Name: ____

NetID: ____

Instructions:

1. You will have 80 minutes to complete the exam.
2. The exam is a total of 5 questions, their respective points values are listed below.
3. Unless stated otherwise, you must justify your answers with proofs.
4. No books, notes, calculators, or electronic devices are permitted.
5. If you require additional space, please use the reverse side of the pages.
6. The exam has a total of 6 pages, please verify that your copy has all 6 pages.

Question	**Score**	**Points**
1.		20
2.		20
3.		20
4.		20
5.		20
Total		100

1. Consider the following matrix
$$A = \begin{pmatrix} 2 & 0 & -2 & -1 & -7 \\ 0 & 1 & 2 & 1 & -2 \\ 1 & 1 & 1 & 0 & -5 \end{pmatrix}$$

   (a) Find a basis for the column space of A.

   (b) Find a basis for the row space of A.

   (c) Find a basis for the kernel of A.

   (d) Compute the dimension of the kernel of A^T.

2. Consider the following vectors in $\mathbb{R}^3$:

$$\mathbf{v}_1 = \begin{pmatrix} 2 \\ 0 \\ 1 \end{pmatrix}, \qquad \mathbf{v}_2 = \begin{pmatrix} 0 \\ 1 \\ 1 \end{pmatrix}, \qquad \mathbf{v}_3 = \begin{pmatrix} -1 \\ 1 \\ 0 \end{pmatrix}.$$

Define a basis $\mathcal{B} := \{\mathbf{v}_1, \mathbf{v}_2, \mathbf{v}_3\}$ for $\mathbb{R}^3$ (you do not need to show it is a basis but you should already know it is one), and let $\mathcal{S} = \{\mathbf{e}_1, \mathbf{e}_2, \mathbf{e}_3\}$ be the standard basis.

(a) Compute the change of coordinate matrix $[I]_{\mathcal{B}}^{\mathcal{S}}$.

(b) Compute the change of coordinate matrix $[I]_{\mathcal{S}}^{\mathcal{B}}$.

(c) Suppose $A \in M_{3\times 3}$ is such that $\mathbf{v}_1$, $\mathbf{v}_2$, and $\mathbf{v}_3$ are each eigenvectors of A with eigenvalues $\lambda_1 = 0$, $\lambda_2 = 3$, and $\lambda_3 = 3$, respectively. Compute A and justify your computation with a proof. [**Hint:** recall how computed the matrix representation for reflection over a line.]

3. Consider the following matrix

$$A = \begin{pmatrix} -3 & -6 & 6 \\ 0 & 3 & 0 \\ -3 & -3 & 6 \end{pmatrix}$$

(a) Compute the characteristic polynomial of A.

(b) Compute $\sigma(A)$, the spectrum of A.

(c) For each $\lambda \in \sigma(A)$, find a basis for the corresponding eigenspace.

4. Let $\mathbf{v}_1, \ldots, \mathbf{v}_n$ be a basis for $\mathbb{R}^n$. Let $A \in M_{m\times n}(\mathbb{R})$ be such that

$$\mathbf{b} = A\mathbf{v}_1 = A\mathbf{v}_2 = \cdots = A\mathbf{v}_n$$

for some non-zero $\mathbf{b} \in \mathbb{R}^m$. Consider the the following $n-1$ vectors in $\mathbb{R}^n$:

$$\mathbf{w}_2 := \mathbf{v}_1 - \mathbf{v}_2, \qquad \mathbf{w}_3 := \mathbf{v}_1 - \mathbf{v}_3, \qquad \ldots \qquad \mathbf{w}_n := \mathbf{v}_1 - \mathbf{v}_n.$$

(a) Show that $\mathbf{w}_2, \mathbf{w}_3, \ldots, \mathbf{w}_n \in \operatorname{Ker}(A)$.

(b) Show that $\mathbf{w}_2, \mathbf{w}_3, \ldots, \mathbf{w}_n$ is a linearly independent system.

(c) Show that $\mathbf{w}_2, \mathbf{w}_3, \ldots, \mathbf{w}_n$ forms a basis for $\operatorname{Ker}(A)$.
[**Hint:** first determine the value of $\operatorname{rank}(A)$.]

5. The following statements are **FALSE**. Turn each of them into a true statement by doing **exactly one** of the following: (i) cross out **one** word; (ii) insert **one** word; or (iii) cross out **one** word and insert a **single** word. (Do not justify your answer.)

   (a) Every invertible matrix can be written as a sum of elementary matrices.

   (b) Any two matrix representations of a linear transformation are equal.

   (c) A generating system for a finite-dimensional vector space V cannot have more than $\dim(V)$ vectors in it.

   (d) A vector space is finite-dimensional if and only if it has a unique basis consisting of finitely many vectors.

   (e) The nullity of a matrix and its transpose are equal.

   (f) The row space of a matrix A is equal to the range of A.

   (g) The determinant is invariant under row reordering.

   (h) The determinant is linear.

   (i) For a linear transformation $T\colon V \to V$, an eigenvector of T is a vector $\mathbf{v} \in V$ such that $T(\mathbf{v}) = \lambda\mathbf{v}$ for some scalar λ.

   (j) For a linear transformation $T\colon V \to V$ with eigenvalue λ, the dimension of $\mathrm{Ker}(T - \lambda I)$ is the multiplicity of λ.