Exercises:

1. Consider the following matrices

$$A = \begin{pmatrix} 0 & 2 & -1 \\ 1 & -1 & 3 \\ 0 & 0 & 4 \end{pmatrix}, \qquad B = \begin{pmatrix} 5 & 0 & -2 \\ 1 & 1 & 1 \\ -3 & -1 & 0 \end{pmatrix}, \qquad C = \begin{pmatrix} 1 & 1 & -2 & 1 \\ 0 & 3 & 0 & 3 \\ -2 & 1 & 1 & 2 \\ 5 & -5 & 0 & -1 \end{pmatrix}.$$

   (a) Compute $\det(A)$ and $\det(B)$ using cofactor expansion along any **row**.

   (b) Compute $\det(A)$ and $\det(B)$ using cofactor expansion along any **column**.

   (c) Compute $\det(C)$ using cofactor expansion along any row or column.

   (d) Compute $\det(C)$ using row reduction.

2. For $A \in M_{n\times m}$, its **adjoint** (or **conjugate transpose**) is the matrix $A^* \in M_{m\times n}$ with entries $(A^*)_{ij} = \overline{(A)_{ji}}$, where $\bar{z}$ is the complex conjugate of $z \in \mathbb{C}$. Prove that $\det(A^*) = \overline{\det(A)}$ for $A \in M_{n\times n}$.

3. In this exercise, you will examine the behavior of the determinant on a variety of special matrices.

   (a) Suppose $A \in M_{n\times n}$ is invertible. Prove that $\det(A^{-1}) = \frac{1}{\det(A)}$.

   (b) Suppose $A, B \in M_{n\times n}$ are similar. Prove that $\det(A) = \det(B)$.

   (c) A matrix $A \in M_{n\times n}$ is called **nilpotent** if $A^k = O$ for some $k \in \mathbb{N}$. Show that if A is nilpotent, then $\det(A) = 0$.

   (d) A matrix $A \in M_{n\times n}(\mathbb{R})$ is called **orthogonal** if $A^T A = AA^T = I_n$. Show that if A is orthogonal, then $\det(A) = \pm 1$.

   (e) A matrix $A \in M_{n\times n}(\mathbb{C})$ is called **unitary** if $A^*A = AA^* = I_n$. Show that if A is unitary, then $|\det(A)| = 1$.

4. Consider the matrix

$$A = \begin{pmatrix} 2 & 0 & -2 \\ 0 & -1 & 0 \\ 1 & 1 & 1 \end{pmatrix}.$$

   (a) Use the determinant to show that A is invertible.

   (b) Use Cramer's rule to solve $A\mathbf{x} = (-4, 0, 8)^T$.

   (c) Use the cofactor inversion formula to compute A^{-1}.

5. For each of the following matrices find: (i) the characteristic polynomial; (ii) the spectrum; and (iii) a basis for each eigenspace.

   (a) $\begin{pmatrix} 1 & 1 \\ -1 & 3 \end{pmatrix}$.

   (b) $\begin{pmatrix} 1 & 1 \\ 1 & 0 \end{pmatrix}$.

   (c) $\begin{pmatrix} 1 & 3 & 3 \\ -3 & -5 & -3 \\ 3 & 3 & 1 \end{pmatrix}$.

6. Let $\theta \in [0, 2\pi)$. Find the spectrum $\sigma(R_\theta)$ for the rotation matrix

$$R_\theta = \begin{pmatrix} \cos(\theta) & -\sin(\theta) \\ \sin(\theta) & \cos(\theta) \end{pmatrix}.$$

   Determine which values of θ yield only real eigenvalues.

Solutions:

1. (a) For A, we expand along the third row:

$$\det(A) = 0 + 0 + (-1)^{3+3}(4)(0-2) = -8.$$

For B, we expand along the first row:

$$\det(B) = (-1)^{1+1}(5)(0-(-1)) + 0 + (-1)^{1+3}(-2)(-1-(-3)) = 5-4 = 1.$$

(b) For A, we expand along the first column:

$$\det(A) = 0 + (-1)^{2+1}(1)(8-0) = -8,$$

which agrees with our answer from part (a). For B, we expand along the second column:

$$\det(B) = 0 + (-1)^{2+2}(1)(0-6) + (-1)^{2+3}(-1)(5-(-2)) = -6+7 = 1,$$

which also agrees with our answer from part (a).

(c) We will expand along the second row

$$\det(C) = 0 + (-1)^{2+2}(3)\det\begin{pmatrix} 1 & -2 & 1 \\ -2 & 1 & 2 \\ 5 & 0 & -1 \end{pmatrix} + 0 + (-1)^{2+4}(3)\det\begin{pmatrix} 1 & 1 & -2 \\ -2 & 1 & 1 \\ -5 & 5 & 0 \end{pmatrix}.$$

We compute the determinant of both 3×3 matrices by expanding along the third row:

$$\det\begin{pmatrix} 1 & -2 & 1 \\ -2 & 1 & 2 \\ 5 & 0 & -1 \end{pmatrix} = (-1)^{3+1}(5)(-4-1) + 0 + (-1)^{3+3}(-1)(1-4) = -25+3 = -22$$

$$\det\begin{pmatrix} 1 & 1 & -2 \\ -2 & 1 & 1 \\ -5 & 5 & 0 \end{pmatrix} = (-1)^{3+1}(-5)(1-(-2)) + (-1)^{3+2}(5)(1-4) + 0 = -15+15 = 0.$$

So continuing our computation of $\det(C)$ we have

$$\det(C) = 3(-22) + 3(0) = -66.$$

(d) We execute the following row operations on C:

$$\begin{pmatrix} 1 & 1 & -2 & 1 \\ 0 & 3 & 0 & 3 \\ -2 & 1 & 1 & 2 \\ 5 & -5 & 0 & -1 \end{pmatrix} \underset{\substack{R3\mapsto R3+2R1 \\ R4\mapsto R4-5R1}}{\longrightarrow} \begin{pmatrix} 1 & 1 & -2 & 1 \\ 0 & 3 & 0 & 3 \\ 0 & 3 & -3 & 4 \\ 0 & -10 & 10 & -6 \end{pmatrix} \underset{R3\mapsto R3-R2}{\overset{R2\mapsto \frac{1}{3}R2}{\longrightarrow}} \begin{pmatrix} 1 & 1 & -2 & 1 \\ 0 & 1 & 0 & 1 \\ 0 & 0 & -3 & 1 \\ 0 & -10 & 10 & -6 \end{pmatrix}$$

$$\underset{R4\mapsto R4+10R2}{\overset{R3\mapsto -\frac{1}{3}R3}{\longrightarrow}} \begin{pmatrix} 1 & 1 & -2 & 1 \\ 0 & 1 & 0 & 1 \\ 0 & 0 & 1 & -\frac{1}{3} \\ 0 & 0 & 10 & 4 \end{pmatrix} \underset{R4\mapsto R4-10R3}{\longrightarrow} \begin{pmatrix} 1 & 1 & -2 & 1 \\ 0 & 1 & 0 & 1 \\ 0 & 0 & 1 & -\frac{1}{3} \\ 0 & 0 & 0 & 22/3 \end{pmatrix}$$

Now, this last matrix is upper triangular so its determinant is simply the product of its diagonal entries: $\frac{22}{3}$. To compute C we need to consider the row operations that got us here. Most of them are row replacement operations, which do not change the determinant, but we did rescale rows twice. Thus we have

$$\det(C) = (3)(-3)\frac{22}{3} = -66,$$

which agrees with our answer from the previous part.

2. Let $A = (a_{i,j})_{i,j=1}^n$. From lecture we have the following explicit formula for the determinant:

$$\begin{aligned}\det(A^T) &= \sum_{\sigma \in \text{Perm}(n)} (A^T)_{\sigma(1),1} \cdots (A^T)_{\sigma(n),n} \text{sign}(\sigma) \\ &= \sum_{\sigma \in \text{Perm}(n)} a_{1,\sigma(1)} \cdots a_{n,\sigma(n)} \text{sign}(\sigma).\end{aligned}$$

Using the formula again we have

$$\begin{aligned}\det(A^*) &= \sum_{\sigma \in \text{Perm}(n)} (A^*)_{\sigma(1),1} \cdots (A^*)_{\sigma(n),n} \text{sign}(\sigma) \\ &= \sum_{\sigma \in \text{Perm}(n)} \overline{a_{1,\sigma(1)}} \cdots \overline{a_{n,\sigma(n)}} \text{sign}(\sigma) \\ &= \overline{\sum_{\sigma \in \text{Perm}(n)} a_{1,\sigma(1)} \cdots a_{n,\sigma(n)} \text{sign}(\sigma)} = \overline{\det(A^T)}.\end{aligned}$$

Since $\det(A) = \det(A^T)$, we therefore have $\det(A^*) = \overline{\det(A)}$. □

3. (a) We know $\det(I_n) = 1$, and since the determinant distributes over products we have

$$1 = \det(I_n) = \det(AA^{-1}) = \det(A)\det(A^{-1}).$$

Dividing both sides by $\det(A)$ yields the desired formula. □

(b) Let Q be an invertible matrix such that $A = Q^{-1}BQ$. Then using the previous part we obtain

$$\det(A) = \det(Q^{-1}BQ) = \det(Q^{-1})\det(B)\det(Q) = \frac{1}{\det(Q)}\det(B)\det(Q) = \det(B).$$

□

(c) Let k be such that $A^k = O$. Note that O is triangular with all zero diagonal entries, so $\det(O) = 0$. So we have

$$0 = \det(O) = \det(A^k) = \det(A)\det(A^{k-1}) = \det(A)^2\det(A^{k-2}) = \cdots = \det(A)^k.$$

So $\det(A)^k = 0$, which implies $\det(A) = 0$. □

(d) We have

$$1 = \det(I_n) = \det(A^TA) = \det(A^T)\det(A) = \det(A)^2.$$

Since the entries of A are real, $\det(A) \in \mathbb{R}$. Thus it is a real number whose square is one, which implies $\det(A)$ is either 1 or -1. □

(e) Using Exercise 4 we have

$$1 = \det(I_n) = \det(A^*A) = \det(A^*)\det(A) = \overline{\det(A)}\det(A) = |\det(A)|^2.$$

So $|\det(A)|^2 = 1$, which implies $|\det(A)| = 1$. □

4. (a) Expanding along the second row we have

$$\det(A) = 0 + (-1)^{2+2}(-1)(2 - (-2)) + 0 = -4.$$

Since this is non-zero, we know A is invertible.

(b) To apply Cramer's rule, we must consider the following matrices, which we obtain from A by replacing each of its columns with $(-4, 0, 8)^T$:

$$B_1 := \begin{pmatrix} -4 & 0 & -2 \\ 0 & -1 & 0 \\ 8 & 1 & 1 \end{pmatrix}, \qquad B_2 := \begin{pmatrix} 2 & -4 & -2 \\ 0 & 0 & 0 \\ 1 & 8 & 1 \end{pmatrix}, \qquad B_3 := \begin{pmatrix} 2 & 0 & -4 \\ 0 & -1 & 0 \\ 1 & 1 & 8 \end{pmatrix}.$$

Each of their determinants are most easily computed by expanding along the second row:

$$\begin{aligned}\det(B_1) &= 0 + (-1)^{2+2}(-1)(-4-(-16)) + 0 = -12\\ \det(B_2) &= 0+0+0 = 0\\ \det(B_3) &= 0 + (-1)^{2+2}(-1)(16-(-4)) = -20\end{aligned}$$

Recalling $\det(A) = -4$ from the previous part, we obtain

$$\mathbf{x} = \begin{pmatrix} \frac{-12}{-4} \\ \frac{0}{-4} \\ \frac{-20}{-4} \end{pmatrix} = \begin{pmatrix} 3 \\ 0 \\ 5 \end{pmatrix}.$$

(c) We first compute all of the cofactors of A

$$\begin{array}{lll} C_{1,1} = -1 & C_{1,2} = 0 & C_{1,3} = 1 \\ C_{2,1} = -2 & C_{2,2} = 4 & C_{2,3} = -2 \\ C_{3,1} = -2 & C_{3,2} = 0 & C_{3,3} = -2 \end{array}$$

Thus

$$A^{-1} = \frac{1}{\det(A)} C^T = \frac{1}{-4}\begin{pmatrix} -1 & -2 & -2 \\ 0 & 4 & 0 \\ 1 & -2 & -2 \end{pmatrix} = \begin{pmatrix} 1/4 & 1/2 & 1/2 \\ 0 & -1 & 0 \\ -1/4 & 1/2 & 1/2 \end{pmatrix}.$$

5. (a) (i) $\det\begin{pmatrix} 1-z & 1 \\ -1 & 3-z \end{pmatrix} = (1-z)(3-z)+1 = z^2-4z+4 = (z-2)^2.$

(ii) $\sigma(A) = \{2\}$

(iii) We do row operations for $(A - 2I \mid \mathbf{0})$:

$$\left(\begin{array}{cc|c} -1 & 1 & 0 \\ -1 & 1 & 0 \end{array}\right) \to \left(\begin{array}{cc|c} 1 & -1 & 0 \\ 0 & 0 & 0 \end{array}\right),$$

which yields a solution of $\mathbf{x} = x_2(1,1)^T$, so that $(1,1)^T$ is a basis for the eigenspace $\ker(A-2I)$.

(b) (i) $\det\begin{pmatrix} 1-z & 1 \\ 1 & -z \end{pmatrix} = (1-z)(-z) - 1 = z^2 - z - 1.$

(ii) We invoke the quadratic formula to find the roots of the above polynomial:

$$\sigma(A) = \left\{ \frac{1-\sqrt{5}}{2}, \frac{1+\sqrt{5}}{2} \right\}$$

We will denote these eigenvalues $\tau := \frac{1-\sqrt{5}}{2}$ and $\varphi := \frac{1+\sqrt{5}}{2}$. ($\varphi$ is sometimes called the golden ratio.)

(iii) During our row operations we will use the fact that τ and φ are both solutions to $z^2-z-1=0$, which implies they also satisfy $1+z-z^2=0$. For $(A - \tau I \mid \mathbf{0})$ we have

$$\left(\begin{array}{cc|c} 1-\tau & 1 & 0 \\ 1 & -\tau & 0 \end{array}\right) \overset{R1\leftrightarrow R2}{\to} \left(\begin{array}{cc|c} 1 & -\tau & 0 \\ 1-\tau & 1 & 0 \end{array}\right)$$
$$\overset{R2\mapsto R2-(1-\tau)R1}{\to} \left(\begin{array}{cc|c} 1 & -\tau & 0 \\ 0 & 1+\tau(1-\tau) & 0 \end{array}\right) = \left(\begin{array}{cc|c} 1 & -\tau & 0 \\ 0 & 0 & 0 \end{array}\right).$$

So the solution is $\mathbf{x} = x_2(\tau,1)^T$, so that $(\tau,1)^T$ is a basis for the eigenspace $\ker(A-\tau I)$. Since φ satisfies the same polynomial equation, the same row operations with φ swapped out for τ yield a solution of $\mathbf{x} = x_2(\varphi,1)^T$ for $(A-\varphi I)\mathbf{x} = \mathbf{0}$. Thus $(\varphi,1)^T$ is a basis for the eigenspace $\ker(A-\varphi I)$.

(c) (i) We will do row and column operations in order to simplify the computation of the determinant in the definition of the characteristic polynomial:

$$\begin{aligned}\det\begin{pmatrix} 1-z & 3 & 3 \\ -3 & -5-z & -3 \\ 3 & 3 & 1-z \end{pmatrix} &= \det\begin{pmatrix} 1-z & 3 & 3 \\ 0 & -2-z & -2-z \\ 3 & 3 & 1-z \end{pmatrix} \qquad (R2 \mapsto R2 + R3) \\ &= \det\begin{pmatrix} 1-z & 0 & 3 \\ 0 & 0 & -2-z \\ 3 & 2+z & 1-z \end{pmatrix}. \qquad (C2 \mapsto C2 - C3)\end{aligned}$$

Now, to compute this determinant we use cofactor expansion along the second row:

$$\det\begin{pmatrix} 1-z & 0 & 3 \\ 0 & 0 & -2-z \\ 3 & 2+z & 1-z \end{pmatrix} = -0 + 0 - (-2-z)((1-z)(2+z) - 0) = -(z+2)^2(z-1)$$

(ii) $\sigma(A) = \{-2, 1\}$. (Note that $m_{-2}(A) = 2$.)

(iii) For $(A - (-2)I \mid \mathbf{0})$ we have

$$\left(\begin{array}{ccc|c} 3 & 3 & 3 & 0 \\ -3 & -3 & -3 & 0 \\ 3 & 3 & 3 & 0 \end{array}\right) \to \left(\begin{array}{ccc|c} 1 & 1 & 1 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{array}\right),$$

which yields a solution of

$$\mathbf{x} = \begin{pmatrix} -x_2 - x_3 \\ x_2 \\ x_3 \end{pmatrix} = x_2\begin{pmatrix} -1 \\ 1 \\ 0 \end{pmatrix} + x_3\begin{pmatrix} -1 \\ 0 \\ 1 \end{pmatrix}.$$

Thus $(-1,1,0)^T, (-1,0,1)^T$ is a basis for the eigenspace $\ker(A - (-2)I)$.
For $(A - 1I \mid \mathbf{0})$ we have

$$\begin{aligned}&\left(\begin{array}{ccc|c} 0 & 3 & 3 & 0 \\ -3 & -6 & -3 & 0 \\ 3 & 3 & 0 & 0 \end{array}\right) \xrightarrow{R2 \mapsto R2 + R1 + R3} \left(\begin{array}{ccc|c} 0 & 3 & 3 & 0 \\ 0 & 0 & 0 & 0 \\ 3 & 3 & 0 & 0 \end{array}\right) \underset{R3 \mapsto \frac{1}{3}R3}{\xrightarrow{R1 \mapsto \frac{1}{3}R1}} \left(\begin{array}{ccc|c} 0 & 1 & 1 & 0 \\ 0 & 0 & 0 & 0 \\ 1 & 1 & 0 & 0 \end{array}\right) \\ &\underset{R1 \mapsto R3 - R1}{\xrightarrow{R1 \leftrightarrow R2}} \left(\begin{array}{ccc|c} 0 & 0 & 0 & 0 \\ 0 & 1 & 1 & 0 \\ 1 & 0 & -1 & 0 \end{array}\right) \xrightarrow{R3 \leftrightarrow R1} \left(\begin{array}{ccc|c} 1 & 0 & -1 & 0 \\ 0 & 1 & 1 & 0 \\ 0 & 0 & 0 & 0 \end{array}\right),\end{aligned}$$

which yields a solution of $\mathbf{x} = x_3(1,-1,1)^T$. Thus $(1,-1,1)^T$ is a basis for the eigenspace $\ker(A - 1I)$.

6. We first compute the characteristic polynomial:

$$\det\begin{pmatrix} \cos(\theta) - z & -\sin(\theta) \\ \sin(\theta) & \cos(\theta) - z \end{pmatrix} = (\cos(\theta) - z)^2 + \sin^2(\theta) = \cos^2(\theta) - 2\cos(\theta)z + z^2 + \sin^2(\theta).$$

Recall that $\sin^2(\theta) + \cos^2(\theta) = 1$. So the above polynomial simplies to $z^2 - 2\cos(\theta)z + 1$. Using the quadratic formula, the spectrum is

$$\sigma(R_\theta) = \left\{\frac{2\cos(\theta) \pm \sqrt{4\cos^2(\theta) - 4}}{2}\right\} = \left\{\cos(\theta) \pm \sqrt{\cos^2(\theta) - 1}\right\}.$$

Now, these values will be real numbers provided the expression under the square root is non-negative: $\cos^2(\theta) - 1 \geq 0$. But recall that $-1 \leq \cos(\theta) \leq 1$, and so $\cos^2(\theta) \leq 1$. Thus $\cos^2(\theta) - 1 \geq 0$ can only hold if $\cos^2(\theta) - 1 = 0$. This is equivalent ot $\cos(\theta) = \pm 1$, which occurs at $\theta = 0, \pi$. Note that for these values of θ we have $R_\theta = I_2, -I_2$ respectively. □