Exercises:

§17 and Nets

1. Let $\mathcal{C}$ be a collection of subsets of X. Assume that $\emptyset, X \in \mathcal{C}$ and that finite unions and arbitrary intersections of sets in $\mathcal{C}$ are in $\mathcal{C}$. Show that the collection $\mathcal{T} := \{X \setminus C \mid C \in \mathcal{C}\}$ is a topology on X and that the collection of closed sets in this topology is $\mathcal{C}$.

2. Let X be a topological space with subset $S \subset X$. Recall that $\overline{S}$ denotes the closure of S and S° denotes the interior of S. We will also denote by $S^c := X \setminus S$ the complement of S.

   (a) Show that $\overline{S} = ((S^c)^\circ)^c$ for all $S \subset X$.

   (b) Show that $S^\circ = (\overline{S^c})^c$ for all $S \subset X$.

3. Let X be a topological space and let $A, B \subset X$ be subsets.

   (a) Show that $A \subset B$ implies $\overline{A} \subset \overline{B}$ and $A^\circ \subset B^\circ$.

   (b) For $A, B \subset X$, show that $\overline{A \cup B} = \overline{A} \cup \overline{B}$.

   (c) For $A, B \subset X$, show that $(A \cap B)^\circ = A^\circ \cap B^\circ$.

   (d) Let $\mathbb{R}$ have the standard topology. Find examples of subsets $A, B \subset \mathbb{R}$ such that $\overline{A \cap B} \neq \overline{A} \cap \overline{B}$ and $(A \cup B)^\circ \neq A^\circ \cup B^\circ$.

4. Let X be a topological space. We say a subset $S \subset X$ is **dense** in X if for every $x \in X$ and every neighborhood U of x one has $U \cap S \neq \emptyset$. Show the following are equivalent:

   (i) S is dense in X.

   (ii) $(S^c)^\circ = \emptyset$.

   (iii) $\overline{S} = X$.

5. Let $(a_n)_{n\in\mathbb{N}}$ be a sequence of real numbers.

   (a) Show that the collection $\mathcal{F}$ of finite subsets of $\mathbb{N}$ ordered by inclusion is a directed set.

   (b) Show the following are equivalent:

      (i) The net
      $$\left(\sum_{n\in F} a_n\right)_{F\in\mathcal{F}}$$
      converges in $\mathbb{R}$ (with the standard topology).

      (ii) For any bijection $\sigma\colon \mathbb{N} \to \mathbb{N}$, the series $\sum_{n=1}^\infty a_{\sigma(n)}$ converges.

      (iii) The series $\sum_{n=1}^\infty |a_n|$ converges.

6*. Let X be a topological space. Define functions $C, K\colon \mathcal{P}(X) \to \mathcal{P}(X)$ by $C(A) := A^c$ and $K(A) = \overline{A}$.

   (a) Given a fixed $A \subset X$, show that successively applying C and K to A yields at most fourteen distinct sets.

   (b) Find a subset of $\mathbb{R}$ (with the standard topology) for which fourteen distinct sets are obtained.

Solutions:

1. First note $X = X \setminus \emptyset \in \mathcal{T}$ and $\emptyset = X \setminus X \in \mathcal{T}$. Next, let $\mathcal{S} \subset \mathcal{T}$ be a subcollection. Then every $U \in \mathcal{S}$ is still of the form $U = X \setminus A$ for some $A \in \mathcal{C}$ and so $\mathcal{D} := \{X \setminus U \mid U \in \mathcal{S}\}$ is a subcollection of $\mathcal{C}$. We have
$$\bigcup_{U\in\mathcal{S}} U = \bigcup_{A\in\mathcal{D}} X \setminus A = X \setminus \bigcap_{A\in\mathcal{D}} A.$$

By assumption we have $\bigcap_{A\in\mathcal{D}} A \in \mathcal{C}$, and so the above set belongs to $\mathcal{T}$. Next, let $U_1, \ldots, U_n \in \mathcal{T}$. Then for each $j = 1, \ldots, n$ there exists $A_j \in \mathcal{C}$ with $U_j = X \setminus A_j$. Consequently

$$U_1 \cap \cdots \cap U_n = (X \setminus A_1) \cap \cdots \cap (X \setminus A_n) = X \setminus (A_1 \cup \cdots \cup A_n).$$

Since $A_1 \cup \cdots \cup A_n \in \mathcal{C}$, the above set is in $\mathcal{T}$. Thus $\mathcal{T}$ is a topology. The closed sets are then the complements of the sets in $\mathcal{T}$, which is precisely the collection $\mathcal{C}$. □

2. (a) Note that $(S^c)^\circ$ is an open contained in S^c, and consequently $((S^c)^\circ)^c$ is a closed set containing S. This implies $\overline{S} \subset ((S^c)^\circ)^c$ since the closure of S is the intersection of all closed sets containing S. Conversely, suppose A is a closed set containing S. Then A^c is an open subset of S^c and hence $A^c \subset (S^c)^\circ$ (the union of all open subsets of S^c). Taking complements again we see that $((S^c)^\circ)^c \subset A$. Taking the intersection over all closed sets containing A yields $((S^c)^\circ)^c \subset \overline{S}$ and hence $((S^c)^\circ)^c = \overline{S}$. □

   (b) Denote $T := S^c$. Then by the previous part we have

$$(\overline{S^c})^c = (\overline{T})^c = (((T^c)^\circ)^c)^c = (T^c)^\circ = (S)^\circ.$$

□

3. (a) Since $A \subset B \subset \overline{B}$, we see that $\overline{B}$ is a closed set containing A. Hence $\overline{A} \subset \overline{B}$. Since $A^\circ \subset A \subset B$, we see that A° is an open subset of B. Hence $A^\circ \subset B^\circ$. □

   (b) First note that $A \cup B \subset \overline{A} \cup \overline{B}$ implies $\overline{A \cup B} \subset \overline{A} \cup \overline{B}$. Conversely, $A \subset A \cup B \subset \overline{A \cup B}$ implies $\overline{A} \subset \overline{A \cup B}$. Similarly, $\overline{B} \subset \overline{A \cup B}$. Consequently, $\overline{A} \subset \overline{B} \subset \overline{A \cup B}$, which implies the desired equality. □

   (c) First note that $A^\circ \cap B^\circ \subset A \cap B$ implies $A^\circ \cap B^\circ \subset (A \cap B)^\circ$. Conversely, $(A \cap B)^\circ \subset A \cap B \subset A$ so that $(A \cap B)^\circ \subset A^\circ$. Similarly, $(A \cap B)^\circ \subset B^\circ$, and so $(A \cap B)^\circ \subset A^\circ \cap B^\circ$, which implies the desired equality. □

   (d) Consider $A = [0, 1)$ and $B = [0, 2]$. Then $A \cap B = \emptyset$, which is closed so $\overline{A \cap B} = \emptyset$. However, $\overline{A} = [0, 1]$ and $\overline{B} = [1, 2]$ so that $\overline{A} \cap \overline{B} = \{1\} \neq \emptyset$.
   Also, $(A \cup B)^\circ = ([0, 2])^\circ = (0, 2)$. But $A^\circ = (0, 1)$ and $B^\circ = (1, 2)$ so that $A^\circ \cup B^\circ = (0, 1) \cup (1, 2) \neq (0, 2)$. □

4. **(i ⇒ ii):** Suppose, towards a contradiction, that there exists $x \in (S^c)^\circ$. But then $(S^c)^\circ$ is a neighborhood of x that fails to contain any elements of S (since $(S^c)^\circ \subset S^c$). Thus we must have $(S^c)^\circ = \emptyset$.

   (ii ⇒ iii): Using Exercise 2.(a) we have

$$\overline{S} = ((S^c)^\circ)^c = (\emptyset)^c = X.$$

   (iii ⇒ i): We saw in lecture that $x \in \overline{S}$ if and only if every neighborhood of x intersects S. Hence for all $x \in X$ and all neighborhoods U of x one has $U \cap S \neq \emptyset$. □

5. (a) For any $F, G \in \mathcal{F}$ write $F \leq G$ if $F \subset G$. We have $F \leq F$ since $F \subset F$. If $F \leq G$ and $G \leq H$ for some $F, G, H \in \mathcal{H}$, then $F \subset G$ and $G \subset H$. Hence $F \subset H$ so that $F \leq H$. Let $F, G \in \mathcal{F}$. Then $H := F \cup G$ is still a finite subset and $F, G \subset H$ so that $F, G \leq H$. Thus $\mathcal{F}$ is a directed set under the relation $\leq$. □

   (b) **(i ⇒ ii):** Fix a bijection $\sigma\colon \mathbb{N} \to \mathbb{N}$. Let S be a limit of the net. Let $\epsilon > 0$, then $(S - \epsilon, S + \epsilon)$ is a neighborhood of S and consequently there exists $F_0 \in \mathcal{F}$ so that

$$\sum_{n\in F} a_n \in (S - \epsilon, S + \epsilon)$$

   for any $F \geq F_0$. This is equivalent to

$$\left|\sum_{n\in F} a_n - S\right| < \epsilon$$

for all $F \subset F_0$. Now, let $N_0 = \max \sigma^{-1}(F_0)$. Then for any $N \geq N_0$ we have $\sigma(\{1, \ldots, N\}) \supset F_0$. Consequently, for any $N \geq N_0$ we have

$$\left| \sum_{n=1}^{N} a_{\sigma(n)} - S \right| = \left| \sum_{k \in \sigma(\{1,\ldots,N\})} a_k - S \right| < \epsilon.$$

Since $\epsilon > 0$ was arbitrary we see that $\sum_{n=1}^{\infty} a_{\sigma(n)} = S$.

(ii ⇒ iii): Note that $\sum_{n=1}^{\infty} a_n$ converges by considering the trivial bijection. Let $I_+ := \{n \in \mathbb{N} \mid a_n \geq 0\}$ and $I_- := \{n \in \mathbb{N} \mid a_n < 0\}$, so that $\mathbb{N} = I_+ \cup I_-$ and $I_+ \cap I_- = \emptyset$. Observe that

$$\sum_{n=1}^{N} |a_n| = \sum_{I_+ \ni n \leq N} a_n + \sum_{I_- \ni n \leq N} -a_n = \sum_{I_+ \ni n \leq N} a_n - \sum_{I_- \ni n \leq N} a_n,$$

and so it suffices to show the series $\sum_{n \in I_+} a_n$ and $\sum_{n \in I_-} a_n$ both converge. First note

$$\sum_{n=1}^{N} a_n = \sum_{I_+ \ni n \leq N} a_n + \sum_{I_- \ni n \leq N} a_n,$$

and so we cannot have only one of the two series converge lest we contradict $\sum_{n=1}^{\infty} a_n$ converging. Thus it remains to rule out the case that both $\sum_{n \in I_+} a_n$ and $\sum_{n \in I_-} a_n$ diverge. Note that this implies I_+ and I_- must be infinite sets, but as subsets of $\mathbb{N}$ they are also countable. Hence there exists bijections $\alpha \colon \mathbb{N} \to I_+$ and $\beta \colon \mathbb{N} \to I_-$. We will now construct a bijection $\sigma \colon \mathbb{N} \to \mathbb{N}$ for which we obtain the contradiction that $\sum_{n=1}^{\infty} a_{\sigma(n)}$ diverges.

Observe that for all $N \in \mathbb{N}$

$$\sum_{n=1}^{N} a_{\alpha(n)} \leq \sum_{n=1}^{N+1} a_{\alpha(n)} \qquad \text{and} \qquad \sum_{n=1}^{N} a_{\beta(n)} \geq \sum_{n=1}^{N+1} a_{\beta(n)}$$

since $\alpha(N+1) \in I_+$ and $\beta(N+1) \in I_-$. Thus the former sequence increases monotonically to $+\infty$ and the latter sequence decreases monotonically to $-\infty$. Let N_1 be the smallest natural number for which

$$\sum_{n=1}^{N_1} a_{\alpha(n)} \geq 1.$$

which exists since $\sum_{n=1}^{N} a_{\alpha(n)}$ increases to $+\infty$. Set $\sigma(n) := \alpha(n)$ for $n = 1, \ldots, N_1$. Let M_1 be the smallest natural number for which

$$\sum_{n=1}^{M_1} a_{\beta(n)} + \sum_{n=1}^{N_1} a_{\sigma(n)} \leq -1,$$

which exists since $\sum_{n=1}^{N} a_{\beta(n)}$ decreases to $-\infty$. Set $\sigma(n + N_1) := \beta(n)$ for $n = 1, \ldots, M_1$. Note that

$$\{\alpha(1), \ldots, \alpha(N_1)\} \cup \{\beta(1), \ldots, \beta(M_1)\} = \{\sigma(1), \ldots, \sigma(N_1 + M_1)\}.$$

Suppose for some $k \in \mathbb{N}$ we have defined $N_1 < N_2 < \cdots < N_{k-1}$, $M_1 < M_2 < \cdots < M_{k-1}$, and $\sigma(n)$ for $n = 1, \ldots, N_{k-1} + M_{k-1}$ so that

$$\sum_{n=1}^{N_{k-1}+M_{k-2}} a_{\sigma(n)} \geq 1 \qquad \text{while} \qquad \sum_{n=1}^{N_{k-1}+M_{k-1}} a_{\sigma(n)} \leq -1,$$

and so that

$$\{\alpha(1), \ldots, \alpha(N_{k-1})\} \cup \{\beta(1), \ldots, \beta(M_{k-1})\} = \{\sigma(1), \ldots, \sigma(N_{k-1} + M_{k-1})\}.$$

Let N_k be the smallest integer satisfying $N_k > N_{k-1}$ and

$$\sum_{n=N_{k-1}+1}^{N_k} a_{\alpha(n)} + \sum_{n=1}^{N_{k-1}+M_{k-1}} a_{\sigma(n)} \geq 1.$$

Set $\sigma(n + N_{k-1} + M_{k-1}) := \alpha(n + N_{k-1})$ for $n = 1, \ldots, N_k - N_{k-1}$. Then let M_k be the smallest integer satisfying $M_k > M_{k-1}$ and

$$\sum_{n=M_{k-1}+1}^{M_k} a_{\beta(n)} + \sum_{n=1}^{N_k+M_{k-1}} a_{\sigma(n)} \leq -1.$$

Set $\sigma(n + N_k + M_{k-1}) := \beta(n + M_{k-1})$ for $n = 1, \ldots, M_k - M_{k_1}$. Cosnequently, we have

$$\{\alpha(1), \ldots, \alpha(N_k)\} \cup \{\beta(1), \ldots, \beta(M_k)\} = \{\sigma(1), \ldots, \sigma(N_k + M_k)\}.$$

So by induction we obtain a map $\sigma \colon \mathbb{N} \to \mathbb{N}$ and strictly increasing sequences $N_1 < \cdots < N_k < \cdots$ and $M_1 < \cdots < M_k < \cdots$ satisfying the above. Note that for all $N \in \mathbb{N}$, there exists $N_1, N_2 \geq N$ so that

$$\sum_{n=1}^{N_1} a_{\sigma(n)} \geq 1 \qquad \text{and} \qquad \sum_{n=1}^{N_2} a_{\sigma(n)} \leq -1.$$

(Take $N_1 = N_N + M_{N-1}$ and $N_2 = N_N + M_N$, for example). Consequently, the series $\sum_{n=1}^{\infty} a_{\sigma(n)}$ diverges. It remains to show σ is a bijection. Note that for each $k \in \mathbb{N}$, the set

$$\{\sigma(1), \ldots, \sigma(N_k + M_k)\} = \{\alpha(1), \ldots, \alpha(N_k)\} \cup \{\beta(1), \ldots, \beta(M_k)\}$$

contains precisely $N_k + M_k$ elements because α and β are each injective and they have disjoint ranges. Thus σ is injective on $\{1, \ldots, N_k + M_k\}$. So given any two distinct $n, m \in \mathbb{N}$, we can find $k \in \mathbb{N}$ so that $n, m \leq N_k + M_k$ and consequently $\sigma(n) \neq \sigma(m)$. That is, σ is injective. To see that σ is surjective, let $m \in \mathbb{N}$. Then, without loss of generality, assume $m \in I_+$ and let $n := \alpha^{-1}(m)$. Let $k \in \mathbb{N}$ be large enough so that $n \leq N_k$. Then

$$m = \alpha(n) \in \{\alpha(1), \ldots, \alpha(N_k)\} \subset \{\sigma(1), \ldots, \sigma(N_k + M_k)\}.$$

So m lies in the range of σ and hence σ is a surjection.
Thus $\sigma \colon \mathbb{N} \to \mathbb{N}$ is a bijection for which $\sum_{n=1}^{\infty} a_{\sigma(n)}$ diverges, a contradiction. Thus we cannot have both $\sum_{n \in I_+} a_n$ and $\sum_{n \in I_-} a_n$ diverge.
(iii $\Rightarrow$ i): Recall from your analysis class that this also implies $\sum_{n=1}^{\infty} a_n$ converges, say to some $S \in \mathbb{R}$. Let U be a neighborhood of S. Then by Exercise 1 on Homework 3 there exists $\epsilon > 0$ so that $(S - \epsilon, S + \epsilon) \subset U$. Let $N_1 \in \mathbb{N}$ be such that for all $N \geq N_1$ one has

$$\left| \sum_{n=1}^{N} a_n - S \right| < \frac{\epsilon}{2}.$$

Then, using the fact that $\sum_{n=1}^{\infty} |a_n|$ converges, let $N_2 \in \mathbb{N}$ be such that for all $N \geq N_2$ one has

$$\sum_{n=N_2+1}^{\infty} |a_n| < \frac{\epsilon}{2}.$$

Let $N_0 = \max\{N_1, N_2\}$ and set $F_0 := \{1, 2, \ldots, N_0\}$. If $F \supset F_0$, then we have...

□

6. (a) We must show that the set of arbitrary words in letter C and K yield at most fourteen sets when applied to A. First note that $KK = K$ and $CC = 1$, where 1 denotes the identify function on $\mathcal{P}(X)$. These rules imply that an arbitrary word can always be reduced to one which is alternating in K and C. Thus it suffices to consider words starting with C or K (as well as the empty word which gives 1):

$$K, KC, KCK, KCKC, \ldots$$
$$C, CK, CKC, CKCK, \ldots.$$

Towards showing that the above lists contain only finitely many distinct functions, we claim that that $KCKCKCK = KCK$. This will follow from showing $KCKCKCKC = KCKC$ and applying C on the right of each side. By Exercise 2.(b), we have $CKC(A) = (\overline{A^c})^c = A^\circ$. Define $I\colon \mathcal{P}(X) \to \mathcal{P}(X)$ by $I(A) = A^\circ$, and so we are trying to prove $KIKI = KI$. We first note

$$KIK(A) = \overline{\overline{A}^\circ} \subset \overline{A} = K(A).$$

Indeed, $\overline{\overline{A}^\circ}$ is the intersection of all closed sets containing $\overline{A}^\circ$, and $\overline{A}$ is one such closed set. Next, note that

$$I(A) = A^\circ \subset \overline{A^\circ}^\circ = IKI(A).$$

Indeed, $\overline{A^\circ}^\circ$ is the union of all open subsets of $\overline{A^\circ}$, and A° is one such open subset. Now, Exercise 3.(a) implies $K(A) \subset K(B)$ whenever $A \subset B$. Piecing all of this together yields

$$KI(A) = K(I(A)) \subset K(IKI(A)) = KIKI(A) = KIK(I(A)) \subset K(I(A)) = KI(A).$$

Consequently, $KIKI = KI$ and so the claimed equality $KCKCKCK = KCK$ is true. This implies that our above lists will start repeating after finitely many steps:

$$K, KC, KCK, KCKC, KCKCK, KCKCKC, KCKCKCK = KCK$$
$$C, CK, CKC, CKCK, CKCKC, CKCKCK, CKCKCKC, CKCKCKCK = CKCK.$$

Hence the fourteen potentially distinct sets are as follows:

1. $1(A) = A$	8. $C(A) = A^c$
2. $K(A) = \overline{A}$	9. $CK(A) = \overline{A}^c$
3. $KC(A) = \overline{A^c}$	10. $CKC(A) = \overline{A^c}^c = A^\circ$
4. $KCK(A) = \overline{\overline{A}^c}$	11. $CKCK(A) = \overline{A}^\circ$
5. $KCKC(A) = \overline{\overline{A^c}^c} = \overline{A^\circ}$	12. $CKCKC(A) = \overline{A^c}^\circ = \overline{A^\circ}^c$
6. $KCKCK(A) = \overline{\overline{A}^\circ}$	13. $CKCKCK(A) = \overline{\overline{A}^\circ}^c$
7. $KCKCKC(A) = \overline{\overline{A^c}^\circ} = \overline{\overline{A^\circ}^c}$	14. $CKCKCKC(A) = \overline{\overline{A^c}^\circ}^c = \overline{A^\circ}^\circ$

Note that aside from A and A^c, every set in the left column is closed and every set in the right column is open. Moreover, the sets in the right column are exactly the complements of the sets directly to their left, while the sets in the left column (aside from A and $K(A)$) are the closures of the sets to the right and up two rows. These observations are useful in computing the example below.

(b) Consider the set

$$A = ([0,1] \setminus \mathbb{Q}) \cup (2,3) \cup (3,4) \cup \{5\}.$$

We then have

1. $([0,1] \setminus \mathbb{Q}) \cup (2,3) \cup (3,4) \cup \{5\}$	8. $(-\infty,0) \cup ((0,1) \setminus \mathbb{Q}) \cup (1,2] \cup \{3\} \cup [4,5) \cup (5,\infty)$
2. $[0,1] \cup [2,4] \cup \{5\}$	9. $(-\infty,0) \cup (1,2) \cup (4,5) \cup (5,\infty)$
3. $(-\infty,2] \cup \{3\} \cup [4,\infty)$	10. $(2,3) \cup (3,4)$
4. $(-\infty,0] \cup [1,2] \cup [4,\infty)$	11. $(0,1) \cup (2,4)$
5. $[2,4]$	12. $(-\infty,2) \cup (4,\infty)$
6. $[0,1] \cup [2,4]$	13. $(-\infty,0) \cup (1,2) \cup (4,\infty)$
7. $(-\infty,2] \cup [4\infty)$	14. $(2,4)$