1. Let $(X, \mathcal{M})$ be a measurable space.
   (a) For each $n \in \mathbb{N}$, let μ_n be a measure on $\mathcal{M}$ and let $\alpha_n \in [0, +\infty)$. Show that
   $$\mu(E) := \sum_{n=1}^{\infty} \alpha_n \mu_n(E)$$
   defines a measure on $\mathcal{M}$.
   (b) Let μ be a measure on $\mathcal{M}$ and $E_0 \in \mathcal{M}$. Show that $\nu(E) := \mu(E \cap E_0)$ defines a measure on $\mathcal{M}$.
   (c) Given a σ-finite measure μ on $\mathcal{M}$, show that there exists a finite measure ν on $\mathcal{M}$ satisfying that $E \in \mathcal{M}$ is μ-null if and only if it is ν-null. (We say μ and ν are **equivalent** in this case.)

Proof. (a) We have
$$\mu(\emptyset) = \sum_{n=1}^{\infty} \alpha_n \cdot 0 = 0.$$
If $E = \bigcup_{k=1}^{\infty} E_k$ is a disjoint union with $A_n \in \mathcal{M}$, then using countable additivity for each μ_n we have
$$\mu(E) = \sum_{n=1}^{\infty} \alpha_n \mu_n(E) = \sum_{n=1}^{\infty} \alpha_n \sum_{k=1}^{\infty} \mu_n(E_k) =^{1} \sum_{k=1}^{\infty} \sum_{n=1}^{\infty} \alpha_n \mu_n(E_k) = \sum_{k=1}^{\infty} \mu(E_k).$$
Hence μ is a measure. □

(b) We have $\nu(\emptyset) = \mu(\emptyset) = 0$, and for a disjoint union $E = \bigcup_{k=1}^{\infty} E_k$ of $\mathcal{M}$-measurable sets one has
$$\nu(E) = \mu(E \cap E_0) = \sum_{k=1}^{\infty} \mu(E_k \cap E_0) = \sum_{k=1}^{\infty} \nu(E_k).$$
Hence ν is a measure. □

(c) Using σ-finiteness of μ, we can write $X = \bigcup_{n=1}^{\infty} E_n$ where $\mu(E_n) < \infty$ for each $n \in \mathbb{N}$. Using parts (a) and (b), the following formula defines a measure:
$$\nu(E) := \sum_{n=1}^{\infty} \frac{2^{-n}}{1 + \mu(E_n)} \mu(E \cap E_n).$$
Since
$$\frac{2^{-n}}{1 + \mu(E_n)} \mu(E \cap E_n) \leq \frac{2^{-n}}{1 + \mu(E_n)} \mu(E_n) \leq 2^{-n},$$
it follows that
$$\nu(E) \leq \sum_{n=1}^{\infty} 2^{-n} = 1$$
for all $E \in \mathcal{M}$. Hence ν is finite. Now, if E is μ-null, then $\mu(E \cap E_n) \leq \mu(E) = 0$ for all $n \in \mathbb{N}$ and hence $\nu(E) = 0$. Conversely, if E is ν-null, then we must have $\mu(E \cap E_n) = 0$ for all $n \in \mathbb{N}$. But then by countable subadditivity we have
$$\mu(E) \leq \sum_{n=1}^{\infty} \mu(E \cap E_n) = 0.$$
Hence E is μ-null. □

2. Suppose $F : \mathbb{R} \to \mathbb{R}$ is an increasing, differentiable function with $\sup_{t \in \mathbb{R}} F'(t) < \infty$, and let μ_F be the associated Lebesgue–Stieltjes measure. Denote the Lebesgue measure by m.

[1] Interchanging series of positive terms without proof was explicitly permitted during the exam.

(a) Show that every m-null set is μ_F-null.

(b) Find an example of such a function F for which the converse is false.

Proof. (a) Denote
$$R := \sup_{t\in\mathbb{R}} F'(t).$$
Note that $0 \le F' \le R$ since F is increasing. For any $a, b \in \mathbb{R}$ with $a < b$, the Mean Value Theorem implies
$$\frac{F(b) - F(a)}{b - a} = F'(c) \le R \tag{1}$$
for some $a < c < b$. Now, suppose $E \subset \mathbb{R}$ is m-null. Then for $\epsilon > 0$ we can find a countable cover $E \subset \bigcup_{n=1}^\infty (a_n, b_n)$ satisfying
$$\sum_{n=1}^\infty (b_n - a_n) < \epsilon.$$
Consequently, by (1) one has
$$\mu_F(E) \le \sum_{n=1}^\infty \mu_F((a_n, b_n)) \le \sum_{n=1}^\infty (F(b_n) - F(a_n)) \le \sum_{n=1}^\infty R(b_n - a_n) < R\epsilon.$$
Thus E is μ_F-null. □

[**Note:** technically the question also asks one to verify that E is μ_F-measurable, but this was not required for grading purposes. The above argument actually shows E is null for the outer measure μ_F^* associated to F. But this implies E is also μ_F^*-measurable:
$$\mu_F^*(A) \le \mu_F^*(A \cap E) + \mu_F^*(A \cap E^c) = \mu_F^*(A \cap E^c) \le \mu_F^*(A).$$
Since μ_F is the complete measure given by restricting μ_F^* to such measurable sets, we see that E is in fact μ_F-measurable with $\mu_F(E) = 0$.]

(b) Consider the constant function $F(t) = 0$ for all $t \in \mathbb{R}$. Then $\mu_F((a, b]) = F(b) - F(a) = 0$ for all h-intervals. Thus $(0, 1]$ is μ_F-null but not m-null. □

3. Suppose $f\colon \mathbb{R} \to \mathbb{R}$ has a countable discontinuity set. Show that f is Borel measurable.
[**Hint:** the ϵ-δ definition of continuity will be more helpful than the sequential definition.]

Proof. By Proposition 2.1 from lecture it suffices to show $f^{-1}(U)$ is Borel measurable for all open sets $U \subset \mathbb{R}$. Fix such an open set and denote the discontinuity set of f by D. Observe that if f is continuous at x and $f(x) \in U$, then x lies in the interior of $f^{-1}(U)$ by definition of continuity. Thus
$$f^{-1}(U) = f^{-1}(U)^\circ \cup [D \cap f^{-1}(U) \setminus f^{-1}(U)^\circ].$$
Since the latter set is at most countable, it is Borel as a countable union of singletons. The interior is Borel as an open set, and thus $f^{-1}(U)$ is Borel. □