Exercises: (Sections 1.5, 2.1)

1. Let μ be a Lebesgue–Stieltjes measure with domain $\mathcal{M}$, and let $E \in \mathcal{M}$ with $\mu(E) < \infty$. Show that for any $\epsilon > 0$ there exists a finite union of open intervals A so that $\mu(E \Delta A) < \epsilon$.

2. Let $N \subset [0, 1)$ be the non-measurable set constructed in Section 1.1 and denote

$$N_q := \{x + q : x \in N \cap [0, 1 - q)\} \cup \{x + q - 1 : x \in N \cap [1 - q, 1)\},$$

for all $q \in \mathbb{Q} \cap [0, 1)$. Let $E \subset \mathbb{R}$ be Lebesgue measurable.

   (a) Show that $E \subset N$ implies $m(E) = 0$.

   (b) Show that $m(E) > 0$ implies E contains a subset that is not Lebesgue measurable. [**Hint:** for $E' \subset [0, 1)$ one has $E' = \bigcup_q E' \cap N_q$.]

3. Let $E \subset \mathbb{R}$ be Lebesgue measurable with $m(E) > 0$.

   (a) Show that for any $0 < \alpha < 1$ there exists an interval I satisfying $m(E \cap I) > \alpha m(I)$.

   (b) Show that the set
$$E - E := \{x - y : x, y \in E\}$$
contains an open interval centered at 0.

4. Let $(X, \mathcal{M})$ be a measurable space and $f, g : X \to \overline{\mathbb{R}}$.

   (a) Show that f is $\mathcal{M}$-measurable if and only if $f^{-1}(\{\infty\}), f^{-1}(\{-\infty\}), f^{-1}(B) \in \mathcal{M}$ for all Borel sets $B \subset \mathbb{R}$.

   (b) Show that f is $\mathcal{M}$-measurable if and only if $f^{-1}((q, \infty]) \in \mathcal{M}$ for all $q \in \mathbb{Q}$.

   (c) Suppose f, g are $\mathcal{M}$-measurable. Fix $a \in \overline{\mathbb{R}}$ and define $h : X \to \overline{\mathbb{R}}$ by
$$h(x) = \begin{cases} a & \text{if } f(x) = -g(x) = \pm\infty \\ f(x) + g(x) & \text{otherwise} \end{cases}.$$
Show that h is $\mathcal{M}$-measurable.

5. Let $(X, \mathcal{M})$ be a measurable space, and let $f_n : X \to \overline{\mathbb{R}}$ be $\mathcal{M}$-measurable for each $n \in \mathbb{N}$. Show that $\{x \in X : \lim_{n\to\infty} f_n(x) \text{ exists}\} \in \mathcal{M}$.

Solutions:

1. By Lemma 1.17, we can find a cover $\{(a_n, b_n) : n \in \mathbb{N}\}$ of E so that

$$\sum_{n=1}^{\infty} \mu((a_n, b_n)) \leq \mu(A) + \frac{\epsilon}{2}.$$

Since the right side is finite, the sum converges and therefore there exists $N \in \mathbb{N}$ so that

$$\sum_{n=N+1}^{\infty} \mu((a_n, b_n)) < \frac{\epsilon}{2}.$$

Let $A := (a_1, b_1) \cup \cdots \cup (a_N, b_N)$ and $B := \bigcup_n (a_n, b_n)$. Then

$$E \Delta A = (E \setminus A) \cup (A \setminus E) \subset (B \setminus A) \cup (B \setminus E).$$

Since $B \setminus A \subset \bigcup_{n=N+1}^{\infty}(a_n, b_n)$, we have by countable subadditivity

$$m(B \setminus A) \leq \sum_{n=N+1}^{\infty} m((a_n, b_n)) < \frac{\epsilon}{2}.$$

Similarly,

$$m(B \setminus E) = \mu(B) - \mu(E) \leq \sum_{n=1}^{\infty} m((a_n, b_n)) - m(E) < \frac{\epsilon}{2}.$$

Thus $m(E \Delta A) < m(B \setminus A) + m(B \setminus E) < \epsilon$. □

2. (a) For each $q \in \mathbb{Q} \cap [0, 1)$, define

$$E_q := \{x + q : x \in E \cap [0, 1-q)\} \cup \{x + q - 1 : x \in E \cap [1-q, 1)\} \subset [0, 1).$$

Note that $E_q = ([E \cap [0, 1-q)] + q) \cup ([E \cap [1-q, 1)] + q - 1)$, and so it is Lebesgue measurable by Theorem 1.21 from lecture with the same measure as E:

$$m(E_q) = m([E\cap[0, 1-q)]+q)+m([E\cap[1-q, 1)]+q-1) = m(E\cap[0, 1-q))+m(E\cap[1-q, 1)) = m(E).$$

Additionally, we have $E_q \subset N_q$ since $E \subset N$, and hence the E_q's are disjoint. Thus

$$1 = m([0, 1)) \geq m\left(\bigcup_{q\in\mathbb{Q}\cap[0,1)} E_q\right) = \sum_{q\in\mathbb{Q}\cap[0,1)} m(E_q) = \sum_{q\in\mathbb{Q}\cap[0,1)} m(E).$$

We therefore must have $m(E) = 0$. □

(b) Since

$$m(E) = \sum_{n\in\mathbb{Z}} m(E \cap [n, n+1)),$$

there exists $n \in \mathbb{Z}$ so that $A := E \cap [n, n+1) - n \subset [0, 1)$ has positive measure. Observe that

$$A = \bigcup_{q\in\mathbb{Q}\cap[0,1)} A \cap N_q.$$

If $A \cap N_q$ is measurable, then the translation invariance of m we have

$$\begin{aligned} m(A \cap N_q) &= m(A \cap [N \cap [0, 1-q) + q]) + m(A \cap [N \cap [1-q, 1) + q - 1)) \\ &= m((A - q) \cap N \cap [0, 1-q)) + m((A - q + 1) \cap N \cap [1-q, 1)) = 0, \end{aligned}$$

where we have used part (a). Thus if $A \cap N_q$ is measurable for all $q \in \mathbb{Q} \cap [0, 1)$, we obtain the contradiction

$$0 < m(A) = \sum_{q\in\mathbb{Q}\cap[0,1)} m(A \cap N_q) = 0.$$

Therefore $A \cap N_q$ must be non-measurable for $q \in \mathbb{Q} \cap [0, 1)$, and therefore $(A \cap N_q) + n$ is a non-measurable subset of E by Theorem 1.21. □

3. (a) Suppose towards a contradiction that there exists $0 < \alpha_0 < 1$ so that $m(E \cap I) \leq \alpha_0 m(I)$ for all intervals I. Then for any cover $\{(a_n, b_n) : n \in \mathbb{N}\}$ of A we have

$$m(A) \leq \sum_{n=1}^{\infty} m(A \cap (a_n, b_n)) \leq \sum_{n=1}^{\infty} \alpha_0 m((a_n, b_n)).$$

Using Lemma 1.17, we can find a cover of A satisfying $\sum_n m((a_n, b_n)) \leq m(A) + \epsilon$ for $\epsilon < \left(\frac{1}{\alpha_0} - 1\right) m(A)$. In this case, the above inequality implies

$$m(A) \leq \alpha_0(m(A) + \epsilon) < \alpha_0 m(A) + (1 - \alpha_0)m(A) = m(A),$$

a contradiction. □

(b) Suppose towards a contradiction that $E-E$ does not contain any intervals centered at 0. This means for all $\delta>0$, there exists $t\in(-\delta,\delta)\setminus(E-E)$. By definition of $E-E$, $t\notin E-E$ implies $E\cap(E+t)=\emptyset$. In particular, for any interval I, we have $(E\cap I)\cap(E\cap I+t)=\emptyset$
For $\alpha=\frac{2}{3}$, let I be the open interval from part (a) satisfying $m(E\cap I)<\frac{2}{3}m(I)$. If $E\cap I$ and $E\cap I+t$ are disjoint, we have

$$m((E\cap I)\cup(E\cap I+t))=m(E\cap I)+m(E\cap I+t)=2m(E\cap I)>\frac{4}{3}m(I).$$

On the other hand, $(E\cap I)\cup(E\cap I+t)\subset I\cup(I+t)$ and for sufficiently small t we have $m(I\cup(I+t))<\frac{4}{3}m(I)$, a contradiction. □

4. (a) $(\Rightarrow)$: Suppose f is $\mathcal{M}$-measurable. Then $\{\infty\},\{-\infty\}\in\mathcal{B}_{\overline{\mathbb{R}}}$ since their intersection with $\mathbb{R}$ is the Borel set $\emptyset$, similarly for a Borel set $B\subset\mathbb{R}$. Hence their preimages under f belong to $\mathcal{M}$.
$(\Leftarrow)$: Let $E\in\mathcal{B}_{\overline{\mathbb{R}}}$. Then $B:=E\cap\mathbb{R}$ is Borel, and E equals one of the following: B, $B\cup\{\infty\}$, $\{-\infty\}\cup B$, or $\{-\infty\}\cup B\cup\{\infty\}$. In each case, the preimage under f is some union of $f^{-1}(B)$, $f^{-1}(\{\infty\})$, and $f^{-1}(\{-\infty\})$, and hence belongs to $\mathcal{M}$. □

(b) $(\Rightarrow)$: Each $(q,\infty]\in\mathcal{B}_{\overline{\mathbb{R}}}$, and so $f^{-1}((q,\infty])\in\mathcal{M}$ by definition of $\mathcal{M}$-measurable.
$(\Leftarrow)$: Let $\mathcal{N}:=\{E\subset\overline{\mathbb{R}}\colon f^{-1}(E)\subset\mathcal{M}\}$, which we note is a σ-algebra since $\mathcal{M}$ is. By assumption $\mathcal{N}$ contains all rays of the form $(q,\infty]$ for $q\in\mathbb{Q}$. It follows that is also contains $[-\infty,q]$ for all $q\in\mathbb{Q}$ and therefore contains

$$\{\infty\}=\bigcap_{n=1}^{\infty}(n,\infty]\qquad\text{and}\qquad\{-\infty\}=\bigcap_{n=1}^{\infty}[-\infty,-n].$$

Now, $(q,\infty)=(q,\infty]\setminus\{-\infty\}\in\mathcal{N}$ for all $q\in\mathbb{Q}$. Also, for any $a\in\mathbb{R}$, we can find a sequence of rational $(q_n)_{n\in\mathbb{N}}$ descending to a, and thus

$$(a,\infty)=\bigcup_{n=1}^{\infty}(q_n,\infty)\in\mathcal{N}.$$

Since these rays generate $\mathcal{B}_{\mathbb{R}}$ by Proposition 1.2, we have $\mathcal{B}_{\mathbb{R}}\subset\mathcal{N}$ by Lemma 1.1. Appealing to part (a) we see that f is $\mathcal{M}$-measurable.

(c) Observe that

$$h^{-1}(\{\infty\})=\begin{cases}\left[f^{-1}(\{\infty\})\setminus g^{-1}(\{-\infty\})\right]\cup\left[g^{-1}(\{\infty\})\setminus f^{-1}(\{-\infty\})\right] & \text{if } a\neq\infty\\ f^{-1}(\{\infty\})\cup g^{-1}(\{\infty\}) & \text{if } a=\infty\end{cases}\in\mathcal{M}.$$

Similarly, $h^{-1}(\{-\infty\})\in\mathcal{M}$. Define $\phi\colon\mathbb{R}^2\to\mathbb{R}$ by $\phi(x,y)=x+y$. Then ϕ is continuous and hence Borel measurable by Corollary 2.2, and so $\phi^{-1}(B)\in\mathcal{B}_{\mathbb{R}}\otimes\mathcal{B}_{\mathbb{R}}$ for all Borel $B\subset\mathbb{R}$. Since $H(x):=(f(x),g(x))$ is $(\mathcal{M},\mathcal{B}_{\overline{\mathbb{R}}}\otimes\mathcal{B}_{\overline{\mathbb{R}}})$-measurable by Proposition 2.4, it follows that $(\phi\circ H)^{-1}(B)=H^{-1}(\phi^{-1}(B))\in\mathcal{M}$ for all $B\subset\mathbb{R}$. Since

$$h^{-1}(B)=\begin{cases}(\phi\circ H)^{-1}(B) & \text{if } a\notin B\\ (\phi\circ H)^{-1}(B)\cup[f^{-1}(\{\infty\})\cap g^{-1}(\{-\infty\})]\cup[f^{-1}(\{-\infty\})\cap g^{-1}(\{\infty\})] & \text{if } a\in B\end{cases},$$

we see that $h^{-1}(B)\in\mathcal{M}$. Thus h is $\mathcal{M}$-measurable by part (a). □

5. Define

$$g(x):=\liminf_{n\to\infty}f_n(x)\qquad\text{and}\qquad h(x):=\limsup_{n\to\infty}f_n(x).$$

Then g and h are $\mathcal{M}$-measurable by Proposition 2.7 and $g(x)\leq h(x)$ for all $x\in X$ with equality if and only if the limit exists. Now, $-g(x)$ is $\mathcal{M}$-measurable since $t\mapsto -t$ is continuous and hence Borel measurable, and consequently

$$k(x)=\begin{cases}0 & \text{if } h(x)=g(x)=\pm\infty\\ h(x)+(-g(x)) & \text{otherwise}\end{cases}$$

is $\mathcal{M}$-measurable by Exercise 4.(c). Thus $k^{-1}(\{0\}) \in \mathcal{M}$, and this set is precisely where $g(x) = h(x)$, and hence the set of points where the limit exists. □