Name: ____

NetID: ____

Instructions:

1. You will have 80 minutes to complete the exam.
2. The exam is a total of 6 questions, their respective points values are listed below.
3. Unless stated otherwise, you must justify your answers with proofs.
4. You may cite any results from lecture or the homework.
5. No books, notes, calculators, or electronic devices are permitted.
6. If you require additional space, please use the reverse side of the pages.
7. The exam has a total of 7 pages, please verify that your copy has all 7 pages.

Question	**Score**	**Points**
1.		25
2.		10
3.		20
4.		15
5.		10
6.		20
Total		100

1. Let $T\colon \mathbb{R}^3 \to \mathbb{R}^3$ be given by

$$T\begin{pmatrix} x \\ y \\ z \end{pmatrix} = \begin{pmatrix} y \\ z \\ 0 \end{pmatrix}.$$

(a) Compute each of the following. (Do not justify your answer.)

- $T \circ T \begin{pmatrix} x \\ y \\ z \end{pmatrix} =$

- $T \circ T \circ T \begin{pmatrix} x \\ y \\ z \end{pmatrix} =$

(b) Prove that T is neither left nor right invertible. [**Hint:** use the previous part.]

(c) Find the matrix representation T; that is, find a matrix A such that $T(\mathbf{v}) = A\mathbf{v}$ for all $\mathbf{v} \in \mathbb{R}^3$.

(d) Find matrix representations for each of the following (Do not justify your answer.)

- $T \circ T$

- $T \circ T \circ T$

2. Let $V = \{x \in \mathbb{R} \colon x > 0\}$. Define an addition operation $\oplus$ on this set by

$$x \oplus y := xy,$$

and a scalar multiplication operation $\otimes$ by

$$\alpha \otimes x := x^{\alpha}.$$

Then V is a vector space with these operations (you do not need to show this).

(a) Determine the zero vector $\mathbf{0}$ of this space, and prove your answer.

(b) Given $x \in V$ find its additive inverse with respect to $\oplus$, and prove your answer.

3. Let $\mathrm{Tr}\colon M_{2\times 2}(\mathbb{R}) \to \mathbb{R}$ denote the trace on the two-by-two matrices:

$$\mathrm{Tr}\begin{pmatrix} a & b \\ c & d \end{pmatrix} = a + d.$$

Recall the null space of Tr is the subspace $\mathrm{Null}(\mathrm{Tr}) = \{A \in M_{2\times 2}(\mathbb{R})\colon \mathrm{Tr}(A) = 0\}$. Consider the following matrices

$$A_1 = \begin{pmatrix} 0 & 1 \\ 0 & 0 \end{pmatrix}, \qquad A_2 = \begin{pmatrix} 0 & 0 \\ 1 & 0 \end{pmatrix}, \qquad A_3 = \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix}.$$

(a) Show that $A_1, A_2, A_3 \in \mathrm{Null}(\mathrm{Tr})$.

(b) Prove that the system A_1, A_2, A_3 is linearly independent.

(c) Prove that the system A_1, A_2, A_3 is a basis for $\mathrm{Null}(\mathrm{Tr})$.

4. Consider the following linear system:

$$\left\{\begin{array}{rrrrrl} x_1 & -x_2 & -2x_3 & & +2x_5 & =0 \\ 3x_1 & -3x_2 & -x_3 & +5x_4 & -4x_5 & =0 \\ -x_1 & +x_2 & +4x_3 & +2x_4 & -3x_5 & =0 \end{array}\right.$$

(a) Find **all** solutions $\mathbf{x} \in \mathbb{R}^5$ of this linear system. If none exist, write "No Solutions." A proof is not required, but you must clearly show your work.

(b) Let $X \subset \mathbb{R}^5$ be the subset of all solutions to the above linear system. Prove that X is a subspace.

5. Consider the following vectors in $\mathbb{R}^3$:

$$\mathbf{v}_1 := \begin{pmatrix} 1 \\ 3 \\ -1 \end{pmatrix}, \quad \mathbf{v}_2 := \begin{pmatrix} -1 \\ -3 \\ 1 \end{pmatrix}, \quad \mathbf{v}_3 := \begin{pmatrix} -2 \\ -1 \\ 4 \end{pmatrix}, \quad \mathbf{v}_4 := \begin{pmatrix} 0 \\ 5 \\ 2 \end{pmatrix}, \quad \mathbf{v}_5 := \begin{pmatrix} 2 \\ -4 \\ -3 \end{pmatrix}.$$

[**Hint:** compare these vectors to the previous page.]

(a) Determine whether the system $\mathbf{v}_1, \mathbf{v}_2, \mathbf{v}_3, \mathbf{v}_4, \mathbf{v}_5$ is linearly dependent or linearly independent. Prove your answer.

(b) Determine whether or not the system $\mathbf{v}_1, \mathbf{v}_2, \mathbf{v}_3, \mathbf{v}_4, \mathbf{v}_5$ spans $\mathbb{R}^3$. Prove your answer.

6. The following statements are **FALSE**. Turn each of them into a true statement by doing **exactly one** of the following: (i) cross out **one** word; (ii) insert **one** word; or (iii) cross out **one** word and insert a **single** word. (Do not justify your answer.)

   (a) A system of vectors $\mathbf{v}_1, \ldots, \mathbf{v}_n$ in a vector space V is a basis if every other vector $\mathbf{v} \in V$ admits a representation as a linear combination of $\mathbf{v}_1, \ldots, \mathbf{v}_n$.

   (b) A system of vectors $\mathbf{v}_1, \ldots, \mathbf{v}_n$ in a vector space V is linearly independent if the trivial linear combination of $\mathbf{v}_1, \ldots, \mathbf{v}_n$ equals the zero vector.

   (c) If a system of vectors $\mathbf{v}_1, \ldots, \mathbf{v}_n \in V$ is linearly dependent, then every $\mathbf{v}_k$ can be written as a linear combination of the other vectors.

   (d) If a system of vectors $\mathbf{v}_1, \ldots, \mathbf{v}_n$ is a spanning system in V, then every $\mathbf{v} \in V$ admits a unique representation as a linear combination of $\mathbf{v}_1, \ldots, \mathbf{v}_n$.

   (e) A transformation $T\colon V \to W$ is quadratic if and only if $T(\alpha\mathbf{v} + \beta\mathbf{w}) = \alpha T(\mathbf{v}) + \beta T(\mathbf{w})$ for all $\mathbf{v}, \mathbf{w} \in V$ and all scalars α, β.

   (f) A linear transformation is an isomorphism if and only if it is injective.

   (g) A linear transformation $A\colon V \to W$ is invertible if there exists a linear transformation $B\colon W \to V$ such that $A \circ B = I_W$.

   (h) Suppose $\mathbf{v}_1, \ldots, \mathbf{v}_n$ is a basis for a vector space V, and suppose $\mathbf{w}_1, \ldots \mathbf{w}_n$ is a system in a vector space W. If $T\colon V \to W$ is a linear transformation satisfying $T(\mathbf{v}_i) = \mathbf{w}_i$ for $i = 1, \ldots, n$, then T is an isomorphism.

   (i) The kernel of a linear transformation $T\colon V \to W$ is the set of all vectors $\mathbf{w} \in W$ for which there exists $\mathbf{v} \in V$ such that $T(\mathbf{v}) = \mathbf{w}$.

   (j) Matrix multiplication is commutative.