Exercises: (Sections 3.3, 3.4)

1. Let ν be a complex measure on $(X, \mathcal{M})$. Show that $\nu = |\nu|$ iff $\nu(X) = |\nu|(X)$.

2. Let ν be a complex measure on $(X, \mathcal{M})$. For $E \in \mathcal{M}$, show

$$\begin{aligned}|\nu|(E) &= \sup\left\{\sum_{j=1}^{n} |\nu(E_j)| \colon E = E_1 \cup \cdots \cup E_n \text{ is a partition}\right\}\\ &= \sup\left\{\sum_{j=1}^{\infty} |\nu(E_j)| \colon E = \bigcup_{j=1}^{\infty} E_j \text{ is a partition}\right\}\\ &= \sup\left\{\left|\int_E f\, d\nu\right| : |f| \leq 1\right\}.\end{aligned}$$

3. Let $f \in L^1(\mathbb{R}^n, m)$ be such that $m(\{x \in \mathbb{R}^n \colon f(x) \neq 0\}) > 0$.
   (a) Show that there exists $C, R > 0$ so that $Hf(x) \geq C|x|^{-n}$ for $|x| > R$.
   (b) Show that there exists $C' > 0$ so that $m(\{x \in \mathbb{R}^n \colon Hf(x) > \epsilon\}) \geq \frac{C'}{\epsilon}$ for all sufficiently small $\epsilon > 0$. [**Note:** this shows that the Hardy–Littlewood maximal inequality is sharp up to the choice of constant.]

4. Let ν be a regular signed or complex Borel measure on $\mathbb{R}^n$ with Lebesgue decomposition $\nu = \lambda + \rho$ with respect to the Lebesgue measure m, where $\lambda \perp m$ and $\rho \ll m$. Show that λ and ρ are both regular.
   [**Hint:** first show $|\nu| = |\lambda| + |\rho|$.]

5. [1] Let $M(X, \mathcal{M})$ denote the set of complex measures on a measurable space $(X, \mathcal{M})$.
   (a) Show that $\|\nu\| := |\nu|(X)$ defines a norm on $M(X, \mathcal{M})$.
   (b) Show that $M(X, \mathcal{M})$ is complete with respect to the metric $\|\nu - \mu\|$.
   (c) Suppose μ is a σ-finite measure on $(X, \mathcal{M})$ and $(\nu_n)_{n\in\mathbb{N}} \subset M(X, \mathcal{M})$ satisfies $\nu_n \ll \mu$ for all $n \in \mathbb{N}$. For $\nu \in M(X, \mathcal{M})$, show that $\|\nu_n - \nu\| \to 0$ if and only if $\nu \ll \mu$ and $\frac{d\nu_n}{d\mu} \to \frac{d\nu}{d\mu}$ in $L^1(\mu)$.

6. [2] For a Borel set $E \subset \mathbb{R}^n$, the **density** of E at a point x is defined as

$$D_E(x) := \lim_{r\to 0} \frac{m(E \cap B(x, r))}{m(B(x, r))}$$

   whenever the limit exists.
   (a) Show that D_E is defined m-almost everywhere and $D_E = 1_E$ m-almost everywhere.
   (b) For $0 < \alpha < 1$, find an example of an E and so that $D_E(0) = \alpha$. [**Hint:** use a sequence of annuli.]
   (c) Find an example of an E so that $D_E(0)$ does not exist. [**Hint:** use another sequence of annuli.]

Solutions:

1. The "only if" direction is immediate. Conversely, suppose $\nu(X) = |\nu|(X)$. We first show $\nu_i \equiv 0$. Let $X = P \cup N$ is a Hahn decomposition of for ν_i. Since $\nu(X) = |\nu|(X)$ is real, we have by Proposition 3.16

$$\begin{aligned}|\nu|(X) &= \nu(X) = \nu_r(X) = \nu_r(P) + \nu_r(N)\\ &\leq |\nu_r(P)| + |\nu_r(N)| \leq |\nu(P)| + |\nu(N)| \leq |\nu|(P) + |\nu|(N) = |\nu|(X).\end{aligned}$$

[1] Not collected
[2] Not collected

Thus all of the above inequalities are actually equality. In particular,

$$|\nu_r(P)| + |\nu_r(N)| = |\nu(P)| + |\nu(N)| = (\nu_r(P)^2 + \nu_i(P)^2)^{1/2} + (\nu_r(N)^2 + \nu_i(N)^2)^{1/2}$$

and so we must have $\nu_i(P) = \nu_i(N) = 0$. But then $\nu_i \equiv 0$ as claimed. Thus $\nu = \nu_r$ is a signed measure, and it remains to show $\nu^- \equiv 0$. Let $X = P' \cup N'$ be a Hahn decomposition for ν. Then

$$|\nu|(X) = \nu(X) = \nu(P') + \nu(N') \leq \nu(P') \leq |\nu(P')| \leq |\nu|(P') \leq |\nu|(X),$$

and so the above inequalities are equalities. In particular, $\nu(N') = 0$ and so $\nu^- \equiv 0$. □

2. Fix $E \in \mathcal{M}$ and denote the three supremums by $\alpha_1, \alpha_2, \alpha_3$, respectively. By letting $E_{n+1} = E_{n+2} = \cdots = \emptyset$, we see that $\alpha_1 \leq \alpha_2$. Next, given a partition $E = \bigcup_{j=1}^\infty E_j$ consider

$$f := \sum_{j=1}^\infty \overline{\text{sgn}(\nu(E_j))} 1_{E_j},$$

which satisfies $|f| \leq 1$ and by the dominated convergence theorem

$$\int_E f d\nu = \sum_{j=1}^\infty \overline{\text{sgn}(\nu(E_j))}\nu(E_j) = \sum_{j=1}^\infty |\nu(E_j)|.$$

Hence $\alpha_2 \leq \alpha_3$. By Proposition 3.16, if $|f| \leq 1$ then

$$\left| \int_E f \; d\nu \right| \leq \int_E |f| \; d|\nu| \leq \int_E 1 \; d|\nu| = |\nu|(E),$$

and so $\alpha_3 \leq |\nu|(E)$.

Finally, let $f := \overline{\frac{d\nu}{d|\nu|}}$. Then Proposition 3.16 implies $f \frac{d\nu}{d|\nu|} = 1$ $|\nu|$-almost everywhere. Thus

$$|\nu|(E) = \int_E 1 \; d|\nu| = \int_E f \frac{d\nu}{d|\nu|} \; d|\nu| = \int_E f \; d\nu.$$

Given $\epsilon > 0$, Proposition 2.10 and the dominated convergence theorem yield a simple function ϕ on E so that $|\phi| \leq |f| = 1$ and $|\int_E (\phi - f) \; d\nu| < \epsilon$. If $\phi = \sum_{j=1}^n \alpha_j 1_{E_j}$ is the standard representation, then $|\alpha_j| \leq 1$ and we have

$$|\nu|(E) = \int_E f \; d\nu \leq \left| \int_E \phi \; d\nu \right| + \epsilon = \left| \sum_{j=1}^n \alpha_j \nu(E_j) \right| + \epsilon \leq \sum_{j=1}^n |\nu(E_j)| + \epsilon.$$

Since $E = E_1 \cup \cdots \cup E_n$ is a finite partition, we can bound the above by $\alpha_1 + \epsilon$. Letting $\epsilon \to 0$, we see that $|\nu|(E) \leq \alpha_1$, and so all quantities are equal. □

3. (a) We have

$$\alpha := \int_{\mathbb{R}^n} f \; dm > 0$$

since otherwise $f = 0$ m-almost everywhere by Proposition 2.16. Now, the dominated convergence theorem,

$$\int_{B(0,n)} |f| \; dm = \int_{\mathbb{R}^n} 1_{B(0,n)} |f| \; dm \to \int_{\mathbb{R}^n} |f| \; dm = \alpha.$$

Thus we can find $R := n \in \mathbb{N}$ large enough so that $\int_{B(0,R)} |f| \; dm \geq \frac{\alpha}{2} > 0$. If $|x| > R$, then for $r := |x| + R$ we have $B(0, R) \subset B(x, r)$ and hence

$$\int_{B(x,r)} |f| \; dm \geq \int_{B(0,R)} |f| \; dm \geq \frac{\alpha}{2}.$$

Also note that since $r = |x| + R < 2|x|$ we have

$$m(B(x,r)) = cr^n \le c2^n|x|^n$$

where c is the measure of the unit ball centered at zero. Thus

$$Hf(x) \ge \frac{1}{m(B(x,r))}\int_{B(x,r)} |f|\ dm \ge \frac{1}{c2^n|x|^n}\frac{\alpha}{2},$$

and so we set $C := \frac{\alpha}{c2^{n+1}}$. □

(b) Let $C, R > 0$ be as in part (a) and let $0 < \epsilon < \frac{C}{R^n}$. Then $R < (C/\epsilon)^{1/n}$ and for $R < |x| < (C/\epsilon)^{1/n}$, part (a) implies

$$Hf(x) \ge \frac{C}{|x|^n} > \epsilon.$$

Hence

$$m(\{x \in \mathbb{R}^n : Hf(x) > \epsilon\}) \ge m(\{x \in \mathbb{R}^n : R < |x| < (C/\epsilon)^{1/n}\}) = c(\frac{C}{\epsilon} - R^n) = c\frac{C - \epsilon R^n}{\epsilon},$$

where c is the measure of the unit ball centered at zero. If we further demand $\epsilon < \frac{C}{2R^n}$, then $C - \epsilon R^n > \frac{C}{2}$. Thus the above is strictly bounded below by $\frac{C'}{\epsilon}$ for $C' = \frac{cC}{2}$. □

4. We first claim $|\nu| = |\lambda| + |\rho|$. Denote $\mu := |\lambda| + |\rho|$. Now, $\rho \ll m \perp \lambda$ implies $\rho \perp \lambda$, and so we can find a partition $\mathbb{R}^n = A \cup B$ so that A is ρ-null and B is λ-null. It follows that $\frac{d\rho}{d\mu} = 0$ μ-almost everywhere on A and $\frac{d\lambda}{d\mu} = 0$ μ-almost everywhere on B. So we have

$$\begin{aligned}|\lambda|(E) + |\rho|(E) = |\lambda|(E \cap A) + |\rho|(E \cap B) &= \int_{E\cap A}\left|\frac{d\lambda}{d\mu}\right|\ d\mu + \int_{E\cap B}\left|\frac{d\rho}{d\mu}\right|\ d\mu \\ &= \int_{E\cap A}\left|\frac{d\nu}{d\mu}\right|\ d\mu + \int_{E\cap B}\left|\frac{d\nu}{d\mu}\right|\ d\mu = \int_{E}\left|\frac{d\nu}{d\mu}\right|\ d\mu = |\nu|(E),\end{aligned}$$

where we have use $|\nu| \le \mu$ and Proposition 3.15. This proves the claim.

Now, for compact $K \subset \mathbb{R}^n$, the claim implies $|\lambda|(K), |\rho|(K) \le |\nu|(K) < \infty$ by regularity of ν. Let $\mathbb{R}^n = A \cup B$ as above, and note that A and B are also $|\rho|$-null and $|\lambda|$-null, respectively. For a Borel set $E \subset \mathbb{R}^n$ and $\epsilon > 0$, the regularity of $|\nu|$ allows us to find $U_1 \supset E \cap A$ and $U_2 \supset E \cap B$ be open sets satisfying

$$\begin{aligned}|\nu|(U_1) &\le |\nu|(E \cap A) + \epsilon = |\lambda|(E) + \epsilon \\ |\nu|(U_2) &\le |\nu|(E \cap B) + \epsilon = |\rho|(E) + \epsilon.\end{aligned}$$

Thus from the claim we have $|\lambda|(U_1) \le |\nu|(U_1) \le |\lambda|(E) + \epsilon$ and $|\rho|(U_2) \le |\nu|(U_2) \le |\rho|(E) + \epsilon$, and so $|\lambda|$ and $|\lambda|$ are both regular. By definition, this means λ and ρ are regular. □

5. (a) By Proposition 3.17, we have $\|\nu + \mu\| = |\nu + \mu|(X) \le |\nu|(X) + |\mu|(X) = \|\nu\| + \|\mu\|$. For $\alpha \in \mathbb{C}$, the first equality in Exercise 5 shows $\|\alpha\nu\| = |\alpha\nu|(X) = |\alpha||\nu|(X) = |\alpha|\|\mathbf{v}\|$. Finally, if $\|\nu\| = 0$, then $|\nu|(E) \le |\nu|(X) = 0$ for all $E \in \mathcal{M}$. Since $\nu \ll |\nu|$ by Proposition 3.16, it follows that $\nu = 0$. Thus $\|\cdot\|$ is a norm. □

(b) Suppose $(\nu_n)_{n\in\mathbb{N}} \subset M(X, \mathcal{M})$ is Cauchy with respect to this metric. Observe that for any $E \in \mathcal{M}$ we have

$$|\nu_n(E) - \nu_m(E)| \le |\nu_n - \nu_m|(E) \le |\nu_n - \nu_m|(X) = \|\nu_n - \nu_m\|.$$

Hence $(\nu_n(E))_{n\in\mathbb{N}} \subset \mathbb{C}$ is a Cauchy sequence, and we will denote its limit by $\nu(E)$. We claim $\nu \in M(X, \mathcal{M})$ and $\|\nu_n - \nu\| \to 0$. First observe that for all $\epsilon > 0$ there exists $N \in \mathbb{N}$ so that for any $n \ge N$ and any partition $E = E_1 \cup \cdots \cup E_d$ one has

$$\sum_{j=1}^{d} |\nu(E_j) - \nu_n(E_j)| < \epsilon.$$

Indeed, let $N \in \mathbb{N}$ is such that $\|\nu_n - \nu_m\| < \frac{\epsilon}{2}$ for all $n, m \geq N$. Then for $n \geq N$ and any partition $E = E_1 \cup \cdots \cup E_d$, Exercise 5 implies

$$\sum_{j=1}^{d} |\nu(E_j) - \nu_n(E_j)| \leq \sum_{j=1}^{d} |\nu(E_j) - \nu_m(E_j)| + \|\nu_m - \nu_n\|$$

for any $m \in \mathbb{N}$. In particular, if we take $m \geq N$ is taken large enough so that $|\nu(E_j) - \nu_m(E_j)| < \frac{\epsilon}{2d}$ for each $j = 1, \ldots, d$, then we obtain the claimed inequality. It now suffices to show ν is a complex measure since then this inequality implies $\|\nu - \nu_n\| \leq \epsilon$ for all $n \geq N$ by Exercise 5, and hence $\|\nu_n - \nu\| \to 0$.

Now, we have $\nu(\emptyset) = \lim_n \nu_n(\emptyset) = 0$. If $(E_k)_{k\in\mathbb{N}} \subset \mathcal{M}$ is a disjoint collection, we first show the series $\sum_k \nu(E_k)$ converges absolutely. For $\epsilon = 1$, let $N \in \mathbb{N}$ be as in our first observation. Then for any $K \in \mathbb{N}$ we have

$$\begin{aligned}\sum_{k=1}^{K} |\nu(E_k)| &\leq \sum_{k=1}^{K} |\nu(E_k) - \nu_N(E_k)| + \sum_{k=1}^{K} |\nu_N(E_k)| \\ &\leq 1 + \sum_{k=1}^{K} |\nu_N|(E_k) = 1 + |\nu_N|\left(\bigcup_{k=1}^{K} E_k\right) \leq 1 + \|\nu_N\|.\end{aligned}$$

Since the bound is independent of K, we see that the series converges absolutely. Now let $\epsilon > 0$ be arbitrary and take $N \in \mathbb{N}$ as in our first observation. Choose $K \in \mathbb{N}$ be large enough so that $\sum_{k>K} |\nu(E_k)| + |\nu_N(E_k)| < \epsilon$. Denoting $E = \bigcup_{k=1}^{\infty} E_k$, we have

$$\begin{aligned}\left|\sum_{k=1}^{\infty} \nu(E_k) - \nu(E)\right| &\leq \left|\sum_{k=1}^{\infty} \nu(E_k) - \nu_N(E_k)\right| + |\nu_N(E) - \nu(E)| \\ &\leq \sum_{k=1}^{K} |\nu(E_k) - \nu_N(E_k)| + \sum_{k>K} |\nu(E_k)| + |\nu_N(E_k)| + |\nu_N(E) - \nu(E)| < 3\epsilon.\end{aligned}$$

Thus we must have $\sum_{k=1}^{\infty} \nu(E_k) = \nu(E)$, and therfore ν is a complex measure. $\square$

(c) Suppose $\|\nu_n - \nu\| \to 0$. If $\mu(E) = 0$ then $\nu(E) = \lim_n \nu_n(E) = 0$. Hence $\nu \ll \mu$. By Proposition 3.15, we have

$$\|\nu_n - \nu\| = |\nu_n - \nu|(X) = \int_X \left|\frac{d(\nu_n - \nu)}{d\mu}\right| \, d\mu = \int_X \left|\frac{d\nu_n}{d\mu} - \frac{d\nu}{d\mu}\right| \, d\mu.$$

Thus $\|\nu_n - \nu\| \to 0$ iff $\frac{d\nu_n}{d\mu} \to \frac{d\nu}{d\mu}$ in L^1. $\square$

6. (a) Fix a Borel set $E \subset \mathbb{R}^n$ and define a Borel measure ν by $\nu(F) := m(E \cap F)$. Equivalently, $d\nu = 1_E dm$. Since m is regular, it follows that $\frac{d\nu}{dm} = 1_E \in L^1_{loc}(\mathbb{R}^n, m)$, and so ν is regular by Lemma 3.25. Consequently Theorem 3.24 implies

$$D_E(x) = \lim_{r\to 0} \frac{\nu(B(x,r))}{m(B(x,r))} = \frac{d\nu}{dm}(x) = 1_E(x)$$

for m-almost every $x \in \mathbb{R}^n$. $\square$

(b) Fix $0 < \alpha < 1$ and let $x = 0$. For each $k \in \mathbb{N}$, let $E_k \subset B(0, \frac{1}{k}) \setminus B(0, \frac{1}{k+1})$ be a Borel subset satisfying

$$m(E_k) = \alpha m\left(B(0, \frac{1}{k}) \setminus B(0, \frac{1}{k+1})\right) = \alpha c\left(\frac{1}{k^n} - \frac{1}{(k+1)^n}\right),$$

where $c = m(B(0,1))$. (E.g. let $E_k = B(0, \frac{1}{k}) \setminus B(0, ((1-\alpha)\frac{1}{k^n} + \alpha\frac{1}{(k+1)^n})^{1/n})$.) Define $E := \bigcup_{k\in\mathbb{N}} E_k$, and note that this union is disjoint. For $r \leq 1$, let $K \in \mathbb{N}$ be the unique integer so that

$\frac{1}{K+1} < r \leq \frac{1}{K}$. It follows that

$$\bigcup_{k=K+1}^{\infty} E_k \subset E \cap B(0,r) \subset \bigcup_{k=K}^{\infty} E_k$$

So using the second inclusion and $\frac{1}{K+1} < r$ we have

$$\frac{m(E \cap B(0,r))}{m(B(0,r))} \leq \sum_{k=K}^{\infty} \frac{\alpha(\frac{1}{k^n} - \frac{1}{(k+1)^n})}{r^n} = \frac{\alpha}{r^n} \sum_{k=K}^{\infty} \frac{1}{k^n} - \frac{1}{(k+1)^n} = \frac{\alpha}{r^n K^n} < \frac{\alpha(K+1)^n}{K^n},$$

while using the first inclusion and $r \leq \frac{1}{K}$ we have

$$\frac{m(E \cap B(0,r))}{m(B(0,r))} \geq \sum_{k=K+1}^{\infty} \frac{\alpha(\frac{1}{k^n} - \frac{1}{(k+1)^n})}{r^n} = \frac{\alpha}{r^n(K+1)^n} \geq \frac{\alpha K^n}{(K+1)^n}.$$

Since $K \to \infty$ as $r \to 0$, we have

$$\limsup_{r \to 0} \frac{m(E \cap B(0,r))}{m(B(0,r))} \leq \limsup_{K \to \infty} \frac{\alpha(K+1)^n}{K^n} = \alpha$$

and

$$\liminf_{r \to 0} \frac{m(E \cap B(0,r))}{m(B(0,r))} \geq \liminf_{K \to \infty} \frac{\alpha K^n}{(K+1)^n} = \alpha.$$

Hence $D_E(0) = \alpha$. □

(c) Fix $\beta \in (0,1)$, and define

$$E := \bigcup_{k=0}^{\infty} B(0, \beta^{2k}) \setminus B(0, \beta^{2k+1}).$$

Also denote

$$E_K := \bigcup_{k=K}^{\infty} B(0, \beta^{2k}) \setminus B(0, \beta^{2k+1}).$$

For $r = \beta^{2K}$, $K \geq 0$, we have

$$\frac{m(E \cap B(0,r))}{m(B(0,r))} = \sum_{k=K} \frac{\beta^{2kn} - \beta^{2kn+n}}{\beta^{2Kn}} = (1-\beta^n) \sum_{k=0}^{\infty} \beta^{2kn} = \frac{1-\beta^n}{1-\beta^{2n}} = \frac{1}{1+\beta^n}.$$

For $r = \beta^{2K+1}$, $K \geq 0$, we have

$$\frac{m(E \cap B(0,r))}{m(B(0,r))} = \sum_{k=K+1} \frac{\beta^{2kn} - \beta^{2kn+n}}{\beta^{2Kn+n}} = (1-\beta^n)\beta^n \sum_{k=0}^{\infty} \beta^{2kn} = \frac{\beta^n}{1+\beta^n}.$$

Since $\beta^{2K}, \beta^{2K+1} \to 0$ as $K \to \infty$, these calculations show

$$\liminf_{r \to 0} \frac{m(E \cap B(0,r))}{m(B(0,r))} \leq \frac{\beta^n}{1+\beta^n} < \frac{1}{1+\beta^n} \leq \limsup_{r \to 0} \frac{m(E \cap B(0,r))}{m(B(0,r))},$$

and the strict inequality implies the limit does not exist. □