1. (a) **(5 pts)**
   - $T \circ T \begin{pmatrix} x \\ y \\ z \end{pmatrix} = \begin{pmatrix} z \\ 0 \\ 0 \end{pmatrix}$
   - $T \circ T \circ T \begin{pmatrix} x \\ y \\ z \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \\ 0 \end{pmatrix}$

   (b) **(10 pts)** Suppose, towards a contradiction, that T is left invertible with left-inverse S. Then $S\colon \mathbb{R}^3 \to \mathbb{R}^3$ and $S \circ T = I$. But from the previous part we see that $T \circ T \circ T = O$, the zero transformation. So composing with S on the left three times yields

$$\begin{aligned} S \circ S \circ S \circ T \circ T \circ T &= S \circ S \circ S \circ O \\ S \circ S \circ I \circ T \circ T &= O \\ S \circ S \circ T \circ T &= O \\ &\vdots \\ I &= O, \end{aligned}$$

   which is a contradiction. Thus T is not left invertible.
   Next, suppose, again towards a contradiction, that T is right invertible with right-inverse S. Taking the equation $T \circ T \circ T = O$ and composing with S three times on the right again yields $I = O$, a contradiction. Thus T is not right invertible either. □

   (c) **(5 pts)** Recall that the columns of A are given by $T(\mathbf{e}_1), T(\mathbf{e}_2), T(\mathbf{e_3})$, where $\mathbf{e}_1, \mathbf{e}_2, \mathbf{e}_3$ is the standard basis for $\mathbb{R}^3$. Thus we compute:

$$T\begin{pmatrix} 1 \\ 0 \\ 0 \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \\ 0 \end{pmatrix}, \qquad T\begin{pmatrix} 0 \\ 1 \\ 0 \end{pmatrix} = \begin{pmatrix} 1 \\ 0 \\ 0 \end{pmatrix}, \qquad T\begin{pmatrix} 0 \\ 0 \\ 1 \end{pmatrix} = \begin{pmatrix} 0 \\ 1 \\ 0 \end{pmatrix}.$$

   Therefore,

$$A = \begin{pmatrix} 0 & 1 & 0 \\ 0 & 0 & 1 \\ 0 & 0 & 0 \end{pmatrix}.$$

   (d) **(5 pts)** Recall that $[T \circ T] = [T][T] = AA$, and similarly $[T \circ T \circ T] = AAA$. Thus we simply compute the matrix multiplication given A above:

$$\begin{aligned} [T \circ T] &= \begin{pmatrix} 0 & 0 & 1 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{pmatrix} \\ [T \circ T \circ T] &= \begin{pmatrix} 0 & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{pmatrix}. \end{aligned}$$

2. (a) **(5 pts)** We claim that $\mathbf{0}$ is the element $1 \in V$. Indeed, for any $x \in V$ we have

$$x \oplus 1 = x1 = x.$$

   Thus 1 satisfies the zero vector property and therefore is THE zero vector. □

   (b) **(5 pts)** Given $x \in V$, we claim that the additive inverse with respect to $\oplus$ is $\frac{1}{x}$. First note that the reciprocal exists since $x > 1$. It then suffices to check $x \oplus \frac{1}{x} = \mathbf{0}$. That is, given the previous part we must show $x \oplus \frac{1}{x} = 1$. Using the definition of $\oplus$ we have

$$x \oplus \frac{1}{x} = x\frac{1}{x} = 1,$$

   as needed. □

3. (a) **(5 pts)** We compute

$$\begin{aligned}\text{Tr}(A_1) &= 0 + 0 = 0\\ \text{Tr}(A_2) &= 0 + 0 = 0\\ \text{Tr}(A_3) &= 1 + -1 = 0.\end{aligned}$$

Thus $A_1, A_2, A_3 \in \text{Null}(\text{Tr})$. □

(b) **(5 pts)** Suppose there are scalars $\alpha_1, \alpha_2, \alpha_3$ such that $\alpha_1 A_1 + \alpha_2 A_2 + \alpha_3 A_3 = \mathbf{0}$. Recalling that the zero vector in $M_{2\times 2}(\mathbb{R})$ is the matrix of all zeros, this implies

$$\alpha_1 \begin{pmatrix} 0 & 1 \\ 0 & 0 \end{pmatrix} + \alpha_2 \begin{pmatrix} 0 & 0 \\ 1 & 0 \end{pmatrix} + \alpha_3 \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix} = \begin{pmatrix} 0 & 0 \\ 0 & 0 \end{pmatrix}$$
$$\begin{pmatrix} \alpha_3 & \alpha_1 \\ \alpha_2 & -\alpha_3 \end{pmatrix} = \begin{pmatrix} 0 & 0 \\ 0 & 0 \end{pmatrix}.$$

So $\alpha_1 = \alpha_2 = \alpha_3 = 0$, and therefore A_1, A_2, A_3 are linearly independent. □

(c) **(10 pts)** It suffices to show that A_1, A_2, A_3 are spanning for Null(Tr). Let $B \in \text{Null}(\text{Tr})$ be arbitrary, say with entries

$$B = \begin{pmatrix} a & b \\ c & d \end{pmatrix}.$$

We must write B as a linear combination of A_1, A_2, A_3. We first observe that

$$B = bA_1 + cA_2 + \begin{pmatrix} a & 0 \\ 0 & d \end{pmatrix}.$$

Now, since $B \in \text{Null}(\text{Tr})$, we have $0 = \text{Tr}(B) = a + b$. Thus $d = -a$, and so the third term above equals aA_3. Thus

$$B = bA_1 + cA_2 + aA_3.$$

Since $B \in \text{Null}(\text{Tr})$ was arbitrary, this shows A_1, A_2, A_3 are spanning for Null(Tr) and therefore a basis. □

4. (a) **(10 pts)** We first translate the linear system into the following augmented matrix:

$$\left(\begin{array}{ccccc|c} 1 & -1 & -2 & 0 & 2 & 0 \\ 3 & -3 & -1 & 5 & -4 & 0 \\ -1 & 1 & 4 & 2 & -3 & 0 \end{array}\right).$$

Next we perform row operations until we arrive at the reduced row echelon form:

$$\begin{aligned}
\left(\begin{array}{ccccc|c} 1 & -1 & -2 & 0 & 2 & 0 \\ 3 & -3 & -1 & 5 & -4 & 0 \\ -1 & 1 & 4 & 2 & -3 & 0 \end{array}\right) &\xrightarrow[R_3\mapsto R_3+R_1]{R_2\mapsto R_2-3R_1} \left(\begin{array}{ccccc|c} 1 & -1 & -2 & 0 & 2 & 0 \\ 0 & 0 & 5 & 5 & -10 & 0 \\ 0 & 0 & 2 & 2 & -1 & 0 \end{array}\right)\\
&\xrightarrow{R_2\mapsto \frac{1}{5}R_2} \left(\begin{array}{ccccc|c} 1 & -1 & -2 & 0 & 2 & 0 \\ 0 & 0 & 1 & 1 & -2 & 0 \\ 0 & 0 & 2 & 2 & -1 & 0 \end{array}\right)\\
&\xrightarrow{R_3\mapsto R_3-2R_2} \left(\begin{array}{ccccc|c} 1 & -1 & -2 & 0 & 2 & 0 \\ 0 & 0 & 1 & 1 & -2 & 0 \\ 0 & 0 & 0 & 0 & 3 & 0 \end{array}\right)\\
&\xrightarrow{R_3\mapsto \frac{1}{3}R_3} \left(\begin{array}{ccccc|c} 1 & -1 & -2 & 0 & 2 & 0 \\ 0 & 0 & 1 & 1 & -2 & 0 \\ 0 & 0 & 0 & 0 & 1 & 0 \end{array}\right)\\
&\xrightarrow[R_2\mapsto R_2+2R_3]{R_1\mapsto R_1-2R_3} \left(\begin{array}{ccccc|c} 1 & -1 & -2 & 0 & 0 & 0 \\ 0 & 0 & 1 & 1 & 0 & 0 \\ 0 & 0 & 0 & 0 & 1 & 0 \end{array}\right)\\
&\xrightarrow{R_1\mapsto R_1+2R_2} \left(\begin{array}{ccccc|c} 1 & -1 & 0 & 2 & 0 & 0 \\ 0 & 0 & 1 & 1 & 0 & 0 \\ 0 & 0 & 0 & 0 & 1 & 0 \end{array}\right)
\end{aligned}$$

From this we see that x_1, x_3, x_5 are pivot variables, while x_2, x_4 are free variables. So we will translate the above RREF back into a linear system and then solve for our pivot variables:

$$\begin{cases} x_1 - x_2 + 2x_4 &= 0 \\ x_3 + x_4 &= 0 \\ x_5 &= 0 \end{cases} \longrightarrow \begin{cases} x_1 &= x_2 - 2x_4 \\ x_3 &= -x_4 \\ x_5 &= 0 \end{cases}$$

Therefore all solutions to this linear system are of the following form:

$$\begin{pmatrix} x_2 - 2x_4 \\ x_2 \\ -x_4 \\ x_4 \\ 0 \end{pmatrix} \qquad x_2, x_4 \in \mathbb{R}.$$

(b) **(5 pts)** Let A be the coefficient matrix of the above linear system. Then $\mathbf{x} \in \mathbb{R}^5$ is an element of X if and only if $A\mathbf{x} = \mathbf{0}$. Therefore $X = \text{Null}(A)$, and we have shown on the homework that this always gives a subspace. Alternatively, one can directly check the following: (i) $\mathbf{0} \in X$; (ii) if $\mathbf{x}, \mathbf{y} \in X$ then $\mathbf{x} + \mathbf{y} \in X$; and (iii) if $\mathbf{x} \in X$ then $t\mathbf{x} \in X$ for any $t \in \mathbb{R}$. The previous part gives us a general description of the elements of X, which can be used to directly verify each of these three conditions. □

5. (a) **(5 pts)** By a theorem from lecture, we know that any linearly independent set in $\mathbb{R}^3$ can contain at most three vectors. Consequently, the system $\mathbf{v}_1, \mathbf{v}_2, \mathbf{v}_3, \mathbf{v}_4, \mathbf{v}_5$ must be linearly dependent. □

(b) **(5 pts)** Let A be the matrix with columns $\mathbf{v}_1, \mathbf{v}_2, \mathbf{v}_3, \mathbf{v}_4, \mathbf{v}_5$:

$$A = \begin{pmatrix} 1 & -1 & -2 & 0 & 2 \\ 3 & -3 & -1 & 5 & -4 \\ -1 & 1 & 4 & 2 & -3 \end{pmatrix}.$$

Observe that this is precisely the coefficient matrix from the previous problem. Therefor its reduced row echelon form is:

$$\begin{pmatrix} 1 & -1 & 0 & 2 & 0 \\ 0 & 0 & 1 & 1 & 0 \\ 0 & 0 & 0 & 0 & 1 \end{pmatrix}.$$

Since there is a pivot in every row, a theorem from lecture tells us that the system $\mathbf{v}_1, \mathbf{v}_2, \mathbf{v}_3, \mathbf{v}_4, \mathbf{v}_5$ spans $\mathbb{R}^3$. □

6. (a) **(2 pts)** A system of vectors $\mathbf{v}_1, \ldots, \mathbf{v}_n$ in a vector space V is a basis if every other vector $\mathbf{v} \in V$ admits a unique representation as a linear combination of $\mathbf{v}_1, \ldots, \mathbf{v}_n$.

(b) **(2 pts)** A system of vectors $\mathbf{v}_1, \ldots, \mathbf{v}_n$ in a vector space V is linearly independent if only the trivial linear combination of $\mathbf{v}_1, \ldots, \mathbf{v}_n$ equals the zero vector.

(c) **(2 pts)** If a system of vectors $\mathbf{v}_1, \ldots, \mathbf{v}_n \in V$ is linearly dependent, then ~~every~~ some $\mathbf{v}_k$ can be written as a linear combination of the other vectors.

(d) **(2 pts)** If a system of vectors $\mathbf{v}_1, \ldots, \mathbf{v}_n$ is a spanning system in V, then every $\mathbf{v} \in V$ admits a ~~unique~~ representation as a linear combination of $\mathbf{v}_1, \ldots, \mathbf{v}_n$.

(e) **(2 pts)** A transformation $T\colon V \to W$ is ~~quadratic~~ linear if and only if $T(\alpha\mathbf{v} + \beta\mathbf{w}) = \alpha T(\mathbf{v}) + \beta T(\mathbf{w})$ for all $\mathbf{v}, \mathbf{w} \in V$ and all scalars α, β.

(f) **(2 pts)** A linear transformation is an isomorphism if and only if it is ~~injective~~ invertible.

(g) **(2 pts)** A linear transformation $A\colon V \to W$ is right invertible if there exists a linear transformation $B\colon W \to V$ such that $A \circ B = I_W$.

(h) **(2 pts)** Suppose $\mathbf{v}_1, \ldots, \mathbf{v}_n$ is a basis for a vector space V, and suppose $\mathbf{w}_1, \ldots \mathbf{w}_n$ is a ~~system~~ basis in a vector space W. If $T\colon V \to W$ is a linear transformation satisfying $T(\mathbf{v}_i) = \mathbf{w}_i$ for $i = 1, \ldots, n$, then T is an isomorphism.

(i) **(2 pts)** The ~~kernel~~ range of a linear transformation $T\colon V \to W$ is the set of all vectors $\mathbf{w} \in W$ for which there exists $\mathbf{v} \in V$ such that $T(\mathbf{v}) = \mathbf{w}$.

(j) **(2 pts)** Matrix multiplication is not commutative.