Exercises: (Sections 2.5, 3.1)

1. On $([0,1], \mathcal{B}_{[0,1]})$, let m be the Lebesgue measure and let ν be the counting measure. For the diagonal set
$$D := \{(t,t) \in [0,1]^2 : 0 \leq t \leq 1\},$$
show that $\iint 1_D \; dmd\nu$, $\iint 1_D \; d\nu dm$, and $\int 1_D \; d(m \times \nu)$ are all distinct.

   [**Note:** this shows the σ-finiteness assumption in the Fubini–Tonelli theorem is necessary.]

2. Let $(X, \mathcal{M}, \mu)$ be a σ-finite measure space and $f \in L^+(X, \mu)$.
   (a) Show that
   $$G_f := \{(x,t) \in X \times [0,\infty) : t \leq f(x)\}$$
   is $\mathcal{M} \otimes \mathcal{B}_{\mathbb{R}}$-measurable with $\mu \times m(G_f) = \int_X f \; d\mu$.
   (b) Prove the **layer cake formula**:
   $$\int_X f \; d\mu = \int_{[0,\infty)} \mu(\{x \in X : f(x) \geq t\}) \; dm(t).$$
   (c) Use part (a) and continuity from below to give an alternate (albeit circular) proof of the monotone convergence theorem.

3. Suppose $f \in L^1((0,1), m)$, and define
$$g(x) := \int_{(x,1)} \frac{1}{t} f(t) \; dm(t) \qquad 0 < x < 1.$$
Show that $g \in L^1((0,1), m)$ with $\int_{(0,1)} g \; dm = \int_{(0,1)} f \; dm$.

4. Let ν and μ be signed measures on a measurable space $(X, \mathcal{M})$.
   (a) Show that E is ν-null if and only if $|\nu|(E) = 0$.
   (b) Show the following are equivalent:
      (i) $\nu \perp \mu$
      (ii) $|\nu| \perp \mu$
      (iii) $\nu^+ \perp \mu$ and $\nu^- \perp \mu$.

5. Let $(X, \mathcal{M}, \mu)$ and $(Y, \mathcal{N}, \nu)$ be measure spaces, let $f : X \to \mathbb{C}$ be $\mathcal{M}$-measurable, and let $g : Y \to \mathbb{C}$ be $\mathcal{N}$-measurable.
   (a) Show that $h : X \times Y \to \mathbb{C}$ defined by $h(x,y) := f(x)g(y)$ is $\mathcal{M} \otimes \mathcal{N}$-measurable.
   (b) Show that if $f \in L^1(X, \mu)$ and $g \in L^1(Y, \nu)$, then $h \in L^1(X \times Y, \mu \times \nu)$ with
   $$\int_{X \times Y} h \; d(\mu \times \nu) = \left(\int_X f \; d\mu \right) \left(\int_Y g \; d\nu \right).$$

Solutions:

1. For $t \in [0,1]$, we have $(1_D)^t(s) = 1_D(s,t) = 1_{\{t\}}(s)$. Since $1_{\{t\}} = 0$ m-almost everywhere, we have $\int 1_D(s,t) \; dm(s) = 0$ for all $t \in [0,1]$ and hence $\iint 1_D \; dm\nu = 0$. On the other hand,
$$\int_{[0,1]} 1_D(s,t) \; d\nu(t) = \int_{[0,1]} 1_{\{s\}}(t) \; d\nu(t) = \nu(\{s\}) = 1,$$

and so $\iint 1_D \, d\nu dm = \int 1 \, dm = 1$. Finally, we have $\int 1_D \, d(m\times\nu) = m\times\nu(D)$. To compute this, recall that $\mu\times\nu$ is by construction the restriction of the outer measure defined by the premeasure π from lecture. Consequently,

$$m\times\nu(D) = \inf\left\{\sum_{n=1}^\infty m(A_n)\nu(B_n)\colon D\subset\bigcup_{n=1}^\infty A_n\times B_n,\ A_n,B_n\in\mathcal{B}_{[0,1]}\right\}.$$

Fix some cover $D\subset\bigcup_{n=1}^\infty A_n\times B_n$. Then $[0,1]\subset\bigcup_{n=1}^\infty A_n$, and so $\sum_{n=1}^\infty m(A_n)\geq 1$. Let $I=\{n\in\mathbb{N}\colon m(A_n)=0\}$ and $E:=[0,1]\setminus\bigcup_{n\in I}A_n$. Then $m(E)=1$ and hence E is necessarily uncountable. We also have

$$\{(t,t)\in[0,1]^2\colon t\in E\}\subset\bigcup_{n\notin I}A_n\times B_n,$$

and so $E\subset\bigcup_{n\notin I}B_n$. Since E is uncountable, at least one B_n, $n\notin I$, must be uncountable. Since $m(A_n)>0$ by definition of I, it follows that $m(A_n)\nu(B_n)=\infty$. Since this was an arbitrary cover of D by measurable rectangles, it follows that $m\times\nu(D)=\infty$. □

2. (a) Consider $F(x,t):=(f(x),t)$, which is $(\mathcal{M}\otimes\mathcal{B}_\mathbb{R},\mathcal{B}_{\overline{\mathbb{R}}}\otimes\mathcal{B}_\mathbb{R})$-measurable by Proposition 2.4, and $[0,\infty]\times[0,\infty)\ni(s,t)\mapsto s-t$ which is continuous and hence $\mathcal{B}_{\overline{\mathbb{R}}}\otimes\mathcal{B}_\mathbb{R}$-measurable. Thus $(x,t)\mapsto f(x)-t$, the composition of these maps, is $\mathcal{M}\otimes\mathcal{B}_\mathbb{R}$ measurable, and G_f is the preimage of $[0,\infty]$ under this map. Note that for $x\in X$, $(G_f)_x=[0,f(x)]$ and so $m((G_f)_x)=f(x)$. Using Theorem 2.35 we have

$$\mu\times m(G_f)\int_X m((G_f)_x)\,d\mu(x)=\int_X f(x)\,d\mu(x)=\int_X f\,d\mu.$$

□

(b) For $t\in[0,\infty)$, observe that $(G_f)^t=\{x\in X\colon(x,t)\in G_f\}=\{x\in X\colon f(x)\geq t\}$. So using the other part of Theorem 2.35, we have

$$\int_X f\,d\mu=\mu\times m(G_f)=\int_{[0,\infty)}\mu((G_f)^t)\,dm(t)=\int_{[0,\infty)}\mu(\{x\in X\colon f(x)\geq t\})\,dm(t).$$

□

(c) Suppose $(f_n)_{n\in\mathbb{N}}\subset L^+(X,\mu)$ increase pointwise to $f\in L^+(X,\mu)$. Then $G_{f_n}\subset G_{f_{n+1}}$ for each $n\in\mathbb{N}$ and

$$\bigcup_{n=1}^\infty G_{f_n}=G_f.$$

Thus by continuity from below and part (a) we have

$$\int_X f\,d\mu=\mu\times m(G_f)=\lim_{n\to\infty}\mu\otimes m(G_{f_n})=\lim_{n\to\infty}\int_X f_n\,d\mu.$$

□

3. Consider $F(x,t):=1_{(x,1)}(t)\frac{1}{t}f(t)$. Note that $1_{(x,1)}(t)=1_{(0,t)}(x)$ for $x,t\in(0,1)$. So by Tonelli's theorem we have

$$\int_{(0,1)^2}|F(x,t)|d(m\times m)=\int_0^1\int_0^1 1_{(0,t)}(x)\frac{1}{t}|f(t)|\,dm(x)dm(t)=\int_0^1|f(t)|\,dm(t)<\infty.$$

So $F\in L^1((0,1)^2,m\times m)$ and therefore $g(x)=\int_{(0,1)}F_x\,dm\in L^1((0,1),m)$ by Fubini's theorem. Fubini's theorem also implies

$$\int_{(0,1)}g(x)=\int_{(0,1)^2}F\,d(m\times m)=\int_0^1\int_0^1 1_{(0,t)}(x)\frac{1}{t}f(t)\,dm(x)dm(t),$$

which equals $\int_0^1 f(t)\,dm(t)$ by the same computation as above. □

4. (a) Suppose E is ν-null. Recall that $\nu^+(E) = \nu(E \cap P)$ and $\nu^-(E) = -\nu(E \cap N)$ where $X = P \cup N$ is a partition of X into positive and negative sets for ν. Since $E \cap P \subset E$, we have $\nu(E \cap P) = 0$ and similarly $\nu(E \cap N) = 0$. Hence

$$|\nu|(E) = \nu^+(E) + \nu^-(E) = \nu(E \cap P) + \nu(E \cap N) = 0.$$

Conversely, suppose $|\nu|(E) = 0$. If there exists a measurable subset $F \subset E$ with $\nu(F) \neq 0$, then without loss of generality we may assume $\nu(F) > 0$. Consequently, $\nu^+(F) = \nu(F) + \nu^-(F) > 0$ and hence $|\nu|(F) \geq \nu^+(F) > 0$. But since $|\nu|$ is a measure, this contradicts monotonicity. Thus we must have $\nu(F) = 0$ for all measurable subsets $F \subset E$, and hence E is ν-null. □

(b) $(i) \Rightarrow (ii)$: Suppose $\nu \perp \mu$ and let $X = E \cup F$ be a partition such that E is μ-null and F is ν-null. By part (a), F satisfies $|\nu|(F) = 0$ and hence is $|\nu|$-null. Consequently $|\nu| \perp \mu$.

$(ii) \Rightarrow (iii)$: Let $X = E \cup F$ be a partition such that E is μ-null and F is $|\nu|$-null. Then $0 = |\nu|(F) = \nu^+(F) + \nu^-(F)$, and so F is ν^+-null so that $\nu^+ \perp \mu$, and F is ν^--null so that $\nu^- \perp \mu$.

$(iii) \Rightarrow (i)$: Let $X = E_+ \cup F_+ = E_- \cup F_-$ be partitions such that $E_\pm$ are μ-null and $F_\pm$ is $\nu^\pm$-null. Then $E := E_+ \cup E_-$ is also μ-null and $F := X \setminus E = F_+ \cap F_-$ which is both ν^+-null and ν^--null. Thus F is ν-null: for any measurable subset $A \subset F$ we have $\nu(A) = \nu^+(A) - \nu^-(A) = 0 - 0 = 0$. Hence $\nu \perp \mu$. □

5. (a) First observe that for $A \in \mathcal{M}$ and $B \in \mathcal{N}$ that $1_A(x)1_B(y) = 1_{A\times B}(x, y)$. Thus products of measurable simple functions on $(X, \mathcal{M})$ and $(Y, \mathcal{N})$ yield measurable simple functions on $(X \times Y, \mathcal{M} \otimes \mathcal{N})$. So, letting $(\phi_n)_{n\in\mathbb{N}}$ and $(\psi_n)_{n\in\mathbb{N}}$ be sequences of simple functions converging pointwise to f and g, respectively, we have

$$h(x, y) = \lim_{n\to\infty} \phi_n(x)\psi_n(y).$$

Since h is a pointwise limit of measurable functions, it is measurable by Proposition 2.7 from lecture. □

(b) We first suppose $f = \sum_{i=1}^m \alpha_i 1_{A_i}$ and $g = \sum_{j=1}^n \beta_j 1_{B_j}$ are non-negative simple functions. Then

$$h = \sum_{i=1}^m \sum_{j=1}^n \alpha_i\beta_j 1_{A_i\times B_j},$$

and

$$\begin{aligned}\int_{X\times Y} h \, d(\mu \times \nu) &= \sum_{i=1}^m \sum_{j=1}^n \alpha_i\beta_j(\mu \times \nu)(A_i \times B_j) \\ &= \sum_{i=1}^m \sum_{j=1}^n \alpha_i\beta_j\mu(A_i)\nu(B_j) \\ &= \left(\sum_{i=1}^m \alpha_i\mu(A_i)\right)\left(\sum_{j=1}^n \beta_j\nu(B_j)\right) = \left(\int_X f \, d\mu\right)\left(\int_Y g \, d\nu\right).\end{aligned}$$

Note that this quantity is finite since $f \in L^1(X, \mu)$ and $g \in L^1(Y, \nu)$, and so $h \in L^1(X \times Y, \mu \times \nu)$. Now let $f \in L^1(X, \mu)$ and $g \in L^1(Y, \nu)$ be arbitrary. Using Theorem 2.10 we can find sequences of simple functions $(\phi_n)_{n\in\mathbb{N}}, (\psi_n)_{n\in\mathbb{N}}$ converging pointwise to f and g, respectively, and satisfying $|\phi_1| \leq |\phi_2| \leq \cdots \leq |f|$ and $|\psi_1| \leq |\psi_2| \leq \cdots \leq |g|$. Then $|\phi_1(x)\psi_1(y)| \leq |\phi_2(x)\psi_2(y)| \leq \cdots \leq |h(x, y)|$ and $\phi_n(x)\psi_n(y) \to h(x, y)$ for all $(x, y) \in X \times Y$. Thus the monotone convergence theorem and the first part of the proof gives us

$$\begin{aligned}\int_{X\times Y} |h| \, d(\mu \times \nu) &= \lim_{n\to\infty} \int_{X\times Y} |\phi_n\psi_n| \, d(\mu \times \nu) \\ &= \lim_{n\to\infty} \left(\int_X |\phi_n| \, d\mu\right)\left(\int_Y |\psi_n| \, d\nu\right) = \left(\int_X |f| \, d\mu\right)\left(\int_Y |g| \, d\nu\right) < \infty.\end{aligned}$$

So $h \in L^1(X \times Y, \mu \times \nu)$. In particular, this implies $\phi_n \psi_n \in L^1(X \times Y, \mu \times \nu)$ and the same computation as in the first part of the proof yields

$$\int_{X \times Y} \phi_n \psi_n \, d(\mu \times \nu) = \left(\int_X \phi_n \, d\mu \right) \left(\int_Y \psi_n \, d\nu \right).$$

Finally, using the dominated convergence theorem three times (with dominating functions $|h|$, $|f|$, and $|g|$ respectively) gives

$$\begin{aligned} \int_{X \times Y} h \, d(\mu \times \nu) &= \lim_{n \to \infty} \int_{X \times Y} \phi_n \psi_n \, d(\mu \times \nu) \\ &= \lim_{n \to \infty} \left(\int_X \phi_n \, d\mu \right) \left(\int_Y \psi_n \, d\nu \right) = \left(\int_X f \, d\mu \right) \left(\int_Y g \, d\nu \right), \end{aligned}$$

as claimed. □