Exercises:

§19, 20

1. Let $(a_n)_{n\in\mathbb{N}}, (b_n)_{\in\mathbb{N}} \in \mathbb{R}^{\mathbb{N}}$ with $a_n > 0$ for all $n \in \mathbb{N}$. Define a map $h\colon \mathbb{R}^{\mathbb{N}} \to \mathbb{R}^{\mathbb{N}}$ by

$$h((x_n)_{n\in\mathbb{N}}) = (a_n x_n + b_n)_{n\in\mathbb{N}}.$$

   (a) Show that h is a bijection.

   (b) Show that if $\mathbb{R}^{\mathbb{N}}$ is given the product topology, then h is a homeomorphism.

   (c) Prove whether or not h is a homeomorphism when $\mathbb{R}^{\mathbb{N}}$ is given the box topology.

2. For $\mathbf{x} = (x_1, \ldots, x_n), \mathbf{y} = (y_1, \ldots, y_n) \in \mathbb{R}^n$, define

$$d_1(\mathbf{x}, \mathbf{y}) := \sum_{j=1}^{n} |x_j - y_j|.$$

   (a) Show that d_1 is a metric on $\mathbb{R}^n$.

   (b) Show that the topology induced by d_1 equals the product topology on $\mathbb{R}^n$.

   (c) For $n = 2$ and $\mathbf{0} = (0, 0) \in \mathbb{R}^2$, draw a picture of $B_{d_1}(\mathbf{0}, 1)$.

3. Let X be a metric space with metric d. For $x \in X$ and $\epsilon > 0$, show that $\{y \in X \mid d(x, y) \le \epsilon\}$ is a closed set.

4. Let X be a metric space with metric d. Show that $d\colon X \times X \to \mathbb{R}$ is continuous.

5. For $\mathbf{x} = (x_1, \ldots, x_n), \mathbf{y} = (y_1, \ldots, y_n) \in \mathbb{R}^n$ and $c \in \mathbb{R}$ define

$$\begin{aligned}
\mathbf{x} + \mathbf{y} &:= (x_1 + y_1, \ldots, x_n + y_n),\\
c\mathbf{x} &:= (cx_1, \ldots, cx_n),\\
\mathbf{x} \cdot \mathbf{y} &:= x_1 y_1 + \cdots + x_n y_n,\\
\|\mathbf{x}\| &:= (x_1^2 + \cdots + x_n^2)^{1/2}.
\end{aligned}$$

   (a) For $\mathbf{x}, \mathbf{y}, \mathbf{z} \in \mathbb{R}^n$ and $a, b \in \mathbb{R}$, prove the following formulas

$$\begin{gathered}
\|\mathbf{x}\|^2 = \mathbf{x} \cdot \mathbf{x}\\
(a\mathbf{x}) \cdot (b\mathbf{y}) = (ab)(\mathbf{x} \cdot \mathbf{y})\\
\mathbf{x} \cdot \mathbf{y} = \mathbf{y} \cdot \mathbf{x}\\
\mathbf{x} \cdot (\mathbf{y} + \mathbf{z}) = \mathbf{x} \cdot \mathbf{y} + \mathbf{x} \cdot \mathbf{z}
\end{gathered}$$

   (b) Show that $|\mathbf{x} \cdot \mathbf{y}| \le \|\mathbf{x}\|\|\mathbf{y}\|$. [**Hint:** for $\mathbf{x}, \mathbf{y} \neq 0$ let $a = \frac{1}{\|\mathbf{x}\|}$ and $b = \frac{1}{\|\mathbf{y}\|}$ and use the fact that $\|a\mathbf{x} \pm b\mathbf{y}\|^2 \ge 0$.]

   (c) Show that $\|\mathbf{x} + \mathbf{y}\| \le \|\mathbf{x}\| + \|\mathbf{y}\|$.

   (d) Prove that the euclidean metric $d(\mathbf{x}, \mathbf{y}) := \|\mathbf{x} - \mathbf{y}\|$ is indeed a metric.

6*. For $\mathbf{x} = (x_1, \ldots, x_n) \in \mathbb{R}^n$ and $1 \le p < \infty$, define

$$\|\mathbf{x}\|_p := (|x_1|^p + \cdots + |x_n|^p)^{1/p},$$

and for $p = \infty$ define

$$\|\mathbf{x}\|_\infty := \max\{|x_1|, \ldots, |x_n|\}.$$

In this exercise you will show $d_p(\mathbf{x}, \mathbf{y}) := \|\mathbf{x} - \mathbf{y}\|_p$ defines a metric for each $1 \le p \le \infty$. Observe that $p = 1, 2, \infty$ yield the metric from Exercise 2, the euclidean metric, and the square metric, respectively.

(a) For $1 < p < \infty$, show that if $q > 0$ satisfies $\frac{1}{p} + \frac{1}{q} = 1$ then $1 < q < \infty$. We call q the **conjugate exponent** to p.

(b) For $a, b \geq 0$ and $0 < \lambda < 1$, show that $a^\lambda b^{1-\lambda} \leq \lambda a + (1-\lambda)b$.

(c) Prove **Hölder's Inequality**: for $1 < p < \infty$ with conjugate exponent q and $\mathbf{x}, \mathbf{y} \in \mathbb{R}^n$ show that

$$|x_1 y_1| + \cdots + |x_n y_n| \leq \|\mathbf{x}\|_p \|\mathbf{y}\|_q.$$

(d) Prove **Minkowski's Inequality**: for $1 < p < \infty$ and $\mathbf{x}, \mathbf{y} \in \mathbb{R}^n$ show that

$$\|\mathbf{x} + \mathbf{y}\|_p \leq \|\mathbf{x}\|_p + \|\mathbf{y}\|_p.$$

[**Hint:** use $|x_j + y_j|^p \leq (|x_j| + |y_j|)|x_j + y_j|^{p-1}$.]

(e) Show that d_p is a metric for $1 < p < \infty$.

(f) Show that the topology induced by d_p equals the product topology on $\mathbb{R}^n$ for $1 < p < \infty$, where $\mathbb{R}$ has the standard topology. [**Hint:** show that $\|\mathbf{x}\|_\infty \leq \|\mathbf{x}\|_p \leq \|\mathbf{x}\|_1$.]

Solutions:

1. (a) Define $g\colon \mathbb{R}^\mathbb{N} \to \mathbb{R}^\mathbb{N}$ by

$$g((x_n)_{n\in\mathbb{N}}) = (\frac{1}{a_n} x_n - \frac{b_n}{a_n})_{n\in N},$$

which is well-defined since $a_n > 0$. Observe that

$$a_n(\frac{1}{a_n} x_n - \frac{b_n}{a_n}) + b_n = x_n - b_n + b_n = x_n$$

and

$$\frac{1}{a_n}(a_n x_n + b_n) - \frac{b_n}{a_n} = x_n + \frac{b_n}{a_n} - \frac{b_n}{a_n} = x_n.$$

Thus $g \circ h((x_n)_{n\in\mathbb{N}}) = h \circ g((x_n)_{n\in\mathbb{N}}) = (x_n)_{n\in\mathbb{N}}$. Thus $g = h^{-1}$ and in particular h is a bijection. $\square$

(b) Let $\prod_{n\in\mathbb{N}} U_n$ be a basis set for the product topology: $U_n \subset \mathbb{R}$ is open for all $n \in \mathbb{N}$ and $U_n = \mathbb{R}$ for all but finitely many $n \in \mathbb{N}$. Then

$$g^{-1}\left(\prod_{n\in\mathbb{N}} U_n\right) = h\left(\prod_{n\in\mathbb{N}} U_n\right) = \prod_{n\in\mathbb{N}} V_n,$$

where

$$V_n := \{a_n x + b_n \mid x \in U_n\}.$$

We claim that V_n is open in $\mathbb{R}$. Indeed, given for $y = a_n x + b_n \in V_n$, there exists $\epsilon > 0$ so that $(x - \epsilon, x + \epsilon) \subset U_n$ since U_n is open. We claim that

$$(y - a_n\epsilon, y + a_n\epsilon) \subset V_n,$$

which implies V_n is open. Indeed, for z in the above interval we have

$$|\left(\frac{1}{a_n} z - \frac{b_n}{a_n}\right) - x| = |\left(\frac{1}{a_n} z - \frac{b_n}{a_n}\right) - \left(\frac{1}{a_n} y - \frac{b_n}{a_n}\right)| = \frac{1}{a_n}|z - y| < \frac{1}{a_n} a_n \epsilon = \epsilon.$$

Thus $\frac{1}{a_n} z - \frac{b_n}{a_n} \in U_n$ and therefore $V_n \ni a_n(\frac{1}{a_n} z - \frac{b_n}{a_n}) + b_n = z$. We also note that if $U_n = \mathbb{R}$, then $V_n = \mathbb{R}$ since for any $y \in \mathbb{R}$ we have $y = a_n(\frac{1}{a_n} y - \frac{b_n}{a_n}) + b_n$. Thus $\prod_{n\in\mathbb{N}} V_n$ is open in the product topology and therefore g is continuous. The same argument where a_n and b_n are swapped with $\frac{1}{a_n}$ and $-\frac{b_n}{a_n}$ shows that h is continuous. Thus h is a homeomorphism. $\square$

(c) When $\mathbb{R}^{\mathbb{N}}$ has the box topology h is still a homeomorphism. The exact same proof as in the previous part works with U_n no longer required to be equal to $\mathbb{R}$ for all but finitely many $n \in \mathbb{N}$.□

2. (a) Clearly $d_1(\mathbf{x}, \mathbf{y}) \geq 0$, and if we have equality then since each $|x_j - y_j|$ is non-negative this must mean we have $|x_j - y_j| = 0$ for each $j = 1, \ldots, n$ and therefore $x_j = y_j$. Hence $\mathbf{x} = \mathbf{y}$. The symmetry $d_1(\mathbf{x}, \mathbf{y}) = d_1(\mathbf{y}, \mathbf{x})$ follows from $|x_j - y_j| = |-(x_j - y_j)| = |-x_j + y_j| = |y_j - x_j|$. Finally, for the triangle inequality we have

$$d_1(\mathbf{x}, \mathbf{z}) = \sum_{j=1}^{n} |x_j - z_j| = \sum_{j=1}^{n} |x_j - y_j + y_j - z_j| \leq \sum_{j=1}^{n} |x_j - y_j| + |y_j - z_j| = d_1(\mathbf{x}, \mathbf{y}) + d_1(\mathbf{y}, \mathbf{z}).$$

Thus d_1 is a metric on $\mathbb{R}^n$. □

(b) Let $U := (a_1, b_1) \times \cdots \times (a_n, b_n)$ be a product of open intervals, which is a basis set for the product topology. For $\mathbf{x} \in U$, define

$$\epsilon := \min\{x_1 - a_1, b_1 - x_1, \ldots, x_n - a_n, b_n - x_n\} > 0.$$

Observe that

$$V := (x_1 - \epsilon, x_1 + \epsilon) \times \cdots (x_n - \epsilon, x_n + \epsilon) \subset U.$$

We claim that $B_{d_1}(\mathbf{x}, \epsilon) \subset\subset U$. Indeed, for $y \in B_{d_1}(\mathbf{x}, \epsilon)$ we have for each $j = 1, \ldots, n$ that

$$|x_j - y_j| \leq \sum_{j=1}^{n} |x_j - y_j| < \epsilon.$$

Thus $y_j \in (x_j - \epsilon, x_j + \epsilon)$, so that $\mathbf{y} \in V \subset U$. Since these ϵ-balls form a basis for the topology induced by d_1, we see by a lemma from §13 that this topology is finer than the product topology. Conversely, given $\epsilon > 0$ and $\mathbf{x} \in \mathbb{R}^n$ we have

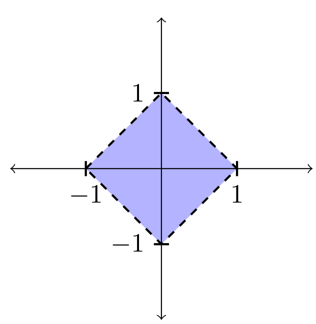
$$(x_1 - \frac{\epsilon}{n}, x_1 + \frac{\epsilon}{n}) \times \cdots (x_n - \frac{\epsilon}{n}, x_n + \frac{\epsilon}{n}) \subset B_{d_1}(\mathbf{x}, \epsilon).$$

Indeed, for $\mathbf{y}$ in the former set we have

$$d_1(\mathbf{x}, \mathbf{y}) = \sum_{j=1}^{n} |x_j - y_j| < \sum_{j=1}^{n} \frac{\epsilon}{n} = \epsilon.$$

Thus $\mathbf{y} \in B_{d_1}(\mathbf{x}, \epsilon)$. By the same lemma in §13, we see that the product topology is finer than the topology induced by d_1, and so these topologies are in fact equal. □

(c) $B_{d_1}(\mathbf{0}, 1)$:



3. Denote $C := \{y \in X \mid d(x, y) \leq \epsilon\}$. If $y \notin C$, then necessarily $d(x, y) > \epsilon$. Set $\delta := d(x, y) - \epsilon > 0$. We claim that $B_d(y, \delta) \subset X \setminus C$. Indeed, suppose, towards a contradiction, that $z \in B_d(y, \delta) \cap C$. Then using the triangle inequality we have

$$d(x, y) \leq d(x, z) + d(z, y) < \epsilon + \delta = \epsilon + d(x, y) - \epsilon = d(x, y),$$

a contradiction. Thus $B_d(y, \delta) \cap C = \emptyset$ and therefore $B_d(y, \delta) \subset X \setminus C$. Since $y \in X \setminus C$ was arbitrary, we see that $X \setminus C$ is open and therefore C is closed. □

4. Let $(a,b) \subset \mathbb{R}$ be an open interval. Then

$$d^{-1}((a,b)) = \{(x,y) \in X \times X \mid a < d(x,y) < b\}.$$

Fix $(x_0,y_0) \in d^{-1}((a,b))$, and set $\epsilon := \min\{d(x_0,y_0)-a, b-d(x_0,y_0)\} > 0$. Note that $U := B_d(x_0,\frac{\epsilon}{2}) \times B_d(y_0,\frac{\epsilon}{2})$ is a neighborhood of (x_0,y_0) in the product topology, and we further claim that it is contained in $d^{-1}((a,b))$. Indeed, if $(x,y) \in U$ then we have $d(x,x_0) < \frac{\epsilon}{2}$ and $d(y,y_0) < \frac{\epsilon}{2}$ and consequently by applying the triangle inequality twice we have

$$\begin{aligned} d(x,y) &\leq d(x,x_0) + d(x_0,y_0) + d(y_0,y) < \frac{\epsilon}{2} + d(x_0,y_0) + \frac{\epsilon}{2} \\ &= \epsilon + d(x_0,y_0) \leq (b - d(x_0,y_0)) + d(x_0,y_0) = b. \end{aligned}$$

Thus $d(x,y) < b$. Next, note that $d(x_0,y) \leq d(x_0,x) + d(x,y)$ implies $d(x,y) \geq d(x_0,y) - d(x_0,x)$, and $d(x_0,y_0) \leq d(x_0,y) + d(y,y_0)$ implies $d(x_0,y) \geq d(x_0,y_0) - d(y_0,y)$. Thus we have

$$\begin{aligned} d(x,y) &\geq d(x_0,y) - d(x_0,x) \geq d(x_0,y_0) - d(y_0,y) - d(x_0,x) \\ &> d(x_0,y_0) - \frac{\epsilon}{2} - \frac{\epsilon}{2} = d(x_0,y_0) - \epsilon \geq d(x_0,y_0) - (d(x_0,y_0) - a) = a. \end{aligned}$$

So $d(x,y) > a$ and therefore $(x,y) \in d^{-1}((a,b))$. Since $(x,y) \in U$ was arbitrary, we see that $U \subset d^{-1}((a,b))$. We have shown that a point $(x_0,y_0) \in d^{-1}((a,b))$ admits a neighborhood contained inside this preimage, and hence the preimage is open as the union of these neighborhoods. Since the open intervals form a basis for the standard topology on $\mathbb{R}$, we obtain that d is continuous. □

5. (a) We have

$$\|\mathbf{x}\|^2 = x_1^2 + \cdots + x_n^2 = x_1x_1 + \cdots + x_nx_n = \mathbf{x} \cdot \mathbf{x}.$$

Also

$$(a\mathbf{x}) \cdot (b\mathbf{y}) = (ax_1)(by_1) + \cdots + (ax_n)(by_n) = (ab)(x_1y_1 + \cdots + x_ny_n) = (ab)(\mathbf{x} \cdot \mathbf{y}).$$

□

(b) We compute

$$\mathbf{x} \cdot \mathbf{y} = x_1y_1 + \cdots + x_ny_n = y_1x_1 + \cdots + y_nx_n = \mathbf{y} \cdot \mathbf{x}.$$

Also

$$\mathbf{x} \cdot (\mathbf{y} + \mathbf{z}) = x_1(y_1+z_1) + \cdots + x_n(y_n+z_n) = (x_1y_1 + \cdots x_ny_n) + (x_1z_1 + \cdots x_nz_n) = \mathbf{x}\cdot\mathbf{y} + \mathbf{x}\cdot\mathbf{z}.$$

□

(c) If either $\|\mathbf{x}\| = 0$ or $\|\mathbf{y}\| = 0$, then all entries of this n-tuple are zero and thus the inequality reduces to $0 \leq 0$. So we may assume both $\|\mathbf{x}\|$ and $\|\mathbf{y}\|$ are non-zero and let $a = \frac{1}{\|\mathbf{x}\|}$ and $b = \frac{1}{\|\mathbf{y}\|}$. Then expanding and using the previous parts yields

$$\begin{aligned} 0 &\leq \|a\mathbf{x} \pm b\mathbf{y}\|^2 = (a\mathbf{x} \pm b\mathbf{y}) \cdot (a\mathbf{x} \pm b\mathbf{y}) = (a\mathbf{x}) \cdot (a\mathbf{x}) \pm (a\mathbf{x}) \cdot (b\mathbf{y}) \pm (b\mathbf{y}) \cdot (a\mathbf{x}) + (b\mathbf{y}) \cdot (b\mathbf{y}) \\ &= a^2\|\mathbf{x}\|^2 \pm (ab)(\mathbf{x}\cdot\mathbf{y}) \pm (ba)(\mathbf{y}\cdot\mathbf{x}) + b^2\|\mathbf{y}\|^2 = 1 \pm 2(ab)(\mathbf{x}\cdot\mathbf{y}) + 1 = 2 \pm 2(ab)(\mathbf{x}\cdot\mathbf{y}). \end{aligned}$$

This is equivalent to $\pm ab(\mathbf{x}\cdot\mathbf{y}) \leq 1$. Multiplying both sides by $\|\mathbf{x}\|\|\mathbf{y}\|$ shows this is further equivalent to $\pm\mathbf{x}\cdot\mathbf{y} \leq \|\mathbf{x}\|\|\mathbf{y}\|$. Hence $|\mathbf{x}\cdot\mathbf{y}| \leq \|\mathbf{x}\|\|\mathbf{y}\|$. □

(d) Using the previous parts we compute

$$\begin{aligned} \|\mathbf{x}+\mathbf{y}\|^2 &= (\mathbf{x}+\mathbf{y}) \cdot (\mathbf{x}+\mathbf{y}) = \mathbf{x}\cdot\mathbf{x} + \mathbf{x}\cdot\mathbf{y} + \mathbf{y}\cdot\mathbf{x} + \mathbf{y}\cdot\mathbf{y} \\ &\leq \|\mathbf{x}\|^2 + \|\mathbf{x}\|\|\mathbf{y}\| + \|\mathbf{y}\|\|\mathbf{x}\| + \|\mathbf{y}\|^2 = (\|\mathbf{x}\| + \|\mathbf{y}\|)^2. \end{aligned}$$

Taking square roots yields the desired inequality. □

(e) We clearly have $d(\mathbf{x},\mathbf{y}) \geq 0$, and equality implies

$$0 = d(\mathbf{x},\mathbf{y})^2 = \sum_{j=1}^n (x_j - y_j)^2.$$

Since each term $(x_j - y_j)^2$ is non-negative, their summing to zero implies they are all zero. Hence $x_j = y_j$ and so $\mathbf{x} = \mathbf{y}$. The symmetry $d(\mathbf{x},\mathbf{y}) = d(\mathbf{y},\mathbf{x})$ follows from $(x_j - y_j)^2 = (-(x_j - y_j))^2 = (-x_j + y_j)^2 = (y_j - x_j)^2$. Finally, for the triangle inequality, using the previous part we have

$$d(\mathbf{x},\mathbf{z}) = \|\mathbf{x}-\mathbf{z}\| = \|\mathbf{x}-\mathbf{y}+\mathbf{y}-\mathbf{x}\| \leq \|\mathbf{x}-\mathbf{y}\| + \|\mathbf{y}-\mathbf{z}\| = d(\mathbf{x},\mathbf{y}) + d(\mathbf{y},\mathbf{z}).$$

Hence d is a metric. □

6. (a) Solving $\frac{1}{p} + \frac{1}{q} = 1$ for q yields

$$q = \frac{p}{p-1}.$$

Since $p > p-1$, we have $q > 1$. Since $p > 1$, we have $q < \infty$. □

(b) Note that $a = 0$ or $b = 0$ makes the inequality trivially true, so we assume $a, b > 0$. Thus the desired inequality is equivalent to the one obtained by dividing both sides by b:

$$a^\lambda b^{-\lambda} \leq \lambda a b^{-1} + 1 - \lambda.$$

Denoting $t := \frac{a}{b}$, this is equivalent to $t^\lambda \leq \lambda t + 1 - \lambda$. Note that $t > 0$, and so it suffices to show $t^\lambda - \lambda t \leq 1 - \lambda$ for all $t > 0$. Set $f(t) := t^\lambda - \lambda t$ and observe that $f'(t) = \lambda t^{\lambda-1} - \lambda$. Thus we have $f'(t) = 0$ if and only if $t = 1$. Noting that $f''(t) = \lambda(\lambda-1)t^{\lambda-2}$ is negative for all $t > 0$ (since $\lambda - 1 < 0$), we see from the second derivative test that f achieves its maximum value at $t = 1$ and that maximum value is $f(1) = 1 - \lambda$. Thus $f(t) \leq 1 - \lambda$ for all $t > 0$ as desired. □

(c) Let $q = \frac{p}{p-1}$ be the conjugate exponent to p. Fix $1 \leq i \leq n$, and consider $a := \frac{|x_i|^p}{\|\mathbf{x}\|_p^p}$, $b := \frac{|y_i|^q}{\|\mathbf{y}\|_q^q}$, and $\lambda := \frac{1}{p}$. Observe that

$$1 - \lambda = \frac{p-1}{p} = \frac{1}{q}.$$

So using part (b) we have

$$\begin{aligned}|x_i y_i| &= \|\mathbf{x}\|_p \|\mathbf{y}\|_q \left(\frac{|x_i|^p}{\|\mathbf{x}\|_p^p}\right)^{1/p} \left(\frac{|y_i|^q}{\|\mathbf{y}\|_q^q}\right)^{1/q} = \|\mathbf{x}\|_p \|\mathbf{y}\|_q a^\lambda b^{1-\lambda} \\ &\leq \|\mathbf{x}\|_p \|\mathbf{y}\|_q (\lambda a + (1-\lambda) b) = \|\mathbf{x}\|_p \|\mathbf{y}\|_q \left(\frac{1}{p}\frac{|x_i|^p}{\|\mathbf{x}\|_p^p} + \frac{1}{q}\frac{|y_i|^q}{\|\mathbf{y}\|_q^q}\right).\end{aligned}$$

Thus

$$\begin{aligned}|x_1 y_1| + \cdots + |x_n y_n| &\leq \|\mathbf{x}\|_p \|\mathbf{y}\|_q \left(\frac{1}{p}\frac{|x_1|^p + \cdots + |x_n|^p}{\|\mathbf{x}\|_p^p} + \frac{1}{q}\frac{|y_1|^q + \cdots + |y_n|^q}{\|\mathbf{y}\|_q^q}\right) \\ &= \|\mathbf{x}\|_p \|\mathbf{y}\|_q \left(\frac{1}{p}\frac{\|\mathbf{x}\|_p^q}{\|\mathbf{x}\|_p^p} + \frac{1}{q}\frac{\|\mathbf{y}\|_q^q}{\|\mathbf{y}\|_q^q}\right) = \|\mathbf{x}\|_p \|\mathbf{y}\|_q \left(\frac{1}{p} + \frac{1}{q}\right) = \|\mathbf{x}\|_p \|\mathbf{y}\|_q.\end{aligned}$$

□

(d) Consider

$$\|\mathbf{x}+\mathbf{y}\|_p^p = \sum_{i=1}^n |x_i + y_i|^p = \sum_{i=1}^n |x_i + y_i||x_i + y_i|^{p-1} \leq \sum_{i=1}^n |x_i||x_i + y_i|^{p-1} + \sum_{i=1}^n |y_i||x_i + y_i|^{p-1}.$$

We apply part (c) to each sum in the last expression to obtain

$$\|\mathbf{x}+\mathbf{y}\|_p^p \leq \|\mathbf{x}\|_p \left(\sum_{i=1}^n |x_i + y_i|^{q(p-1)}\right)^{1/q} + \|\mathbf{y}\|_p \left(\sum_{i=1}^n |x_i + y_i|^{q(p-1)}\right)^{1/q}$$

Using $q = \frac{p}{p-1}$, this becomes

$$\|\mathbf{x} + \mathbf{y}\|_p^p \leq (\|\mathbf{x}\|_p + \|y\|_p)\|\mathbf{x} + \mathbf{y}\|_p^{p-1}.$$

Dividing both sides by $\|\mathbf{x} + \mathbf{y}\|_p^{p-1}$ (an noting that the inequality is trivially true when this is zero) yields the desired inequality. □

(e) The positive, non-degeneracy, and symmetry of d_p are all clear, while the triangle inequality follows from the previous part:

$$d_p(\mathbf{x}, \mathbf{z}) = \|\mathbf{x} - \mathbf{z}\|_p = \|\mathbf{x} - \mathbf{y} + \mathbf{y} - \mathbf{z}\|_p \leq \|\mathbf{x} - \mathbf{y}\|_p + \|\mathbf{y} - \mathbf{z}\|_p = d_p(\mathbf{x}, \mathbf{y}) + d_p(\mathbf{y}, \mathbf{z}).$$

□

(f) Observe that for each $i = 1, \ldots, n$ we have

$$|x_i| = (|x_i|^p)^{1/p} \leq (|x_1|^p + \cdots + |x_n|^p)^{1/p} = \|\mathbf{x}\|_p.$$

Hence

$$\|\mathbf{x}\|_\infty = \max_{1 \leq i \leq n} |x_i| \leq \|\mathbf{x}\|_p.$$

Thus $d_\infty(\mathbf{x}, \mathbf{y}) \leq d_p(\mathbf{x}, \mathbf{y})$ for all $\mathbf{x}, \mathbf{y} \in \mathbb{R}^n$ and it follows that

$$B_{d_p}(\mathbf{x}, \epsilon) \subset B_{d_\infty}(x, \epsilon).$$

This shows the topology induced by d_p is finer than the topology induced by d_∞. Since d_∞ is the square metric, we know from lecture that this induces the product topology on $\mathbb{R}^n$.

We next observe that $\|\mathbf{x}\|_p \leq \|\mathbf{x}\_1$ follows from taking the pth power of each side:

$$\|\mathbf{x}_p\|^p = |x_1|^p + \cdots + |x_n|^p \leq (|x_1| + \cdots + |x_n|)^p.$$

Consequently, by the same argument as above we have that the topology induced by d_1 is finer than the topology induced by d_p. From Exercise 2, we know that the former topology is nothing more than the product topology. Hence the topology induced by d_p is the product topology. □