0(B)$ is compact and so has a minimal element. Hence $f_0(B) \subseteq [\alpha+\varepsilon, \infty)$ for some $\varepsilon > 0$. Take $f(x) = f_0(x) - i f_0(ix)$. $\square$

Cor Let X be a LCS. A subspace $Y \subseteq X$ is dense if and only if the only continuous linear functional on X that vanishes on Y is identically zero.

Cor Let X be a LCS. For $Y \subseteq X$ a closed subspace and $x_0 \in X \setminus Y$, there exists a continuous linear functional $f: X \to \mathbb{F}$ such that $f|_Y \equiv 0$ and $f(x_0) = 1$.

Rem Requiring X be an LCS instead of merely a TVS is to ensure an abundance of non-empty open convex sets (e.g. $p^{-1}([0,1))$ for any cts seminorm p). For a general TVS, X itself may be the only non-empty open convex set as the following example demonstrates.

Ex Fix $0<p<1$ and let $L^p(0,1)$ be the equivalence classes of measurable functions $f: (0,1) \to \mathbb{R}$ satisfying

$$\int_0^1 |f(x)|^p \, dx < \infty.$$

One can show that

$$d(f,g) := \int_0^1 |f(x) - g(x)|^p \, dx$$

defines a metric on $L^p(0,1)$ making it into a Fréchet space (Exercise), hence a TVS. However, we claim if $G \subseteq L^p(0,1)$ is nonempty, open, and convex then $G = L^p(0,1)$. (By the last Proposition in Section IV.1, this means $L^p(0,1)$ is not an LCS). Indeed, fix $R>0$ and let $f \in B(0,R)$. Say $d(f,0) = r$. Note that

$$t \mapsto \int_0^t |f(x)|^p \, dx$$

is cts, equals 0 at $t=0$, and equals r at $t=1$. By the IVT $\exists\, 0<t_0<1$ such that

$$\int_0^{t_0} |f(x)|^p \, dx = \frac{r}{2}.$$

Define

$$g(x) := \begin{cases} f(x) & \text{if } x \leq t_0 \\ 0 & \text{otherwise} \end{cases} \qquad \text{and} \qquad h := f - g.$$

Then

$$f = g + h = \tfrac{1}{2}(2g + 2h)$$

and

$$d(2g, 0) = \int_0^1 |2g(x)|^p dx = 2^p \int_0^{t_0} |f(x)|^p dx = 2^{p-1} r < 2^{p-1} R$$
$$d(2h, 0) = \int_0^1 |g(x)|^p dx = 2^p \int_{t_0}^1 |f(x)|^p dx = 2^{p-1} r < 2^{p-1} R$$

So $f \in \operatorname{co} B(0, 2^{p-1}R)$. We have shown $B(0,R) \subseteq \operatorname{co} B(0, 2^{p-1}R)$, which implies

$$B(0, 2^{1-p} R) \subseteq \operatorname{co} B(0,R).$$

But then

$$B(0, 2^{2(1-p)} R) \subseteq \operatorname{co} B(0, 2^{1-p} R) \subseteq \operatorname{co} B(0,R)$$

and iterating yields

$$L^p(0,1) = \bigcup_{n=1}^{\infty} B(0, 2^{n(1-p)} R) \subseteq \operatorname{co} B(0,R).$$

Now, let $f_0 \in G$ and let $R>0$ be such that $B(f_0, R) \subset G \Rightarrow B(0,R) \subset G - f_0$. Since G (and hence $G - f_0$) is convex, we obtain

$$L^p(0,1) \subseteq \operatorname{co} B(0,R) \subseteq G - f_0 \Rightarrow G = L^p(0,1).$$

Another consequence of this is that if $\phi : L^p(0,1) \to \mathbb{R}$ is continuous, then for all $a<b$, $\phi^{-1}((a,b))$ is either empty or all of $L^p(0,1)$. If $a<0<b$, the linearity of ϕ implies $\phi(0) = 0 \in (a,b)$. Thus $\forall \varepsilon > 0$, $L^p(0,1) = \phi^{-1}((-\varepsilon, \varepsilon))$ which implies $\phi \equiv 0$. $\square$